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LYAPUNOV-TYPE INEQUALITIES FOR NONLINEAR SYSTEMS WITH PRABHAKAR FRACTIONAL DERIVATIVES

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Abstract. This article is devoted to the study of a fractional nonlinear system of differential equations including the Prabhakar fractional derivative. We present some new Lyapunov-type inequalities for this system and consider some special cases of the system.

1. Introduction

In the recent years, many researchers have studied mathematical inequalities of fractional differential equations (equations with non-integer integral and derivatives). Most of these inequalities have been presented for the linear fractional differential equations with different boundary conditions. One of the important discussed inequality in literature is the Lyapunov inequality which various representations of it have been given in [1, 2], [8, 10–12], [19–21, 28, 34, 35]. In this paper, we intend to extend the Lyapunov inequality for the nonlinear fractional differential equations with a generalized fractional derivative (Prabhakar derivative). For this purpose, we consider the following fractional nonlinear system of differential equations (a generalization of the Emden-Fowler-type and half-linear equations) [32]

$$\begin{cases} x'(t) = \alpha_1(t)x(t) + \beta_1(t)|u(t)|^{\gamma-2}u(t), & \gamma > 1, \\ u'(t) = -\beta_2(t)|x(t)|^{\beta-2}x(t) - \alpha_1(t)u(t), & \beta > 1, \end{cases}$$

with initial conditions

$$x(a) = x(b) = 0, \qquad a, b \in \mathbb{R}, a < b,$$

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and intend to fractionalize this system in the sense of Prabhakar fractional derivative for $\rho, \mu, \omega, \lambda \in \mathbb{C}$, $\Re(\rho) > 0$ and $0 < \mu < 1$

$$(1) \qquad \begin{cases} D^{\lambda}_{\rho,\mu,\omega,t} x(t) = \alpha_1(t)x(t) + \beta_1(t)|u(t)|^{\gamma-2}u(t), & \gamma > 1, \\ D^{\lambda}_{\rho,\mu,\omega,t} u(t) = -\beta_2(t)|x(t)|^{\beta-2}x(t) - \alpha_1(t)u(t), & \beta > 1. \end{cases}$$

The functions α_1, β_1 and β_2 are continuous such that $\beta_1(t) > 0$ for $t \in [t_0, \infty)$ and initial conditions are given in terms of the Prabhakar integral

$$x(a) = x(b) = 0, \quad (E^{-\lambda}_{\rho,1-\mu,\omega,a^+} u)(b) = (E^{-\lambda}_{\rho,1-\mu,\omega,b^-} u)(a) = 0, \quad a, b \in \mathbb{R}, a < b.$$

Throughout this paper, we suppose that the nontrivial solution $(x(t), u(t))$ of the fractional nonlinear system (1) exists and the following hypotheses hold:

- **i):** $\beta_1, \beta_2 : [t_0, \infty) \subset \mathbb{R} \to \mathbb{R}$ are continuous functions such that $\beta_1(t) > 0$ for $t \in [t_0, \infty)$.
- **ii):** $\alpha_1 : [t_0, \infty) \to \mathbb{R}$ is a continuous function.

The rest of the paper is organized as follows. In Section 2, we recall some fundamental preliminaries of the Prabhakar integrals and derivatives. In Section 3, we state our main theorem for the fractional nonlinear system (1) and deduce some special cases for the obtained Lyapunov inequality.

2. Preliminaries

In this section, we recall some basic definitions and properties of fractional Prabhakar integral and derivative, and generalized Mittag-Leffler function.

2.1. **The generalized Mittag-Leffler function.** In year 1971, Prabhakar introduced the generalized Mittag-Leffler function (Mittag-Leffler function with three parameters) on his study in singular integral equations as follows [29]

$$(2) \qquad E^{\lambda}_{\rho,\mu}(z) := \sum_{k=0}^{\infty} \frac{(\lambda)_k}{\Gamma(\rho k + \mu)} \frac{z^k}{k!}, \qquad \lambda, \rho, \mu \in \mathbb{C}, \ \Re(\rho) > 0,$$

where $(\lambda)_k$ is the Pochhammer symbol

$$(\lambda)_0 = 1, \qquad (\lambda)_k = \lambda(\lambda+1)\cdots(\lambda+k-1), \qquad k = 1, 2, \ldots.$$

For $\lambda = 1$, we get the two-parameter Mittag-Leffler function $E_{\rho,\mu}(z)$ defined by

$$E_{\rho,\mu}(z) := E^1_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}, \qquad \rho, \mu \in \mathbb{C}, \ \Re(\rho) > 0.$$

For the special case $\lambda = \rho = 1$, this function coincides with the classical Mittag-Leffler function [26, 27]

$$E_{\rho}(z) := E^1_{\rho,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + 1)}, \qquad \rho \in \mathbb{C}, \ \Re(\rho) > 0.$$

For $\lambda = 0$ we have

$$E^{0}_{\rho,\mu}(z) = \frac{1}{\Gamma(\mu)}.$$

Many researchers have established many contributions on the generalized Mittag-Leffler function, especially in the theory of fractional calculus, and detected some applications in physics, engineering and applied sciences. For example, these functions are applied in the theory of fractional and operational calculus such as fractional order differential and integral equations and Cauchy-type initial and boundary value problems [3, 4, 6–9, 13–18, 22–25, 30, 31, 33].

Lemma 1. *For $\lambda, \rho, \mu, \omega \in \mathbb{C}$ and $\Re(\rho) > 0$, differentiation of the generalized Mittag-Leffler function (2) is given by [23]*

$$\left(\frac{d}{dz}\right)^{n}\left[z^{\mu-1}E^{\lambda}_{\rho,\mu}(\omega z^{\rho})\right] = z^{\mu-n-1}E^{\lambda}_{\rho,\mu-n}(\omega z^{\rho}), \quad n \in \mathbb{N}.$$

Theorem 1. *For $\lambda, \rho, \mu, \nu, \sigma, \omega \in \mathbb{C}$ $(\Re(\rho), \Re(\mu), \Re(\nu) > 0)$, the following relations hold [23]*

$$\int_{a}^{t}(t-\eta)^{\mu-1}E^{\lambda}_{\rho,\mu}(\omega(t-\eta)^{\rho})\eta^{\nu-1}E^{\sigma}_{\rho,\nu}(\omega\eta^{\rho})d\eta = (t-a)^{\mu+\nu-1}E^{\lambda+\sigma}_{\rho,\mu+\nu}(\omega(t-a)^{\rho}),$$

$$\int_{t}^{b}(\eta-t)^{\mu-1}E^{\lambda}_{\rho,\mu}(\omega(\eta-t)^{\rho})\eta^{\nu-1}E^{\sigma}_{\rho,\nu}(\omega\eta^{\rho})d\eta = (b-t)^{\mu+\nu-1}E^{\lambda+\sigma}_{\rho,\mu+\nu}(\omega(b-t)^{\rho}),$$

and in the special case $\sigma = 0$ and $\nu = 1$, we have

$$\int_{a}^{t}(t-\eta)^{\mu-1}E^{\lambda}_{\rho,\mu}(\omega(t-\eta)^{\rho})d\eta = (t-a)^{\mu}E^{\lambda}_{\rho,\mu+1}(\omega(t-a)^{\rho}),$$

$$\int_{a}^{t}(t-\eta)^{\mu-1}E^{\lambda}_{\rho,\mu}(\omega(t-\eta)^{\rho})d\eta = (t-a)^{\mu}E^{\lambda}_{\rho,\mu+1}(\omega(t-a)^{\rho}).$$

2.2. **Fractional Prabhakar Integrals and Derivatives.**

Definition 1. (Prabhakar integral) The left-sided and right-sided Prabhakar integral operators of $f \in L^{1}[a, b]$ are defined as follows [13]

$$E^{\lambda}_{\rho,\mu,\omega,a^{+}}f(t)dt = \int_{a}^{t}(t-\xi)^{\mu-1}E^{\lambda}_{\rho,\mu}\big(\omega(t-\xi)^{\rho}\big)f(\xi)d\xi,$$

$$E^{\lambda}_{\rho,\mu,\omega,b^{-}}f(t)dt = \int_{t}^{b}(\xi-t)^{\mu-1}E^{\lambda}_{\rho,\mu}\big(\omega(\xi-t)^{\rho}\big)f(\xi)d\xi,$$

where $\rho, \mu, \omega, \lambda \in \mathbb{C}$, $\Re(\rho)$, $\Re(\mu) > 0$.

Remark 1. We note that for $\lambda = 0$, the Prabhakar integral operators coincide with the Riemann-Liouville fractional integrals of order μ

$$E^{0}_{\rho,\mu,\omega,a^{+}}f(t) = I^{\mu}_{a^{+}}f(t), \qquad E^{0}_{\rho,\mu,\omega,b^{-}}f(t) = I^{\mu}_{b^{-}}f(t).$$

Definition 2. (Prabhakar derivative) The left-sided and right-sided Prabhakar derivatives of $f \in L^1[a,b]$ are defined by [13]

$$D^{\lambda}_{\rho,\mu,\omega,a^+} f(t) = \frac{d^m}{dt^m} E^{-\lambda}_{\rho,m-\mu,\omega,a^+} f(t),$$

$$D^{\lambda}_{\rho,\mu,\omega,b^-} f(t) = -\frac{d^m}{dt^m} E^{-\lambda}_{\rho,m-\mu,\omega,b^-} f(t),$$

where $\rho, \mu, \omega, \lambda \in \mathbb{C},\ \Re(\rho) > 0,\ \Re(\mu) > 0$ and $m-1 < \mu < m$.

Remark 2. It is obvious that the Prabhakar derivative generalizes the Riemann-Liouville fractional derivative.

Lemma 2. *For $0 < \mu < 1$, the Prabhakar fractional derivatives of a constant are given by*

$$(D^{\lambda}_{\rho,\mu,\omega,a^+} 1)(t) = (t-a)^{-\mu} E^{-\lambda}_{\rho,1-\mu}(\omega(t-a)^{\rho}), \tag{3}$$

$$(D^{\lambda}_{\rho,\mu,\omega,b^-} 1)(t) = (b-t)^{-\mu} E^{-\lambda}_{\rho,1-\mu}(\omega(b-t)^{\rho}). \tag{4}$$

which are not identically equal to zero.

Proof. Applying Definition 2 and using Theorem 1, and Lemma 1 for $0 < \mu < 1$, we can easily obtain the relations (3) and (4). □

Lemma 3. *Let $m-1 < \mu < m,\ m \in \mathbb{N}$, then the following relations hold*

$$\int_a^b f(t) E^{\lambda}_{\rho,\mu,\omega,a^+} g(t) dt = \int_a^b g(t) E^{\lambda}_{\rho,\mu,\omega,b^-} f(t) dt, \tag{5}$$

$$\int_a^b f(t) D^{\lambda}_{\rho,\mu,\omega,a^+} g(t) dt = \int_a^b g(t) D^{\lambda}_{\rho,\mu,\omega,b^-} f(t) dt. \tag{6}$$

Proof. To prove the relation (5), we use Definition 1 and change the order of integration □

$$\begin{aligned}
\int_a^b f(t) E^{\lambda}_{\rho,\mu,\omega,a^+} g(t) dt &= \int_a^b f(t) \int_a^t (t-\xi)^{\mu-1} E^{\lambda}_{\rho,\mu}\big(\omega(t-\xi)^{\rho}\big) g(\xi) d\xi dt \\
&= \int_a^b g(\xi) \int_{\xi}^b (t-\xi)^{\mu-1} E^{\lambda}_{\rho,\mu}\big(\omega(t-\xi)^{\rho}\big) f(t) dt d\xi \\
&= \int_a^b g(\xi) E^{\lambda}_{\rho,\mu,\omega,b^-} f(\xi) d\xi.
\end{aligned}$$

Now, by using Definition 2 and the relation (5), we observe that

$$\begin{aligned}\int_a^b f(t) D^{\lambda}_{\rho,\mu,\omega,a^+} g(t)dt &= \int_a^b f(t) \frac{d^m}{dt^m} E^{-\lambda}_{\rho,m-\mu,\omega,a^+} g(t)dt \\ &= \int_a^b g(t) \frac{d^m}{dt^m} E^{-\lambda}_{\rho,m-\mu,\omega,b^-} f(t)dt \\ &= \int_a^b g(t) D^{\lambda}_{\rho,\mu,\omega,b^-} f(t)dt,\end{aligned}$$

which completes the proof.

Lemma 4. *If $f(t) \in C(a,b) \cap L(a,b)$, then*

$$D^{\lambda}_{\rho,\mu,\omega,a^+} E^{\lambda}_{\rho,\mu,\omega,a^+} f(t) = f(t),$$

$$D^{\lambda}_{\rho,\mu,\omega,b^-} E^{\lambda}_{\rho,\mu,\omega,b^-} f(t) = f(t),$$

and if $f(t)$ and its fractional Prabhakar derivatives belong to $C(a,b) \cap L(a,b)$, then for $c_j \in \mathbb{R}$ and $m-1 < \mu \leq m$, we have [8]

$$E^{\lambda}_{\rho,\mu,\omega,a^+} D^{\lambda}_{\rho,\mu,\omega,a^+} f(t) = f(t) + \sum_{j=1}^{m} c_j (t-a)^{\mu-j} E^{\lambda}_{\rho,\mu-j+1}\big(\omega(t-a)^{\rho}\big),$$

$$E^{\lambda}_{\rho,\mu,\omega,b^-} D^{\lambda}_{\rho,\mu,\omega,b^-} f(t) = f(t) - \sum_{j=1}^{m} (-1)^{m-j} c_j (b-t)^{\mu-j} E^{\lambda}_{\rho,\mu-j+1}\big(\omega(b-t)^{\rho}\big).$$

3. Main Theorems

In this section, we consider the fractional nonlinear system (1) with the Prabhakar derivatives and establish some new Lyapunov-type inequalities for this system. We also derive some particular cases of the obtained Lyapunov-type inequalities.

Theorem 2. *For the fractional nonlinear system* (1) *with the Prabhakar derivative the following inequality holds*

$$\begin{aligned}2 \leq &\int_a^b |t-\tau|^{\mu-1} E^{\lambda}_{\rho,\mu}(\omega|t-\tau|^{\rho}) |\alpha_1(t)| dt \\ &+ M^{\frac{\beta}{\alpha}-1} \Big(\int_a^b |t-\tau|^{\gamma(\mu-1)} \big(E^{\lambda}_{\rho,\mu}(\omega|t-\tau|^{\rho})\big)^{\gamma} \beta_1(t) dt \Big)^{\frac{1}{\gamma}} \Big(\int_a^b \beta_2^{+}(t)| dt \Big)^{\frac{1}{\alpha}},\end{aligned}$$

where $\frac{1}{\gamma} + \frac{1}{\alpha} = 1$, $M = \max|x(t)|_{a<t<b}$ and $\beta_2^{+}(t) = \max\{\beta_2(t), 0\}$.

Proof. Since $x(a) = x(b) = 0$ and x is not identically zero on $[a,b]$, there exists $\tau \in (a,b)$ such that $M = |x(\tau)| = \max|x(t)|_{a<t<b} > 0$. By separating the interval $[a,b]$ into two subintervals $[a,\tau]$ and $[\tau,b]$ and applying the left-sided and right-sided Prabhakar fractional integral operator $E^{\lambda}_{\rho,\mu,\omega,a^+}$ and $E^{\lambda}_{\rho,\mu,\omega,b^-}$

on the both sides of the first equation of system (1) in the subintervals $[a,\tau]$ and $[\tau,b]$, respectively, we have

$$\Big(E^{\lambda}_{\rho,\mu,\omega,a^+}D^{\lambda}_{\rho,\mu,\omega,a^+}x(t)\Big)(\tau)=\Big(E^{\lambda}_{\rho,\mu,\omega,a^+}\big[\alpha_1(t)x(t)+\beta_1(t)|u(t)|^{\gamma-2}u(t)\big]\Big)(\tau),$$

and

$$\Big(E^{\lambda}_{\rho,\mu,\omega,b^-}D^{\lambda}_{\rho,\mu,\omega,b^-}x(t)\Big)(\tau)=\Big(E^{\lambda}_{\rho,\mu,\omega,b^-}\big[\alpha_1(t)x(t)+\beta_1(t)|u(t)|^{\gamma-2}u(t)\big]\Big)(\tau).$$

By employing Lemma 4 for some real constants c_1 and d_1, we obtain

$$\begin{aligned}x(\tau)&-c_1(\tau-a)^{\mu-1}E^{\lambda}_{\rho,\mu}\big(\omega(\tau-a)^{\rho}\big)\\&=\int_a^{\tau}(\tau-t)^{\mu-1}E^{\lambda}_{\rho,\mu}\big(\omega(\tau-t)^{\rho}\big)\Big[\alpha_1(t)x(t)+\beta_1(t)|u(t)|^{\gamma-2}u(t)\Big]dt,\end{aligned}$$

and

$$\begin{aligned}x(\tau)&-d_1(b-\tau)^{\mu-1}E^{\lambda}_{\rho,\mu}\big(\omega(b-\tau)^{\rho}\big)\\&=\int_{\tau}^{b}(t-\tau)^{\mu-1}E^{\lambda}_{\rho,\mu}\big(\omega(t-\tau)^{\rho}\big)\Big[\alpha_1(t)x(t)+\beta_1(t)|u(t)|^{\gamma-2}u(t)\Big]dt.\end{aligned}$$

Since $x(a)=x(b)=0$ and $0<\mu<1$, the coefficients c_1 and d_1 must be zero. So, by using the triangle inequality, the above relations lead to the following inequalities

$$\begin{aligned}|x(\tau)|\le&\int_a^{\tau}|\tau-t|^{\mu-1}E^{\lambda}_{\rho,\mu}\big(\omega|\tau-t|^{\rho}\big)|\alpha_1(t)||x(t)|dt\\&+\int_a^{\tau}|\tau-t|^{\mu-1}E^{\lambda}_{\rho,\mu}\big(\omega|\tau-t|^{\rho}\big)\beta_1(t)|u(t)|^{\gamma-1}dt,\end{aligned}$$

and

$$\begin{aligned}|x(\tau)|\le&\int_{\tau}^{b}|t-\tau|^{\mu-1}E^{\lambda}_{\rho,\mu}\big(\omega|t-\tau|^{\rho}\big)|\alpha_1(t)||x(t)|dt\\&+\int_{\tau}^{b}|t-\tau|^{\mu-1}E^{\lambda}_{\rho,\mu}\big(\omega|t-\tau|^{\rho}\big)\beta_1(t)|u(t)|^{\gamma-1}dt.\end{aligned}$$

Now, by summing up the two previous inequalities, we get

$$\begin{aligned}2|x(\tau)|\le&\int_a^{b}|\tau-t|^{\mu-1}E^{\lambda}_{\rho,\mu}\big(\omega|\tau-t|^{\rho}\big)|\alpha_1(t)||x(t)|dt\\&+\int_a^{b}|\tau-t|^{\mu-1}E^{\lambda}_{\rho,\mu}\big(\omega|\tau-t|^{\rho}\big)\beta_1(t)|u(t)|^{\gamma-1}dt,\end{aligned}\tag{7}$$

whence, by using Hölder inequality on the second integral of the right hand side of (7) for $\frac{1}{\alpha}+\frac{1}{\gamma}=1$, we obtain

$$\begin{aligned}
&\int_a^b |\tau-t|^{\mu-1}E^{\lambda}_{\rho,\mu}\big(\omega|\tau-t|^{\rho}\big)\beta_1(t)|u(t)|^{\gamma-1}dt\\
&\quad=\int_a^b |\tau-t|^{\mu-1}E^{\lambda}_{\rho,\mu}\big(\omega|\tau-t|^{\rho}\big)\beta_1^{\frac{1}{\gamma}}(t)\beta_1^{\frac{1}{\alpha}}(t)|u(t)|^{\gamma-1}dt\\
&\quad\le\Big(\int_a^b |\tau-t|^{\gamma(\mu-1)}\big(E^{\lambda}_{\rho,\mu}\big(\omega|\tau-t|^{\rho}\big)\big)^{\gamma}\beta_1(t)dt\Big)^{\frac{1}{\gamma}}\Big(\int_a^b \beta_1(t)|u(t)|^{\alpha(\gamma-1)}dt\Big)^{\frac{1}{\alpha}}\\
&\quad=\Big(\int_a^b |\tau-t|^{\gamma(\mu-1)}\big(E^{\lambda}_{\rho,\mu}\big(\omega|\tau-t|^{\rho}\big)\big)^{\gamma}\beta_1(t)dt\Big)^{\frac{1}{\gamma}}\Big(\int_a^b \beta_1(t)|u(t)|^{\gamma}dt\Big)^{\frac{1}{\alpha}},
\end{aligned}$$

and subsequently

$$\begin{aligned}
2|x(\tau)|\le&\int_a^b |\tau-t|^{\mu-1}E^{\lambda}_{\rho,\mu}\big(\omega|\tau-t|^{\rho}\big)|\alpha_1(t)||x(t)|dt\\
&+\Big(\int_a^b |\tau-t|^{\gamma(\mu-1)}\big(E^{\lambda}_{\rho,\mu}\big(\omega|\tau-t|^{\rho}\big)\big)^{\gamma}\beta_1(t)dt\Big)^{\frac{1}{\gamma}}\\
&\times\Big(\int_a^b \beta_1(t)|u(t)|^{\gamma}dt\Big)^{\frac{1}{\alpha}}.
\end{aligned}\tag{8}$$

At this point, without loss of generality, we assume that the operator $D^{\lambda}_{\rho,\mu,\omega,t}$ of system (1) is the left-sided Prabhakar derivative. If we multiply the first equation of system (1) by $u(t)$ and the second one by $x(t)$ and then add the results, we have

$$u(t)D^{\lambda}_{\rho,\mu,\omega,a^+}x(t)+x(t)D^{\lambda}_{\rho,\mu,\omega,a^+}u(t)=\beta_1(t)|u(t)|^{\gamma}-\beta_2(t)|x(t)|^{\beta}.$$

By integrating the above equation on interval $[a,b]$, using the relation (6), Definition 2 for $0<\mu<1$ and $\tau,s\in(a,b)$, we deduce

$$\begin{aligned}
&\int_a^b \beta_1(t)|u(t)|^{\gamma}dt-\int_a^b \beta_2(t)|x(t)|^{\beta}dt\\
&\qquad=\int_a^b u(t)D^{\lambda}_{\rho,\mu,\omega,a^+}x(t)dt+\int_a^b x(t)D^{\lambda}_{\rho,\mu,\omega,a^+}u(t)dt\\
&\qquad=\int_a^b u(t)D^{\lambda}_{\rho,\mu,\omega,a^+}x(t)dt+\int_a^b u(t)D^{\lambda}_{\rho,\mu,\omega,b^-}x(t)dt\\
&\qquad=\int_a^b u(t)\Big[D^{\lambda}_{\rho,\mu,\omega,a^+}x(t)+D^{\lambda}_{\rho,\mu,\omega,b^-}x(t)\Big]dt
\end{aligned}$$

$$\le \int_a^b u(t)\Big[\frac{d}{dt}\int_a^t |t-s|^{-\mu}E^{-\lambda}_{\rho,1-\mu}\big(\omega|t-s|^{\rho}\big)|x(s)|ds$$
$$+\frac{d}{dt}\int_t^b |s-t|^{-\mu}E^{-\lambda}_{\rho,1-\mu}\big(\omega|s-t|^{\rho}\big)|x(s)|ds\Big]dt$$
$$\le \int_a^b u(t)\Big[\frac{d}{dt}\int_a^t |t-s|^{-\mu}E^{-\lambda}_{\rho,1-\mu}\big(\omega|t-s|^{\rho}\big)\max|x(s)|ds$$
$$+\frac{d}{dt}\int_t^b |s-t|^{-\mu}E^{-\lambda}_{\rho,1-\mu}\big(\omega|s-t|^{\rho}\big)\max|x(s)|ds\Big]dt$$
$$=\int_a^b u(t)|x(\tau)|\Big[\frac{d}{dt}\int_a^t |t-s|^{-\mu}E^{-\lambda}_{\rho,1-\mu}\big(\omega|t-s|^{\rho}\big)ds$$
$$+\frac{d}{dt}\int_t^b |s-t|^{-\mu}E^{-\lambda}_{\rho,1-\mu}\big(\omega|t-s|^{\rho}\big)ds\Big]dt$$
$$=(\pm 1)^{\rho-\mu}M\int_a^b u(t)\Big[(D^{\lambda}_{\rho,\mu,\omega,a^+}1)(t)+(D^{\lambda}_{\rho,\mu,\omega,b^-}1)(t)\Big]dt,$$

where $M=|x(\tau)|=\max|x(s)|_{a<s<b}$. Using the relations (3) and (4), also taking into account $(E^{-\lambda}_{\rho,1-\mu,\omega,a^+}u)(b)=(E^{-\lambda}_{\rho,1-\mu,\omega,b^-}u)(a)=0$, we get

$$(\pm 1)^{\rho-\mu}M\int_a^b u(t)\Big[(D^{\lambda}_{\rho,\mu,\omega,a^+}1)(t)+(D^{\lambda}_{\rho,\mu,\omega,b^-}1)(t)\Big]dt$$
$$=(\pm 1)^{\rho-\mu}M\Big[\int_a^b (t-a)^{-\mu}E^{-\lambda}_{\rho,1-\mu}\big(\omega|t-a|^{\rho}\big)u(t)dt$$
$$+\int_a^b (b-t)^{-\mu}E^{-\lambda}_{\rho,1-\mu}\big(\omega|b-t|^{\rho}\big)u(t)dt\Big]$$
$$=(\pm 1)^{\rho-\mu}M\Big[(E^{-\lambda}_{\rho,1-\mu,\omega,a^+}u)(a)+(E^{-\lambda}_{\rho,1-\mu,\omega,b^-}u)(b)\Big]=0,$$

so

$$\int_a^b \beta_1(t)|u(t)|^{\gamma}dt\le\int_a^b \beta_2(t)|x(t)|^{\beta}dt.$$

Therefore, from (8) we get

$$2|x(\tau)|\le\int_a^b |\tau-t|^{\mu-1}E^{\lambda}_{\rho,\mu}\big(\omega|\tau-t|^{\rho}\big)|\alpha_1(t)||x(t)|dt$$
$$+\Big(\int_a^b |\tau-t|^{\gamma(\mu-1)}\big(E^{\lambda}_{\rho,\mu}\big(\omega|\tau-t|^{\rho}\big)\big)^{\gamma}\beta_1(t)dt\Big)^{\frac{1}{\gamma}}$$
$$\times\Big(\int_a^b \beta_2(t)|x(t)|^{\beta}dt\Big)^{\frac{1}{\alpha}}. \tag{9}$$

Since $M = |x(\tau)| = \max|x(t)|_{a<t<b}$ and $\beta_2^+(t) = \max\{\beta_2(t), 0\}$, by (9), the following inequality holds

$$\begin{aligned}2|x(\tau)| \le |x(\tau)| &\int_a^b |\tau - t|^{\mu-1} E_{\rho,\mu}^{\lambda}\big(\omega|\tau - t|^{\rho}\big)|\alpha_1(t)|dt \\ &+ |x(\tau)|^{\frac{\beta}{\alpha}} \Big(\int_a^b |\tau - t|^{\gamma(\mu-1)} \big(E_{\rho,\mu}^{\lambda}\big(\omega|\tau - t|^{\rho}\big)\big)^{\gamma} \beta_1(t)dt\Big)^{\frac{1}{\gamma}} \Big(\int_a^b \beta_2^+(t)dt\Big)^{\frac{1}{\alpha}},\end{aligned}$$

and finally

$$\begin{aligned}2 \le &\int_a^b |\tau - t|^{\mu-1} E_{\rho,\mu}^{\lambda}\big(\omega|\tau - t|^{\rho}\big)|\alpha_1(t)|dt \\ &+ M^{\frac{\beta}{\alpha}-1} \Big(\int_a^b |\tau - t|^{\gamma(\mu-1)} \big(E_{\rho,\mu}^{\lambda}\big(\omega|\tau - t|^{\rho}\big)\big)^{\gamma} \beta_1(t)dt\Big)^{\frac{1}{\gamma}} \Big(\int_a^b \beta_2^+(t)dt\Big)^{\frac{1}{\alpha}}.\end{aligned}$$

□

Corollary 1. *In the case $\beta = \alpha$ for the system* (1), *i.e.,*

$$\begin{aligned}&D_{\rho,\mu,\omega,t}^{\lambda} x(t) = \alpha_1(t)x(t) + \beta_1(t)|u(t)|^{\gamma-2}u(t),\\ &D_{\rho,\mu,\omega,t}^{\lambda} u(t) = -\beta_2(t)|x(t)|^{\alpha-2}x(t) - \alpha_1(t)u(t),\\ &x(a) = x(b) = 0, \quad (E_{\rho,1-\mu,\omega,a^+}^{-\lambda} u)(b) = (E_{\rho,1-\mu,\omega,b^-}^{-\lambda} u)(a) = 0, \quad a, b \in \mathbb{R}, a < b,\end{aligned}$$

where $\frac{1}{\alpha} + \frac{1}{\gamma} = 1$, the following inequality holds for $0 < \mu < 1$

$$\begin{aligned}2 \le &\int_a^b |\tau - t|^{\mu-1} E_{\rho,\mu}^{\lambda}\big(\omega|\tau - t|^{\rho}\big)|\alpha_1(t)|dt \\ &+ \Big(\int_a^b |\tau - t|^{\gamma(\mu-1)} \big(E_{\rho,\mu}^{\lambda}\big(\omega|\tau - t|^{\rho}\big)\big)^{\gamma} \beta_1(t)dt\Big)^{\frac{1}{\gamma}} \Big(\int_a^b \beta_2^+(t)dt\Big)^{\frac{1}{\alpha}}.\end{aligned}$$

Corollary 2. *In the subcase $\beta = 2$ and $\gamma = 2$ for the system* (1), *i.e., for the fractional linear system*

$$\begin{aligned}&D_{\rho,\mu,\omega,t}^{\lambda} x(t) = \alpha_1(t)x(t) + \beta_1(t)u(t),\\ &D_{\rho,\mu,\omega,t}^{\lambda} u(t) = -\beta_2(t)x(t) - \alpha_1(t)u(t),\\ &x(a) = x(b) = 0, \quad (E_{\rho,1-\mu,\omega,a^+}^{-\lambda} u)(b) = (E_{\rho,1-\mu,\omega,b^-}^{-\lambda} u)(a) = 0, \quad a, b \in \mathbb{R}, a < b,\end{aligned}$$

the following inequality holds for $0 < \mu < 1$

$$\begin{aligned}2 \le &\int_a^b |\tau - t|^{\mu-1} E_{\rho,\mu}^{\lambda}\big(\omega|\tau - t|^{\rho}\big)|\alpha_1(t)|dt \\ &+ \Big(\int_a^b |\tau - t|^{2(\mu-1)} \big(E_{\rho,\mu}^{\lambda}\big(\omega|\tau - t|^{\rho}\big)\big)^{2} \beta_1(t)dt\Big)^{\frac{1}{2}} \Big(\int_a^b \beta_2^+(t)dt\Big)^{\frac{1}{2}}.\end{aligned}$$

Theorem 3. *In the special case of system* (1), *for the fractional system*

$$\begin{cases} D_{\rho,\mu,\omega,t}^{\lambda} x(t) = \beta_1(t)|u(t)|^{\gamma-2}u(t),\\ D_{\rho,\mu,\omega,t}^{\lambda} u(t) = -\beta_2(t)|x(t)|^{\beta-2}x(t),\end{cases} \tag{10}$$

with initial conditions

$$x(a)=x(b)=0,\quad (E^{-\lambda}_{\rho,1-\mu,\omega,a^+}u)(b)=(E^{-\lambda}_{\rho,1-\mu,\omega,b^-}u)(a)=0,\quad a,b\in\mathbb{R}, a<b,$$

the following inequalities hold for $0<\mu<1$ *and* $\tau\in(a,b)$

$$1\le M^{\beta-\alpha}\Big(\int_a^\tau |t-\tau|^{\gamma(\mu-1)}\big(E^\lambda_{\rho,\mu}\big(\omega|t-\tau|^\rho\big)\big)^\gamma\beta_1(t)dt\Big)^{\alpha-1}\Big(\int_a^\tau \beta_2^+(t)dt\Big),$$

$$1\le M^{\beta-\alpha}\Big(\int_\tau^b |t-\tau|^{\gamma(\mu-1)}\big(E^\lambda_{\rho,\mu}\big(\omega|t-\tau|^\rho\big)\big)^\gamma\beta_1(t)dt\Big)^{\alpha-1}\Big(\int_\tau^b \beta_2^+(t)dt\Big),$$

$$2^\alpha\le M^{\beta-\alpha}\Big(\int_a^b |t-\tau|^{\gamma(\mu-1)}\big(E^\lambda_{\rho,\mu}\big(\omega|t-\tau|^\rho\big)\big)^\gamma\beta_1(t)dt\Big)^{\alpha-1}\Big(\int_a^b \beta_2^+(t)dt\Big).$$

Proof. Since $x(a)=x(b)=0$ and x is not identically zero on $[a,b]$, there exists $\tau\in(a,b)$ such that $M=|x(\tau)|=\max|x(t)|_{a<t<b}>0$. Applying the left-sided Prabhakar fractional integral operator $E^\lambda_{\rho,\mu,\omega,a^+}$ on the both sides of the first equation of system (10) in the interval $[a,\tau]$ and using Lemma 4, also taking into account $x(a)=0$ and $0<\mu<1$, we get

$$x(\tau)=\int_a^\tau(\tau-t)^{\mu-1}E^\lambda_{\rho,\mu}\big(\omega(\tau-t)^\rho\big)\beta_1(t)|u(t)|^{\gamma-1}dt,$$

and so

$$|x(\tau)|\le\int_a^\tau|\tau-t|^{\mu-1}E^\lambda_{\rho,\mu}\big(\omega|\tau-t|^\rho\big)\beta_1(t)|u(t)|^{\gamma-1}dt. \tag{11}$$

Hence, by using the Hölder inequality on the second integral of the right side of (11), we obtain

$$\begin{aligned}
&\int_a^\tau|\tau-t|^{\mu-1}E^\lambda_{\rho,\mu}\big(\omega|\tau-t|^\rho\big)\beta_1(t)|u(t)|^{\gamma-1}dt\\
&\quad=\int_a^\tau|\tau-t|^{\mu-1}E^\lambda_{\rho,\mu}\big(\omega|\tau-t|^\rho\big)\beta_1^{\frac{1}{\gamma}}(t)\beta_1^{\frac{1}{\alpha}}(t)|u(t)|^{\gamma-1}dt\\
&\quad\le\Big(\int_a^\tau|\tau-t|^{\gamma(\mu-1)}\big(E^\lambda_{\rho,\mu}\big(\omega|\tau-t|^\rho\big)\big)^\gamma\beta_1(t)dt\Big)^{\frac{1}{\gamma}}\Big(\int_a^\tau\beta_1(t)|u(t)|^{\alpha(\gamma-1)}dt\Big)^{\frac{1}{\alpha}}\\
&\quad=\Big(\int_a^\tau|\tau-t|^{\gamma(\mu-1)}\big(E^\lambda_{\rho,\mu}\big(\omega|\tau-t|^\rho\big)\big)^\gamma\beta_1(t)dt\Big)^{\frac{1}{\gamma}}\Big(\int_a^\tau\beta_1(t)|u(t)|^{\gamma}dt\Big)^{\frac{1}{\alpha}},
\end{aligned}$$

where $\frac{1}{\alpha}+\frac{1}{\gamma}=1$. Therefore, the relation (11) implies that

$$|x(\tau)|\le\Big(\int_a^\tau|\tau-t|^{\gamma(\mu-1)}\big(E^\lambda_{\rho,\mu}\big(\omega|\tau-t|^\rho\big)\big)^\gamma\beta_1(t)dt\Big)^{\frac{1}{\gamma}}\Big(\int_a^\tau\beta_1(t)|u(t)|^{\gamma}dt\Big)^{\frac{1}{\alpha}}. \tag{12}$$

Now, without loss of generality, let the operator $D^\lambda_{\rho,\mu,\omega,t}$ of the system (10) be left-sided Prabhakar derivative. If we multiply the first equation of system (10) by $u(t)$ and the second one by $x(t)$ and then adding the results, we get

$$u(t)D^\lambda_{\rho,\mu,\omega,a^+}x(t)+x(t)D^\lambda_{\rho,\mu,\omega,a^+}u(t)=\beta_1(t)|u(t)|^\gamma-\beta_2(t)|x(t)|^\beta.$$

At this point, by integrating the above equation on interval $[a,\tau]$, using the relation (6) and Definition 2, we obtain for $y \in (a,\tau)$

$$\begin{aligned}
&\int_a^\tau \beta_1(t)|u(t)|^\gamma dt - \int_a^\tau \beta_2(t)|x(t)|^\beta dt \\
&\quad= \int_a^\tau u(t) D^\lambda_{\rho,\mu,\omega,a^+} x(t)dt + \int_a^\tau x(t) D^\lambda_{\rho,\mu,\omega,a^+} u(t)dt \\
&\quad= \int_a^\tau u(t) D^\lambda_{\rho,\mu,\omega,a^+} x(t)dt + \int_a^\tau u(t) D^\lambda_{\rho,\mu,\omega,\tau^-} x(t)dt \\
&\quad= \int_a^\tau u(t)\Big[D^\lambda_{\rho,\mu,\omega,a^+} x(t) + D^\lambda_{\rho,\mu,\omega,\tau^-} x(t)\Big]dt \\
&\quad\leq \int_a^\tau |u(t)|\Big[\frac{d}{dt}\int_a^t |t-y|^{-\mu} E^{-\lambda}_{\rho,1-\mu}\big(\omega|t-y|^\rho\big)|x(y)|dy \\
&\qquad + \frac{d}{dt}\int_t^\tau |y-t|^{-\mu} E^{-\lambda}_{\rho,1-\mu}\big(\omega|t-y|^\rho\big)|x(y)|dy\Big]dt \\
&\quad\leq \int_a^\tau |u(t)|\Big[\frac{d}{dt}\int_a^t |t-y|^{-\mu} E^{-\lambda}_{\rho,1-\mu}\big(\omega|t-y|^\rho\big)\max|x(y)|dy \\
&\qquad + \frac{d}{dt}\int_t^\tau |y-t|^{-\mu} E^{-\lambda}_{\rho,1-\mu}\big(\omega|t-y|^\rho\big)\max|x(y)|dy\Big]dt \\
&\quad= \int_a^\tau |u(t)||x(\tau)|\Big[\frac{d}{dt}\int_a^t |t-y|^{-\mu} E^{-\lambda}_{\rho,1-\mu}\big(\omega|t-y|^\rho\big)dy \\
&\qquad + \frac{d}{dt}\int_t^\tau |y-t|^{-\mu} E^{-\lambda}_{\rho,1-\mu}\big(\omega|\tau-t|^\rho\big)dy\Big]dt \\
&\quad\leq \int_a^b |u(t)||x(\tau)|\Big[\frac{d}{dt}\int_a^t |t-y|^{-\mu} E^{-\lambda}_{\rho,1-\mu}\big(\omega|t-y|^\rho\big)dy \\
&\qquad + \frac{d}{dt}\int_t^b |y-t|^{-\mu} E^{-\lambda}_{\rho,1-\mu}\big(\omega|t-y|^\rho\big)dy\Big]dt \\
&\quad=(\pm 1)^{1+\rho-\mu} M\int_a^b u(t)\Big[(D^\lambda_{\rho,\mu,\omega,a^+}1)(t) + (D^\lambda_{\rho,\mu,\omega,b^-}1)(t)\Big]dt,
\end{aligned}$$

where $M = |x(\tau)| = \max|x(y)|_{a<y<\tau<b}$. Using the relations (3) and (4), also taking into account $(I^{1-\mu}_{a^+}u)(b) = (I^{1-\mu}_{b^-}u)(a) = 0$, we get

$$\begin{aligned}
&(\pm 1)^{1+\rho-\mu} M\int_a^b u(t)\Big[(D^\lambda_{\rho,\mu,\omega,a^+}1)(t) + (D^\lambda_{\rho,\mu,\omega,b^-}1)(t)\Big]dt \\
&\quad=(\pm 1)^{1+\rho-\mu} M\Big[\int_a^b (t-a)^{-\mu} E^{-\lambda}_{\rho,1-\mu}\big(\omega|t-y|^\rho\big)u(t)dt \\
&\qquad + \int_a^b (b-t)^{-\mu} E^{-\lambda}_{\rho,1-\mu}\big(\omega|t-y|^\rho\big)u(t)dt
\end{aligned}$$

$$= (\pm 1)^{1+\rho-\mu} M\Big[(I_{b^-}^{1-\mu}u)(a) + (I_{a^+}^{1-\mu}u)(b)\Big] = 0,$$

so

$$\int_a^\tau \beta_1(t)|u(t)|^\gamma dt \le \int_a^\tau \beta_2(t)|x(t)|^\beta dt.$$

Thus, we find from (12)

$$|x(\tau)| \le \Big(\int_a^\tau |\tau-t|^{\gamma(\mu-1)}\big(E_{\rho,\mu}^\lambda(\omega|\tau-t|^\rho)\big)^\gamma \beta_1(t)dt\Big)^{\frac{1}{\gamma}}\Big(\int_a^\tau \beta_2(t)|x(t)|^\beta dt\Big)^{\frac{1}{\alpha}}. \tag{13}$$

Since $M = |x(\tau)| = \max|x(t)|_{a<t<b}$ and $\beta_2^+(t) = \max\{\beta_2(t), 0\}$, thus (13) yields

$$|x(\tau)| \le |x(\tau)|^{\frac{\beta}{\alpha}}\Big(\int_a^\tau |\tau-t|^{\gamma(\mu-1)}\big(E_{\rho,\mu}^\lambda(\omega|\tau-t|^\rho)\big)^\gamma \beta_1(t)dt\Big)^{\frac{1}{\gamma}}\Big(\int_a^\tau \beta_2^+(t)dt\Big)^{\frac{1}{\alpha}},$$

and hence we have

$$1 \le M^{\frac{\beta}{\alpha}-1}\Big(\int_a^\tau |\tau-t|^{\gamma(\mu-1)}\big(E_{\rho,\mu}^\lambda(\omega|\tau-t|^\rho)\big)^\gamma \beta_1(t)dt\Big)^{\frac{1}{\gamma}}\Big(\int_a^\tau \beta_2^+(t)dt\Big)^{\frac{1}{\alpha}}. \tag{14}$$

Taking the α-th power of both sides of inequality (14), we get

$$1 \le M^{\beta-\alpha}\Big(\int_a^\tau |\tau-t|^{\gamma(\mu-1)}\big(E_{\rho,\mu}^\lambda(\omega|\tau-t|^\rho)\big)^\gamma \beta_1(t)dt\Big)^{\alpha-1}\Big(\int_a^\tau \beta_2^+(t)dt\Big).$$

Now, by applying the right-sided Prabhakar fractional integral operator $I_{b^-}^\mu$ on both sides of the first equation of system (10) in the interval $[\tau, b]$ and using Lemma 4, also taking into account $x(b) = 0$, we get

$$|x(\tau)| \le \int_\tau^b |t-\tau|^{\mu-1}E_{\rho,\mu}^\lambda(\omega|t-\tau|^\rho)\beta_1(t)|u(t)|^{\gamma-1}dt,$$

and similarly by repeating the same process, we conclude

$$1 \le M^{\beta-\alpha}\Big(\int_\tau^b |t-\tau|^{\gamma(\mu-1)}\big(E_{\rho,\mu}^\lambda(\omega|t-\tau|^\rho)\big)^\gamma \beta_1(t)dt\Big)^{\alpha-1}\Big(\int_\tau^b \beta_2^+(t)dt\Big).$$

Since for $t > 0$ the function $J(t) = t^{1-\alpha}$ is convex, due to the Jensen inequality $J(\frac{y+z}{2}) \le \frac{J(y)+J(z)}{2}$ with

$$y = \int_a^\tau |\tau-t|^{\gamma(\mu-1)}\big(E_{\rho,\mu}^\lambda(\omega|\tau-t|^\rho)\big)^\gamma \beta_1(t)dt,$$

and

$$z = \int_\tau^b |t-\tau|^{\gamma(\mu-1)}\big(E_{\rho,\mu}^\lambda(\omega|t-\tau|^\rho)\big)^\gamma \beta_1(t)dt,$$

we can obtain

$$
\begin{aligned}
\int_a^b \beta_2^+(t)dt &= \int_a^\tau \beta_2^+(t)dt + \int_\tau^b \beta_2^+(t)dt \\
&\geq \frac{1}{M^{\beta-\alpha}}\Bigg[\Big(\int_a^\tau |\tau-t|^{\gamma(\mu-1)}\big(E^{\lambda}_{\rho,\mu}(\omega|\tau-t|^{\rho})\big)^{\gamma}\beta_1(t)dt\Big)^{1-\alpha} \\
&\quad + \Big(\int_\tau^b |t-\tau|^{\gamma(\mu-1)}\big(E^{\lambda}_{\rho,\mu}(\omega|t-\tau|^{\rho})\big)^{\gamma}\beta_1(t)dt\Big)^{1-\alpha}\Bigg] \\
&\geq \frac{1}{M^{\beta-\alpha}}2\Bigg[\frac{1}{2}\Big(\int_a^\tau |\tau-t|^{\gamma(\mu-1)}\big(E^{\lambda}_{\rho,\mu}(\omega|\tau-t|^{\rho})\big)^{\gamma}\beta_1(t)dt \\
&\quad + \int_\tau^b |t-\tau|^{\gamma(\mu-1)}\big(E^{\lambda}_{\rho,\mu}(\omega|t-\tau|^{\rho})\big)^{\gamma}\beta_1(t)dt\Big)\Bigg]^{1-\alpha} \\
&= \frac{2^{\alpha}}{M^{\beta-\alpha}}\Big(\int_a^b |t-\tau|^{\gamma(\mu-1)}\big(E^{\lambda}_{\rho,\mu}(\omega|t-\tau|^{\rho})\big)^{\gamma}\beta_1(t)dt\Big)^{1-\alpha},
\end{aligned}
$$

and accordingly

$$
2^{\alpha} \leq M^{\beta-\alpha}\Big(\int_a^b |t-\tau|^{\gamma(\mu-1)}\big(E^{\lambda}_{\rho,\mu}(\omega|t-\tau|^{\rho})\big)^{\gamma}\beta_1(t)dt\Big)^{\alpha-1}\Big(\int_a^b \beta_2^+(t)dt\Big).
$$

□

Corollary 3. *In the special case* $\beta = \alpha$ *for the system* (10)*, the following inequalities hold*

$$
1 \leq \Big(\int_a^\tau |t-\tau|^{\gamma(\mu-1)}\big(E^{\lambda}_{\rho,\mu}(\omega|t-\tau|^{\rho})\big)^{\gamma}\beta_1(t)dt\Big)^{\alpha-1}\Big(\int_a^\tau \beta_2^+(t)dt\Big),
$$

$$
1 \leq \Big(\int_\tau^b |t-\tau|^{\gamma(\mu-1)}\big(E^{\lambda}_{\rho,\mu}(\omega|t-\tau|^{\rho})\big)^{\gamma}\beta_1(t)dt\Big)^{\alpha-1}\Big(\int_\tau^b \beta_2^+(t)dt\Big),
$$

and

$$
2^{\alpha} \leq \Big(\int_a^b |t-\tau|^{\gamma(\mu-1)}\big(E^{\lambda}_{\rho,\mu}(\omega|t-\tau|^{\rho})\big)^{\gamma}\beta_1(t)dt\Big)^{\alpha-1}\Big(\int_a^b \beta_2^+(t)dt\Big).
$$

Corollary 4. *In the case* $\beta = 2$ *and* $\gamma = 2$ *for the system* (10)

$$
\begin{aligned}
&D^{\lambda}_{\rho,\mu,\omega,t}x(t) = \beta_1(t)u(t), \\
&D^{\lambda}_{\rho,\mu,\omega,t}u(t) = -\beta_2(t)x(t), \\
&x(a) = x(b) = 0, \quad (E^{-\lambda}_{\rho,1-\mu,\omega,a^+}u)(b) = (E^{-\lambda}_{\rho,1-\mu,\omega,b^-}u)(a) = 0, \quad a,b \in \mathbb{R}, a < b,
\end{aligned}
$$

the following inequalities hold

$$
1 \leq \Big(\int_a^\tau |t-\tau|^{2(\mu-1)}\big(E^{\lambda}_{\rho,\mu}(\omega|t-\tau|^{\rho})\big)^{2}\beta_1(t)dt\Big)\Big(\int_a^\tau \beta_2^+(t)dt\Big),
$$

$$1 \leq \Big(\int_\tau^b |t-\tau|^{2(\mu-1)} \big(E_{\rho,\mu}^{\lambda}\big(\omega|t-\tau|^{\rho}\big)\big)^2 \beta_1(t)dt \Big) \Big(\int_\tau^b \beta_2^+(t)dt \Big),$$

$$4 \leq \Big(\int_a^b |t-\tau|^{2(\mu-1)} \big(E_{\rho,\mu}^{\lambda}\big(\omega|t-\tau|^{\rho}\big)\big)^2 \beta_1(t)dt \Big) \Big(\int_a^b \beta_2^+(t)dt \Big).$$

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