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ON S-3 LIKE FOUR-DIMENSIONAL FINSLER SPACES

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Abstract. In 1977, M. Matsumoto and R. Miron [9] constructed an orthonormal frame for an n-dimensional Finsler space, called 'Miron frame'. The present authors [1, 2, 3, 10, 11] discussed four-dimensional Finsler spaces equipped with such frame. M. Matsumoto [7, 8] proved that in a three-dimensional Berwald space, all the main scalars are h-covariant constants and the h-connection vector vanishes. He also proved that in a three-dimensional Finsler space satisfying T-condition, all the main scalars are functions of position only and the v-connection vector vanishes [6, 7]. The purpose of the present paper is to generalize these results for an S-3 like four-dimensional Finsler space.

1. Preliminaries

Let M^4 be a four-dimensional smooth manifold and $F^4 = (M^4, L)$ be a four-dimensional Finsler space equipped with a metric function $L(x, y)$ on M^4. The normalized supporting element, the metric tensor, the angular metric tensor and Cartan tensor are defined by $l_i = \dot\partial_i L$, $g_{ij} = \frac{1}{2}\dot\partial_i\dot\partial_j L^2$, $h_{ij} = L\dot\partial_i\dot\partial_j L$ and $C_{ijk} = \frac{1}{2}\dot\partial_k g_{ij}$ respectively. The torsion vector C^i is defined by $C^i = C^i_{jk}g^{jk}$. Throughout this paper, we use the symbols $\dot\partial_i$ and ∂_i for $\partial/\partial y^i$ and $\partial/\partial x^i$ respectively. The Cartan connection in the Finsler space is given as $C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$. The h- and v-covariant derivatives of a covariant vector $X_i(x, y)$ with respect to the Cartan connection are given by

$$X_{i|j} = \partial_j X_i - (\dot\partial_h X_i)G^h_j - F^r_{ij}X_r, \tag{1.1}$$

and

$$X_i|_j = \dot\partial_j X_i - C^r_{ij}X_r. \tag{1.2}$$

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The Miron frame for a four-dimensional Finsler space is constructed by the unit vectors $(e^i_{1)}, e^i_{2)}, e^i_{3)}, e^i_{4)})$. The first vector $e^i_{1)}$ is the normalized supporting element l^i and the second $e^i_{2)}$ is the normalized torsion vector $m^i = C^i/\widetilde{c}$, where $\widetilde{c}$ is the length of the torsion vector C^i. The third $e^i_{3)} = n^i$ and the fourth $e^i_{4)} = p^i$ are constructed by the method of Matsumoto and Miron [9]. With respect to this frame, the scalar components of an arbitrary tensor T^i_j are defined by

$$T_{\alpha\beta} = T^i_j e_{\alpha)i} e^j_{\beta)}. \tag{1.3}$$

From this, we get

$$T^i_j = T_{\alpha\beta} e^i_{\alpha)} e_{\beta)j}, \tag{1.4}$$

where summation convention is also applied to Greek indices. The scalar components of the metric tensor g_{ij} are $\delta_{\alpha\beta}$. Therefore we get

$$g_{ij} = l_i l_j + m_i m_j + n_i n_j + p_i p_j. \tag{1.5}$$

Let $H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}/L$ be scalar components of the h- and v-covariant derivatives $e^i_{\alpha)|j}$ and $e^i_{\alpha)}|_j$ respectively of the vectors $e^i_{\alpha)}$, then

$$e^i_{\alpha)|j} = H_{\alpha)\beta\gamma} e^i_{\beta)} e_{\gamma)j}, \tag{1.6}$$

and

$$L e^i_{\alpha)}|_j = V_{\alpha)\beta\gamma} e^i_{\beta)} e_{\gamma)j}. \tag{1.7}$$

$H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}$ are called h- and v-connection scalars respectively and are positively homogeneous of degree 0 in y.

Orthogonality of the Miron frame yields

$$H_{\alpha)\beta\gamma} = -H_{\beta)\alpha\gamma} \text{ and } V_{\alpha)\beta\gamma} = -V_{\beta)\alpha\gamma}.$$

Also we have $H_{1)\beta\gamma} = 0$ and $V_{1)\beta\gamma} = \delta_{\beta\gamma} - \delta_{1\beta}\delta_{1\gamma}$ [7].

Now we define Finsler vector fields :

$$h_i = H_{2)3\gamma} e_{\gamma)i},\ j_i = H_{4)2\gamma} e_{\gamma)i},\ k_i = H_{3)4\gamma} e_{\gamma)i},$$

and

$$u_i = V_{2)3\gamma} e_{\gamma)i},\ v_i = V_{4)2\gamma} e_{\gamma)i},\ w_i = V_{3)4\gamma} e_{\gamma)i}.$$

The vector fields h_i, j_i, k_i are called h-connection vectors and the vector fields u_i, v_i, w_i are called v-connection vectors. The scalars $H_{2)3\gamma}$, $H_{4)2\gamma}$, $H_{3)4\gamma}$ and $V_{2)3\gamma}$, $V_{4)2\gamma}$, $V_{3)4\gamma}$ are considered as the scalar components h_γ, j_γ, k_γ and u_γ, v_γ, w_γ of the h- and v-connection vectors respectively with respect to the orthonormal frame.

Let $C_{\alpha\beta\gamma}$ are the scalar components of LC_{ijk} then

$$LC_{ijk} = C_{\alpha\beta\gamma} e_{\alpha)i} e_{\beta)j} e_{\gamma)k}. \tag{1.8}$$

The main scalars of a four-dimensional Finsler space are given by [1, 3, 11]

$$C_{222} = A,\ C_{233} = B,\ C_{244} = C,\ C_{322} = D,$$
$$C_{333} = E,\ C_{422} = F,\ C_{433} = G,\ C_{234} = H.$$

We also have $C_{344} = -(D + E)$, $C_{444} = -(F + G)$ and

$$A + B + C = L\widetilde{c}. \tag{1.9}$$

$L\widetilde{c}$ is called the unified main scalar.

Taking h-covariant differentiation of (1.4), we get

$$T^i_{j|k} = (\delta_k T_{\alpha\beta}) e^i_{\alpha)} e_{\beta)j} + T_{\alpha\beta} e^i_{\alpha)|k} e_{\beta)j} + T_{\alpha\beta} e^i_{\alpha)} e_{\beta)j|k}, \tag{1.10}$$

where $\delta_k = \partial_k - G^r_k \dot{\partial}_r$. If $T_{\alpha\beta,\gamma}$ are scalar components of $T^i_{j|k}$, i.e.

$$T^i_{j|k} = T_{\alpha\beta,\gamma} e^i_{\alpha)} e_{\beta)j} e_{\gamma)k}, \tag{1.11}$$

then we obtain

$$T_{\alpha\beta,\gamma} = (\delta_k T_{\alpha\beta}) e^k_{\gamma)} + T_{\mu\beta} H_{\mu)\alpha\gamma} + T_{\alpha\mu} H_{\mu)\beta\gamma}. \tag{1.12}$$

Similarly, if $T_{\alpha\beta;\gamma}$ are scalar components of $LT^i_j|_k$, i.e.

$$LT^i_j|_k = T_{\alpha\beta;\gamma} e^i_{\alpha)} e_{\beta)j} e_{\gamma)k}, \tag{1.13}$$

then we get

$$T_{\alpha\beta;\gamma} = L(\dot{\partial}_k T_{\alpha\beta}) e^k_{\gamma)} + T_{\mu\beta} V_{\mu)\alpha\gamma} + T_{\alpha\mu} V_{\mu)\beta\gamma}. \tag{1.14}$$

The scalar components $T_{\alpha\beta,\gamma}$ and $T_{\alpha\beta;\gamma}$ are respectively called h- and v-scalar derivatives of scalar components $T_{\alpha\beta}$ of T.

2. T-condition

The tensor T_{hijk} defined by

$$T_{hijk} = LC_{hij}|_k + C_{hij} l_k + C_{hik} l_j + C_{hkj} l_i + C_{kij} l_h, \tag{2.1}$$

is called T-tensor in a Finsler space. It is completely symmetric in its indices. A Finsler space is said to satisfy T-condition if the T-tensor T_{hijk} vanishes identically.

We are concerned with the tensor $C_{hij}|_k$. From (1.8) and (1.13), it follows that

$$L^2 C_{hij}|_k + LC_{hij} l_k = C_{\alpha\beta\gamma;\delta} e_{\alpha)h} e_{\beta)i} e_{\gamma)j} e_{\delta)k},$$

which implies

$$L^2 C_{hij}|_k = (C_{\alpha\beta\gamma;\delta} - C_{\alpha\beta\gamma}\delta_{1\delta}) e_{\alpha)h} e_{\beta)i} e_{\gamma)j} e_{\delta)k}. \tag{2.2}$$

Therefore the scalar components $T_{\alpha\beta\gamma\delta}$ of LT_{hijk} are given by

$$T_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma;\delta} + \delta_{1\alpha} C_{\beta\gamma\delta} + \delta_{1\beta} C_{\alpha\gamma\delta} + \delta_{1\gamma} C_{\alpha\beta\delta}.$$

From $T_{hijk}l^k = 0$, we have $T_{\alpha\beta\gamma 1} = 0$. Thus the surviving components $T_{\alpha\beta\gamma\delta}$ are only

$$T_{\alpha\beta\gamma\delta} = C_{\alpha\beta\gamma;\delta}; \qquad \alpha,\ \beta,\ \gamma,\ \delta = 2, 3, 4. \tag{2.3}$$

Using (1.14), the explicit forms of $C_{\alpha\beta\gamma;\delta}$ are obtained as follows:

$$\begin{cases} a)\ C_{222;\delta} = A_{;\delta} - 3Du_\delta + 3Fv_\delta, \\ b)\ C_{233;\delta} = B_{;\delta} + (2D - E)u_\delta + Gv_\delta - 2Hw_\delta, \\ c)\ C_{244;\delta} = C_{;\delta} + (D + E)u_\delta - (3F + G)v_\delta + 2Hw_\delta, \\ d)\ C_{322;\delta} = D_{;\delta} + (A - 2B)u_\delta + 2Hv_\delta - Fw_\delta, \\ e)\ C_{333;\delta} = E_{;\delta} + 3Bu_\delta - 3Gw_\delta, \\ f)\ C_{422;\delta} = F_{;\delta} - 2Hu_\delta - (A - 2C)v_\delta + Dw_\delta, \\ g)\ C_{433;\delta} = G_{;\delta} + 2Hu_\delta - Bv_\delta + (2D + 3E)w_\delta, \\ h)\ C_{234;\delta} = H_{;\delta} + (F - G)u_\delta - (2D + 3E)v_\delta + (B - C)w_\delta, \\ i)\ C_{344;\delta} = -D_{;\delta} - E_{;\delta} + Cu_\delta - 2Hv_\delta + (F + 3G)w_\delta, \\ j)\ C_{444;\delta} = -F_{;\delta} - G_{;\delta} - 3Cv_\delta - (3D + 3E)w_\delta, \\ k)\ C_{1\beta\gamma;\delta} = -C_{\beta\gamma\delta}, \end{cases} \tag{2.4}$$

where $A_{;\delta} = L(\dot{\partial}_k A)e^k_{\delta)}$. From (1.9) and (2.4), we get

$$\begin{cases} C_{222;\delta} + C_{233;\delta} + C_{244;\delta} = A_{;\delta} + B_{;\delta} + C_{;\delta} = (A + B + C)_{;\delta} = (L\widetilde{c})_{;\delta}, \\ C_{322;\delta} + C_{333;\delta} + C_{344;\delta} = L\widetilde{c}\, u_\delta, \\ C_{422;\delta} + C_{433;\delta} + C_{444;\delta} = -L\widetilde{c}\, v_\delta. \end{cases} \tag{2.5}$$

Thus from (2.3), (2.4) and (2.5), we have

Theorem 2.1. *In a four-dimensional Finsler space satisfying T-condition, the v-connection vectors u_i and v_i vanish identically. Also main scalar A and the unified main scalar $L\widetilde{c}$ are v-covariant constants (functions of position only). Furthermore, if v-connection vector w_i vanishes then all the main scalars are functions of position only.*

3. Berwald space

A Berwald space is characterized by $C_{hij|k} = 0$. From (1.8) and (1.11), it follows that

$$LC_{hij|k} = C_{\alpha\beta\gamma,\delta} e_{\alpha)h} e_{\beta)i} e_{\gamma)j} e_{\delta)k}, \tag{3.1}$$

where $C_{\alpha\beta\gamma,\delta}$ are given by

$$C_{\alpha\beta\gamma,\delta} = (\delta_k C_{\alpha\beta\gamma})e^k_{\delta)} + C_{\mu\beta\gamma}H_{\mu)\alpha\delta} + C_{\alpha\mu\gamma}H_{\mu)\beta\delta} + C_{\alpha\beta\mu}H_{\mu)\gamma\delta}.$$

The explicit forms of $C_{\alpha\beta\gamma,\delta}$ are obtained as follows:

$$
(3.2)\qquad \begin{cases}
a)\ C_{222,\delta} = A_{,\delta} - 3Dh_\delta + 3Fj_\delta, \\
b)\ C_{233,\delta} = B_{,\delta} + (2D - E)h_\delta + Gj_\delta - 2Hk_\delta, \\
c)\ C_{244,\delta} = C_{,\delta} + (D + E)h_\delta - (3F + G)j_\delta + 2Hk_\delta, \\
d)\ C_{322,\delta} = D_{,\delta} + (A - 2B)h_\delta + 2Hj_\delta - Fk_\delta, \\
e)\ C_{333,\delta} = E_{,\delta} + 3Bh_\delta - 3Gk_\delta, \\
f)\ C_{422,\delta} = F_{,\delta} - 2Hh_\delta - (A - 2C)j_\delta + Dk_\delta, \\
g)\ C_{433,\delta} = G_{,\delta} + 2Hh_\delta - Bj_\delta + (2D + 3E)k_\delta, \\
h)\ C_{234,\delta} = H_{,\delta} + (F - G)h_\delta - (2D + 3E)j_\delta + (B - C)k_\delta, \\
i)\ C_{344,\delta} = -D_{,\delta} - E_{,\delta} + Ch_\delta - 2Hj_\delta + (F + 3G)k_\delta, \\
j)\ C_{444,\delta} = -F_{,\delta} - G_{,\delta} - 3Cj_\delta - (3D + 3E)k_\delta, \\
k)\ C_{1\beta\gamma,\delta} = 0.
\end{cases}
$$

From (1.9) and (3.2), we get

$$
(3.3)\qquad \begin{aligned}
&C_{322,\delta} + C_{333,\delta} + C_{344,\delta} = (A + B + C)h_\delta = L\widetilde{c}h_\delta, \\
&C_{422,\delta} + C_{433,\delta} + C_{444,\delta} = -(A + B + C)j_\delta = -L\widetilde{c}j_\delta, \\
&C_{222,\delta} + C_{233,\delta} + C_{244,\delta} = (A_{,\delta} + B_{,\delta} + C_{,\delta}) = (A + B + C)_{,\delta}.
\end{aligned}
$$

Thus from (3.2) and (3.3), we have:

Theorem 3.1 ([11])**.** *In a four-dimensional Berwald space, the* h*-connection vectors h_i and j_i vanish identically. Also main scalar A and the unified main scalar $L\widetilde{c}$ are* h*-covariant constants. Furthermore, if* h*-connection vector k_i vanishes then all the main scalars are* h*-covariant constants.*

4. *v*-Curvature tensor

The *v*-curvature tensor is defined by

$$
(4.1)\qquad S_{hijk} = C^r_{hk} C_{ijr} - C^r_{hj} C_{ikr}.
$$

The scalar components $S_{\alpha\beta\gamma\delta}$ of $L^2 S_{hijk}$ are given by

$$
(4.2)\qquad L^2 S_{hijk} = S_{\alpha\beta\gamma\delta}\, e_{\alpha)h} e_{\beta)i} e_{\gamma)j} e_{\delta)k}.
$$

Since S_{hijk} is skew-symmetric in h and i as well as j and k and $S_{0ijk} = S_{hi0k} = 0$, the surviving independent components of $S_{\alpha\beta\gamma\delta}$ are only six, which are given by

$$\begin{aligned}
S_{2323} &= C_{23\mu}C_{\mu32} - C_{22\mu}C_{\mu33} = D^2 + B^2 + H^2 - AB - DE - FG,\\
S_{2424} &= C_{24\mu}C_{\mu42} - C_{22\mu}C_{\mu44} = 2F^2 + H^2 + C^2 + D^2 - AC + DE + FG,\\
S_{3434} &= C_{34\mu}C_{\mu34} - C_{33\mu}C_{\mu44} = H^2 + 2G^2 + D^2 + 2E^2 + 3DE - BC + FG,\\
S_{2334} &= C_{24\mu}C_{\mu33} - C_{23\mu}C_{\mu34} = BF + 2EH + CG - BG,\\
S_{2434} &= C_{24\mu}C_{\mu34} - C_{23\mu}C_{\mu44} = 2FH + 2GH - 2CD - CE + BD + BE,\\
S_{2324} &= C_{24\mu}C_{\mu23} - C_{22\mu}C_{\mu34} = 2FD + BH + CH - AH - DG + EF.
\end{aligned}$$

A Finsler space $F^n (n \geq 4)$ is called S-3 like, if there exists a scalar S such that the curvature tensor S_{hijk} of F^n is written in the form

$$L^2 S_{hijk} = S(h_{hj}h_{ik} - h_{hk}h_{ij}). \tag{4.3}$$

Let us consider a four-dimensional S-3 like Finsler space. Then

$$\begin{aligned}
L^2 S_{hijk} &= S(h_{hj}h_{ik} - h_{hk}h_{ij})\\
&= S[(m_h m_j + n_h n_j + p_h p_j)(m_i m_k + n_i n_k + p_i p_k)\\
&\qquad - (m_h m_k + n_h n_k + p_h p_k)(m_i m_j + n_i n_j + p_i p_j)]\\
&= S[(m_h n_i - m_i n_h)(m_j n_k - m_k n_j) + (m_h p_i - m_i p_h)(m_j p_k - m_k p_j)\\
&\qquad + (n_h p_i - n_i p_h)(n_j p_k - n_k p_j)].
\end{aligned}$$

This implies that the scalar components are

$$S_{2323} = S, \quad S_{2324} = 0, \quad S_{2334} = 0, \quad S_{2424} = S, \quad S_{2434} = 0, \quad S_{3434} = S.$$

M. Matsumoto [5] proved that the v-curvature S of an S-3 like Finsler space is function of position only. Therefore in S-3 like four-dimensional Finsler space, six functions $D^2 + B^2 + H^2 - AB - DE - FG$, $2F^2 + H^2 + C^2 + D^2 - AC + DE + FG$, $H^2 + 2G^2 + D^2 + 2E^2 + 3DE - BC + FG$, $BF + 2EH + CG - BG$, $2FH + 2GH - 2CD - CE + BD + BE$ and $2FD + BH + CH - AH - DG + EF$ are functions of position only. In view of theorem 2.1 and equation (1.9), functions A and $A + B + C$ are functions of position only in a four-dimensional Finsler space satisfying T-condition. Thus, in an S-3 like Finsler space satisfying T-condition, eight functions A, $A+B+C$, $D^2+B^2+H^2-AB-DE-FG$, $2F^2+H^2+C^2+D^2-AC+DE+FG$, $H^2+2G^2+D^2+2E^2+3DE-BC+FG$, $BF+2EH+CG-BG$, $2FH+2GH-2CD-CE+BD+BE$ and $2FD+BH+CH-AH-DG+EF$ are functions of position only. These eight functions are clearly independent and therefore the main scalars A, B, C, D, E, F, G and H are functions of position only. Thus, we have:

Theorem 4.1. *In an S-3 like four-dimensional Finsler space satisfying* T-*condition, all the main scalars are functions of position only.*

It is clear from (2.4) that if all the main scalars are functions of position only in a Finsler space satisfying T-condition, then the v-connection vectors u_i, v_i, and w_i vanish. This leads to:

Theorem 4.2. *In an S-3 like four-dimensional Finsler space satisfying T-condition, the* v*-connection vectors u_i, v_i, and w_i vanish identically.*

A Landsberg space is characterized by $C_{hij|k} = C_{hik|j}$. H. Yasuda [12] proved that in an S-3 like Landsberg space, the v-curvature S is constant. In view of this result, in an S-3 like four-dimensional Landsberg space, six independent functions $D^2+B^2+H^2-AB-DE-FG$, $2F^2+H^2+C^2+D^2-AC+DE+FG$, $H^2+2G^2+D^2+2E^2+3DE-BC+FG$, $BF+2EH+CG-BG$, $2FH+2GH-2CD-CE+BD+BE$ and $2FD+BH+CH-AH-DG+EF$ are constants. Since every Berwald space is a Landsberg space, these six functions are constant in an S-3 like Berwald space. From theorem 3.1 and equation (1.9), functions A and $A+B+C$ are h-covariant constants in a four-dimensional Berwald space. Therefore in an S-3 like Berwald space, eight independent functions A, $A+B+C$, $D^2+B^2+H^2-AB-DE-FG$, $2F^2+H^2+C^2+D^2-AC+DE+FG$, $H^2+2G^2+D^2+2E^2+3DE-BC+FG$, $BF+2EH+CG-BG$, $2FH+2GH-2CD-CE+BD+BE$ and $2FD+BH+CH-AH-DG+EF$ are h-covariant constants and therefore the main scalars A, B, C, D, E, F, G and H are h-covariant constants.
Thus, we have:

Theorem 4.3. *In an S-3 like four-dimensional Berwald space, all the main scalars are h-covariant constants.*

It is clear from (3.2) that if all the main scalars are h-covariant constants in a Berwald space, then the h-connection vectors h_i, j_i and k_i vanish.
This leads to:

Theorem 4.4. *In an S-3 like four-dimensional Berwald space, the* h*-connection vectors h_i, j_i and k_i vanish identically.*

In view of theorems 4.1, 4.2, 4.3 and 4.4, we can say

Theorem 4.5. *In an S-3 like four-dimensional Berwald space satisfying T-condition, all the main scalars are constants and the h- and* v*-connection vectors vanish.*

F. Ikeda [4] proved that a Landsberg space satisfying T-condition is a Berwald space. Thus, we may conclude:

Theorem 4.6. *In an S-3 like four-dimensional Landsberg space satisfying T-condition, all the main scalars are constants and the h- and* v*-connection vectors vanish.*

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