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# NOTE ON THE RIEMANNIAN GEOMETRY OF IMAGE PROCESSING

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*Dedicated to the distinguished Finsler geometer Prof. Dr. Lajos Tamássy on his 90th birthday*

Abstract. A brief review is given of the Riemannian geometry of image processing. Methods are exposited for treating images as Riemannian manifolds in general embedding spaces for application to linear and nonlinear scale-space image processing. In particular, the Beltrami operator approach to obtaining a nonlinear differential equation for processing of gray-scale images is treated in some detail. A minor error is found in earlier work appearing in the literature.

## 1. Introduction

In the Riemannian geometry of image processing, differential geometric methods are exploited for treating images as Riemannian manifolds in general Euclidean embedding spaces for applications in linear and nonlinear scale-space image processing, and for developing improved mathematical procedures for enhancement, smoothing, and segmentation of multi-spectral and texture images [4]-[2]. In the present work, the Beltrami operator approach is reviewed in some detail for obtaining a nonlinear differential equation for processing of gray-scale images. A minor error is found in earlier work appearing in the literature [3]

## 2. Gray-scale image processing

A gray-scale image can be represented by a two-dimensional curved surface in which the light intensity $I(x, y)$ is given as a function of coordinates $x$ and $y$. The surface is referred to as the image manifold $\Sigma$ which is taken to be a Riemannian manifold embedded in a higher dimensional spatial-feature manifold $M$. One has the following embedding map:

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(1) $$(X^1(x^1,x^2)=x^1, X^2(x^1,x^2)=x^2, X^3(x^1,x^2)=I(x^1,x^2)) \equiv (x^1,x^2,I(x^1,x^2)) \equiv (x,y,I(x,y))).$$

Here, $X^i, i=1,2,3$ are the embedding-space coordinates, and $x^1 \equiv x$ and $x^2 \equiv y$ are the image-space coordinates. The line element in the isometric three-dimensional embedding space is

(2) $$ds^2 = \eta_{ij} dX^i dX^j = \eta_{ij}\partial_\mu X^i \partial_\nu X^j d\sigma^\mu d\sigma^\nu = g_{\mu\nu} dx^\mu dx^\nu, \quad \{i,j=1-3\},\ \{\mu,\nu=1,2\},$$

in which $\eta_{ij}$ is the metric in the embedding-space manifold, $\{\partial_\mu\} \equiv \{\partial_1,\partial_2\} \equiv \{\partial/\partial x^1, \partial/\partial x^2\} = \{\partial/\partial x, \partial/\partial y\}$, and $g_{\mu\nu}$ is the metric in the two-dimensional image-space manifold. For the rectilinear flat Euclidean embedding-space manifold which is treated below, one has

(3) $$\eta_{ij} = \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases},$$

where $\delta_{ij}$ is the Kronecker delta. In this case then the image-space metric is

(4) $$g_{\mu\nu} = \delta_{ij}\partial_\mu X^i \partial_\nu X^j = \partial_\mu x^1 \partial_\nu x^1 + \partial_\mu x^2 \partial_\nu x^2 + \partial_\mu I(x,y)\partial_\nu I(x,y).$$

Then

(5) $$g_{11} = \partial_1 x^1 \partial_1 x^1 + \partial_1 x^2 \partial_1 x^2 + \partial_1 I(x,y)\partial_1 I(x,y) = 1 + (\partial_x I)^2,$$

(6) $$g_{12} = g_{21} = \partial_1 x^1 \partial_2 x^1 + \partial_1 x^2 \partial_2 x^2 + \partial_1 I(x,y)\partial_2 I(x,y) = \partial_x I \partial_y I,$$

(7) $$g_{22} = \partial_2 x^1 \partial_2 x^1 + \partial_2 x^2 \partial_2 x^2 + \partial_2 I(x,y)\partial_2 I(x,y) = 1 + (\partial_y I)^2,$$

or in matrix form:

(8) $$[g_{\mu\nu}] = \begin{bmatrix} 1+I_x^2 & I_x I_y \\ I_x I_y & 1+I_y^2 \end{bmatrix},$$

where $I_x \equiv \frac{\partial I}{\partial x}$ and $I_y \equiv \frac{\partial I}{\partial y}$.

Given an input image $I(0)$ for scale parameter $t=0$, image information is to be processed locally and forwarded to $t+dt$. The output image $I(t)$ will be a solution to a differential equation of the form:

(9) $$\partial_t I = OI,$$

where $O$ is a local (generally nonlinear) differential operator, and the initial condition is given by the input image $I(0)$. The measure on the space of embedding manifolds is given by the following functional integral [4], [3]:

(10) $$S\left[X^i, g_{\mu\nu}, h_{ij}\right] = \int d^2x \sqrt{g} g^{\mu\nu}\partial_\mu X^i \partial_\nu X^j h_{ij},$$

where $d^2x \equiv dxdy$, and for a given gray-scale image in a rectilinear embedding space, one has Eq. (3).

In the following, the normal vector to the two-dimensional image surface is needed. A two-dimensional surface embedded in three-dimensional rectangular Cartesian space with coordinates $X, Y, Z$ can be represented by

$$G(X.Y.Z) = 0. \tag{11}$$

The normal to a surface plane at any point is given by

$$\left(\partial G/\partial x, \partial G/\partial y, \partial G/\partial z\right). \tag{12}$$

The gray-scale image, expressed in terms of the image-space coordinates, is given by

$$I = I(x, y), \tag{13}$$

and representing the image as a two-dimensional surface embedded in three-dimensional space with $(X, Y, Z) \equiv (x, y, I)$, one has

$$G(X, Y, Z) = I - I(x, y) = 0. \tag{14}$$

Therefore, the normal to the surface is given by the following vector:

$$(\partial G/\partial x, \partial G/\partial y, \partial G/\partial z) = (-I_x, -I_y, \partial I/\partial I) = (-I_x, -I_y, 1). \tag{15}$$

One proceeds by varying the functional integral Eq. (10) with respect to the embedding, keeping the image metric $g_{\mu\nu}$ constant since it is completely determined by the image $I(x, y)$, and is given by Eq. (8). One easily obtains the corresponding inverse of the image metric, namely,

$$[g^{\mu\nu}] = \frac{1}{g}\begin{bmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{bmatrix} = \frac{1}{g}\begin{bmatrix} 1+I_y^2 & -I_xI_y \\ -I_xI_y & 1+I_x^2 \end{bmatrix}, \tag{16}$$

where the determinant of the image metric is given by

$$g = 1 + I_x^2 + I_y^2. \tag{17}$$

Also, one has the following:

$$\partial_x\sqrt{g} \;=\; \frac{1}{\sqrt{g}}\left(I_xI_{xx} + I_yI_{yx}\right) \tag{18}$$

and

$$\partial_y\sqrt{g} \;=\; \frac{1}{\sqrt{g}}\left(I_xI_{xy} + I_yI_{yy}\right). \tag{19}$$

Next, varying the functional integral Eq. (10) with respect to the embedding. One has

$$\frac{\delta}{\delta X^k(x')}S = \int d^2x\sqrt{g}g^{\mu\nu}\left[\frac{\delta}{\delta X^k}\left(\partial_\mu X^i\right)\partial_\nu X^j h_{ij} + \partial_\mu X^i\frac{\delta}{\delta X^k}\left(\partial_\nu X^j\right)h_{ij} \right. \\ \left. + \partial_\mu X^i\partial_\nu X^j\frac{\delta}{\delta X^k}h_{ij}\right], \tag{20}$$

in which the metric $g_{\mu\nu}$ in the image space is given and fixed, and the metric in the embedding space $h_{ij}$ is taken to be constant. Equation (20) becomes

$$\frac{\delta}{\delta X^k(x')}S = \int d^2x\sqrt{g}g^{\mu\nu}\partial_\mu\left(\delta_{ik}\delta^2\left(x-x'\right)\right)\partial_\nu X^j h_{ij} + \partial_\mu X^i\partial_\nu\left(\delta_{kj}\delta^2\left(x-x'\right)\right)h_{ij} + \partial_\mu X^i\partial_\nu X^j\partial_k h_{ij}\delta^2\left(x-x'\right)\Big], \tag{21}$$

where $\delta^2\left(x\right)$ is the two-dimensional Dirac delta function. Integrating by parts, with the support for integration vanishing at the boundaries of the image, Eq. (21) becomes

$$\begin{aligned}\frac{\delta}{\delta X^k(x')}S = \int d^2x\,&\big[\partial_\mu\left\{\sqrt{g}g^{\mu\nu}\partial_\nu X^j h_{kj}\delta^2\left(x-x'\right)\right\}\\ &-\partial_\mu\left\{\sqrt{g}g^{\mu\nu}\partial_\nu X^j h_{kj}\right\}\delta^2\left(x-x'\right)\\ &+\partial_\nu\left\{\sqrt{g}g^{\mu\nu}\partial_\mu X^j h_{kj}\delta^2\left(x-x'\right)\right\}\\ &-\partial_\nu\left\{\sqrt{g}g^{\mu\nu}\partial_\mu X^i h_{ik}\right\}\delta^2\left(x-x'\right)\\ &+\sqrt{g}g^{\mu\nu}\partial_\mu X^i\partial_\nu X^j\partial_k h_{ij}\delta^2\left(\sigma-\sigma'\right)\big]\\ = 0 &- \partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu X^j\right)h_{kj} - \sqrt{g}g^{\mu\nu}\partial_\nu X^j\partial_\mu X^l\partial_l h_{kj}\\ &+0 - \partial_\nu\left(\sqrt{g}g^{\mu\nu}\partial_\mu X^i\right)h_{ik} - \sqrt{g}g^{\mu\nu}\partial_\mu X^i\partial_\nu X^l\partial_l h_{ik}\\ &+\sqrt{g}g^{\mu\nu}\partial_\mu X^i\partial_\nu X^j\partial_k h_{ij}\\ = &-\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu X^j\right)h_{kj} - \sqrt{g}g^{\mu\nu}\partial_\nu X^j\partial_\mu X^l\partial_l h_{kj}\\ &-\partial_\nu\left(\sqrt{g}g^{\mu\nu}\partial_\mu X^j\right)h_{kj} - \sqrt{g}g^{\mu\nu}\partial_\mu X^j\partial_\nu X^l\partial_l h_{jk}\\ &+\sqrt{g}g^{\mu\nu}\partial_\mu X^i\partial_\nu X^j\partial_k h_{ij}.\end{aligned} \tag{22}$$

Next using the symmetries of the image-space and embedding-space metrics, and renaming dummy indices, then Eq. (22) becomes

$$\begin{aligned}\frac{\delta}{\delta X^k(\sigma')}S &= -2\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu X^j\right)h_{kj}\\ &\quad-\sqrt{g}g^{\mu\nu}\partial_\mu X^j\partial_\nu X^l\left(\partial_l h_{kj} + \partial_l h_{kj} - \partial_k h_{jl}\right)\\ &= -2\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu X^j\right)h_{kj}\\ &\quad-\sqrt{g}g^{\mu\nu}\partial_\mu X^j\partial_\nu X^l\left(\partial_l h_{jk} + \partial_j h_{kl} - \partial_k h_{jl}\right)\\ &= -2\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu X^j\right)h_{kj}\\ &\quad-\sqrt{g}g^{\mu\nu}\partial_\mu X^j\partial_\nu X^l\left(\partial_l h_{jk} + \partial_j h_{lk} - \partial_k h_{jl}\right).\end{aligned} \tag{23}$$

Therefore from Eq. (23), with both metrics positive definite, and substituting Eq. (3), one conveniently has

$$\begin{aligned}\frac{1}{2\sqrt{g}}h^{nk}\frac{\delta}{\delta X^k(x')}S &= \frac{1}{2\sqrt{g}}\big[-2\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu X^j\right)h^{nk}h_{kj}\\ &\quad-h^{nk}\left(\partial_l h_{jk} + \partial_j h_{lk} - \partial_k h_{jl}\right)\sqrt{g}g^{\mu\nu}\partial_\mu X^j\partial_\nu X^l\big]\\ &= -\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu X^j\right)\delta^n_j\end{aligned} \tag{24}$$

$$-\frac{1}{2}h^{nk}\left(\partial_l h_{jk}+\partial_j h_{kl}-\partial_k h_{jl}\right)g^{\mu\nu}\partial_\mu X^j\partial_\nu X^l.$$

Next in accord with Eqs. (9) and (24), and following convention, one writes [3]

$$\frac{dX^n}{dt}=-\frac{1}{2\sqrt{g}}h^{nk}\frac{\delta}{\delta X^k(\sigma')}S\ , \tag{25}$$

and substituting Eq. (24) in Eq. (25), one obtains for the scale-space evolution of the image processing in terms of the scale-space parameter $t$:

$$\frac{dX^n}{dt}=\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu X^n\right)+\Gamma^n_{jl}\partial_\mu X^j\partial_\nu X^l\ g^{\mu\nu}, \tag{26}$$

where $\Gamma^n_{jl}$ is the Christoffel symbol of the second kind in the embedding space, namely,

$$\Gamma^n_{jl}=\frac{1}{2}h^{nk}\left(\partial_l h_{jk}+\partial_j h_{lk}-\partial_k h_{jl}\right). \tag{27}$$

For the gray-scale image surface embedded in three-dimensional Euclidean space with rectangular Cartesian coordinates, with the embedding-space metric given by Eq. (3), one sees by substituting Eq. (3) in Eq. (27) that the Christoffel symbols in the embedding space are vanishing, namely,

$$\Gamma^n_{jl}=0. \tag{28}$$

In this case one obtains using Eqs. (26) and (28) the so-called Beltrami operator equation for the evolution of the image processing [3]

$$\frac{dX^n}{dt}=\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu X^n\right),\ n=1,2,3, \tag{29}$$

where it is to be understood that

$$\{\partial_1,\partial_2,\}=\{\partial_x,\partial_y\}. \tag{30}$$

It is well to note that the associated map is harmonic. Then according to Eq. (29) for $X^1=x$, in accord with Eq. (1), one has for the evolution of the $x$ coordinate of the image:

$$\begin{aligned}\frac{dx}{dt}&=\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu X^1\right)\\&=\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu x\right)=\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu1}\partial_1 x\right)\\&=\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu1}\partial_x x\right)\\&=\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu1}\right)=\frac{1}{\sqrt{g}}\left[\partial_x\left(\sqrt{g}g^{11}\right)+\partial_y\left(\sqrt{g}g^{21}\right)\right]\\&=\frac{1}{\sqrt{g}}\left[\partial_x\sqrt{g}g^{11}+\sqrt{g}\partial_x g^{11}+\partial_y\sqrt{g}g^{21}+\sqrt{g}\partial_y g^{21}\right].\end{aligned} \tag{31}$$

Next using Eqs. (16)-(19) in Eq. (31), one obtains

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{\sqrt{g}}\left[\frac{1}{\sqrt{g}}\left(I_xI_{xx}+I_yI_{yx}\right)\left(\frac{1+I_y^2}{g}\right)+\sqrt{g}\partial_x\left(\frac{1+I_y^2}{g}\right)\right.\\ &\quad\left.+\frac{1}{\sqrt{g}}\left(I_xI_{xy}+I_yI_{yy}\right)\left(\frac{-I_xI_y}{g}\right)+\sqrt{g}\partial_y\left(\frac{-I_xI_y}{g}\right)\right]\\ &= \frac{1}{g^{3/2}}\left[\left(I_xI_{xx}+I_yI_{xy}\right)\left(1+I_y^2\right)-2\left(I_xI_{xx}+I_yI_{xy}\right)\left(1+I_y^2\right)\right.\\ &\quad+2gI_yI_{xy}-\left(I_xI_{xy}+I_yI_{yy}\right)I_xI_y\\ &\quad\left.+2\left(I_xI_{xy}+I_yI_{yy}\right)I_xI_y-g\left(I_yI_{xy}+I_xI_{yy}\right)\right]\frac{1}{\sqrt{g}},\end{aligned} \tag{32}$$

or substituting Eq. (17) and simplifying, one obtains

$$\begin{aligned}\frac{dx}{dt} &= \frac{1}{g^{3/2}}\left[\left(I_xI_{xx}+I_yI_{xy}\right)\left(1+I_y^2\right)-2\left(I_xI_{xx}+I_yI_{xy}\right)\left(1+I_y^2\right)\right.\\ &\quad+2\left(1+I_x^2+I_y^2\right)I_yI_{xy}-\left(I_xI_{xy}+I_yI_{yy}\right)I_xI_y+2\left(I_xI_{xy}+I_yI_{yy}\right)I_xI_y\\ &\quad\left.-\left(1+I_x^2+I_y^2\right)\left(I_yI_{xy}+I_xI_{yy}\right)\right]\frac{1}{\sqrt{g}}\\ &= \frac{1}{g^{3/2}}\left[I_xI_{xx}+I_yI_{xy}+I_xI_y^2I_{xx}+I_y^3I_{xy}-2I_xI_{xx}-2I_yI_{xy}\right.\\ &\quad-2I_xI_y^2I_{xx}-2I_y^3I_{xy}\\ &\quad+2I_yI_{xy}+2I_x^2I_yI_{xy}+2I_y^3I_{xy}-I_x^2I_yI_{xy}-I_xI_y^2I_{yy}+2I_x^2I_yI_{xy}\\ &\quad+2I_xI_y^2I_{yy}-I_yI_{xy}\\ &\quad\left.-I_xI_{yy}-I_x^2I_yI_{xy}-I_x^3I_{yy}-I_y^3I_{xy}-I_xI_y^2I_{yy}\right]\frac{1}{\sqrt{g}}\\ &= \frac{1}{g^{3/2}}\left[-I_xI_{xx}-I_xI_y^2I_{xx}+2I_x^2I_yI_{xy}-I_xI_{yy}-I_x^3I_{yy}\right]\frac{1}{\sqrt{g}}\\ &= \frac{1}{g^{3/2}}\left[-I_xI_{xx}\left(1+I_y^2\right)+2I_x^2I_yI_{xy}-I_xI_{yy}\left(1+I_x^2\right)\right]\frac{1}{\sqrt{g}}.\end{aligned} \tag{33}$$

Thus one has for the evolution of the $x$ coordinate of the image surface:

$$\frac{dx}{dt}=\frac{1}{g^{3/2}}\left[I_{xx}\left(1+I_y^2\right)-2I_xI_yI_{xy}+I_{yy}\left(1+I_x^2\right)\right]\frac{(-I_x)}{\sqrt{g}}. \tag{34}$$

Next according to Eq. (29), one has for the evolution of the $X_2 = y$ coordinate of the image:

$$\begin{aligned}\frac{dy}{dt} &= \frac{1}{\sqrt{g}}\left[\partial_x\sqrt{g}g^{12}+\sqrt{g}\partial_xg^{12}+\partial_y\sqrt{g}g^{22}+\sqrt{g}\partial_yg^{22}\right]\\ &= \frac{1}{\sqrt{g}}\left[\frac{1}{\sqrt{g}}\left(I_xI_{xx}+I_yI_{xy}\right)\left(\frac{-I_xI_y}{g}\right)+\sqrt{g}\partial_x\left(\frac{-I_xI_y}{g}\right)\right.\end{aligned} \tag{35}$$

$$+\frac{1}{\sqrt{g}}\left(I_xI_{xy}+I_yI_{yy}\right)\left(\frac{1+I_x^2}{g}\right)+\sqrt{g}\partial_y\left(\frac{1+I_x^2}{g}\right)\Bigg]$$

$$=\frac{1}{\sqrt{g}}\left[\frac{1}{\sqrt{g}}\left(I_xI_{xx}+I_yI_{xy}\right)\left(\frac{-I_xI_y}{g}\right)\right.$$

$$-\frac{\sqrt{g}}{g^2}2\left(I_xI_{xx}+I_yI_{xy}\right)\left(-I_xI_y\right)$$

$$+\frac{1}{\sqrt{g}}\left(-I_{xx}I_y-I_xI_{xy}\right)+\frac{1}{\sqrt{g}}\left(I_xI_{xy}+I_yI_{yy}\right)\left(\frac{1+I_x^2}{g}\right)$$

$$\left.-\frac{\sqrt{g}}{g^2}2\left(I_xI_{xy}+I_yI_{yy}\right)\left(1+I_x^2\right)+\frac{1}{\sqrt{g}}\left(2I_xI_{xy}\right)\right],$$

or

$$\frac{dy}{dt}=\frac{1}{g^{3/2}}\left[\left(I_xI_{xx}+I_yI_{xy}\right)\left(-I_xI_y\right)-2\left(I_xI_{xx}+I_yI_{xy}\right)\left(-I_xI_y\right)\right. \tag{36}$$

$$+\left(1+I_x^2+I_y^2\right)\left(-I_{xx}I_y-I_xI_{xy}\right)+\left(I_xI_{xy}+I_yI_{yy}\right)\left(1+I_x^2\right)$$

$$\left.-2\left(I_xI_{xy}+I_yI_{yy}\right)\left(1+I_x^2\right)+\left(1+I_x^2+I_y^2\right)\left(2I_xI_{xy}\right)\right]\frac{1}{\sqrt{g}}$$

$$=\frac{1}{g^{3/2}}\left[\left(I_xI_{xx}+I_yI_{xy}\right)I_xI_y-\left(1+I_x^2+I_y^2\right)\left(I_{xx}I_y+I_xI_{xy}\right)\right.$$

$$\left.-\left(I_xI_{xy}+I_yI_{yy}\right)\left(1+I_x^2\right)+2\left(1+I_x^2+I_y^2\right)I_xI_{xy}\right]\frac{1}{\sqrt{g}}$$

$$=\frac{1}{g^{3/2}}\left[I_x^2I_yI_{xx}+I_xI_y^2I_{xy}-I_{xx}I_y-I_xI_{xy}-I_x^2I_yI_{xx}-I_x^3I_{xy}-I_y^3I_{xx}\right.$$

$$-I_xI_y^2I_{xy}-I_xI_{xy}-I_yI_{yy}-I_x^3I_{xy}-I_x^2I_yI_{yy}$$

$$\left.+2I_xI_{xy}+2I_x^3I_{xy}+2I_xI_y^2I_{xy}\right]\frac{1}{\sqrt{g}},$$

or further simplifying, then

$$\frac{dy}{dt}=\frac{1}{g^{3/2}}\left[-I_{xx}I_y-I_y^3I_{xx}-I_yI_{yy}-I_x^2I_yI_{yy}+2I_xI_y^2I_{xy}\right]\left(\frac{1}{\sqrt{g}}\right) \tag{37}$$

$$=\frac{1}{g^{3/2}}\left[-I_{xx}I_y(1+I_y^2)-I_{yy}I_y(1+I_x^2)+2I_xI_y^2I_{xy}\right]\left(\frac{1}{\sqrt{g}}\right)$$

$$=\frac{1}{g^{3/2}}\left[(1+I_y^2)I_{xx}-2I_xI_yI_{xy}+(1+I_x^2)I_{yy}\right]\left(\frac{-I_y}{\sqrt{g}}\right).$$

Thus one has for the evolution of the $y$ coordinate of the image:

$$\frac{dy}{dt}=\frac{1}{g^{3/2}}\left[(1+I_y^2)I_{xx}-2I_xI_yI_{xy}+(1+I_x^2)I_{yy}\right]\left(\frac{-I_y}{\sqrt{g}}\right). \tag{38}$$

Next according to Eq. (29), one has for $X^3 = I$ , the equation whose solution yields the evolution of the image surface:

$$
\begin{aligned}
(39)\quad \frac{d}{dt}I &= \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu I\right)\\
&= \frac{1}{\sqrt{g}}\left[\partial_x\left(\sqrt{g}g^{11}I_x\right)+\partial_x\left(\sqrt{g}g^{12}I_y\right)\right]\\
&\quad+\frac{1}{\sqrt{g}}\left[\partial_y\left(\sqrt{g}g^{21}I_x\right)+\partial_y\left(\sqrt{g}g^{22}I_y\right)\right]\\
&= \frac{1}{\sqrt{g}}\left[\partial_x\left(\frac{1}{\sqrt{g}}\left(1+I_y^2\right)I_x\right)-\partial_x\left(\frac{1}{\sqrt{g}}I_xI_y^2\right)\right]\\
&\qquad+\frac{1}{\sqrt{g}}\left[\partial_y\left(\frac{1}{\sqrt{g}}(-I_xI_y)I_x\right)+\partial_y\left(\frac{1}{\sqrt{g}}(1+I_x^2)I_y\right)\right],
\end{aligned}
$$

or

$$
\begin{aligned}
(40)\quad \frac{dI}{dt} &= \frac{1}{\sqrt{g}}\Bigg[-\frac{1}{2g^{3/2}}(2I_xI_{xx}+2I_yI_{xy})\left[\left(1+I_y^2\right)I_x-I_xI_y^2\right]\\
&\quad+\frac{1}{\sqrt{g}}\left[I_{xx}+I_y^2I_{xx}+2I_xI_yI_{xy}-I_{xx}I_y^2-2I_xI_yI_{xy}\right]\\
&\quad+\frac{1}{\sqrt{g}}\left[-\frac{1}{2g^{3/2}}(2I_xI_{xy}+2I_yI_{yy})(-I_x^2I_y+I_y+I_x^2I_y)\right]\\
&\quad-\frac{1}{\sqrt{g}}\left[2I_xI_{xy}I_y+I_x^2I_{yy}-I_{yy}-2I_xI_{xy}I_y-I_x^2I_{yy}\right]\Bigg]\\
&= -\frac{1}{\sqrt{g}}\frac{1}{g^{3/2}}\left[(I_xI_{xx}+I_yI_{xy})[\left(1+I_y^2\right)I_x-I_xI_y^2]-\left(1+I_x^2+I_y^2\right)I_{xx}\right]\\
&\quad-\frac{1}{\sqrt{g}}\frac{1}{g^{3/2}}(I_xI_{xy}+I_yI_{yy})(-I_x^2I_y+I_y+I_x^2I_y)\\
&\quad+\left(1+I_x^2+I_y^2\right)(-I_{yy})],
\end{aligned}
$$

or

$$
\begin{aligned}
\frac{dI}{dt} &= -\frac{1}{\sqrt{g}}\frac{1}{g^{3/2}}\left(I_x^2I_{xx}+I_xI_yI_{xy}+I_x^2I_y^2I_{xx}+I_xI_y^3I_{xy}\right.\\
&\quad-I_x^2I_y^2I_{xx}-I_xI_y^3I_{xy}-I_{xx}-I_x^2I_{xx}-I_{xx}I_y^2-I_x^3I_yI_{xy}\\
&\qquad-I_x^2I_y^2I_{yy}+I_xI_yI_{xy}+I_y^2I_{yy}+I_x^3I_yI_{xy}+I_x^2I_y^2I_{yy}\\
&\qquad\left.-I_{yy}-I_x^2I_{yy}-I_y^2I_{yy}\right)\\
&= -\frac{1}{\sqrt{g}}\frac{1}{g^{3/2}}(2I_xI_yI_{xy}-I_{xx}-I_{xx}I_y^2-I_{yy}-I_{yy}I_x^2).
\end{aligned}
$$

Thus one has for the evolution of the image:

$$\frac{dI}{dt} = \frac{1}{g^{3/2}}\left[\left(1+I_x^2\right)I_{yy} - 2I_xI_yI_{xy} + \left(1+I_y^2\right)I_{xx}\right]\frac{1}{\sqrt{g}}, \tag{41}$$

which agrees with Eq. (36) of [2].

Summarizing Eqs. (34), (38), and (41), one has the following nonlinear differential equation for the gray-scale image flow:

$$\frac{d\overrightarrow{X}}{dt} = H\overrightarrow{N}, \tag{42}$$

where the nonlinear operator $H$ is given by

$$H \equiv \frac{1}{g^{3/2}}\left[\left(1+I_x^2\right)I_{yy} - 2I_xI_yI_{xy} + \left(1+I_y^2\right)I_{xx}\right], \tag{43}$$

and the vector $\overrightarrow{X}$ is defined in terms of the $x$ and $y$ coordinates of the image, and the image $I$ itself:

$$\overrightarrow{X} = \begin{bmatrix} x \\ y \\ I \end{bmatrix}, \tag{44}$$

and the vector $\overrightarrow{N}$ is defined by

$$\overrightarrow{N} = \frac{1}{\sqrt{g}}\begin{bmatrix} -I_x \\ -I_y \\ 1 \end{bmatrix}. \tag{45}$$

Comparing Eq. (45) with Eq. (15), and using Eq. (17), one sees that $\overrightarrow{N}$ is the unit normal vector orthogonal to the image surface at any point. Also, it can be shown that $H$ in Eq. (43) is proportional to the mean curvature of the image surface at any point. Equation (45) is in disagreement with Eq. (27) of [3], since the latter erroneously has $I_x$ interchanged with $I_y$ in Eq. (45) above. The error would of course result in erroneous processing of the image (unless the image is such that $I_x = I_y$, which, of course, is not generally the case).

## 3. Conlusion

A brief review has been given of the Riemannian geometry of Image processing. Images are treated as Riemannian manifolds in general embedding spaces for application to linear and nonlinear scale-space image processing. The Beltrami operator approach to obtaining a nonlinear differential equation for processing of gray-scale images is treated in some detail, and a minor error is found in earlier work appearing in the literature.

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