

REGULARITY OF WEAK SOLUTIONS OF DEGENERATE ELLIPTIC EQUATIONS

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ABSTRACT. In this article we establish the existence of higher order weak derivatives of weak solutions of the Dirichlet problem for a class of degenerate elliptic equations.

1. INTRODUCTION

In this paper we shall study the existence of higher order weak derivatives (see Theorem 3.9) of weak solutions of degenerate elliptic equations $Lu = g$, where L is an elliptic operator

$$(1.1) \quad Lu = - \sum_{i,j=1}^n D_j(a_{ij}(x)D_i u)(x)$$

whose coefficients a_{ij} are measurable, real-valued functions, and whose coefficient matrix $A = (a_{ij})$ is symmetric and satisfies the degenerate ellipticity condition

$$(1.2) \quad \omega(x)|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \leq v(x)|\xi|^2$$

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for all $\xi \in \mathbb{R}^n$ and almost every $x \in \Omega \subset \mathbb{R}^n$, where Ω is a bounded open set, ω and v are weight functions (that is, ω and v are locally integrable and nonnegative functions on \mathbb{R}^n).

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occurs as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients it is natural to look for solutions in weighted Sobolev spaces (see [1], [2], [3], [4], [5] and [8]).

2. DEFINITIONS AND BASIC RESULTS

By a *weight*, we shall mean a locally integrable function ω on \mathbb{R}^n such that $0 < \omega(x) < \infty$ for a.e. $x \in \mathbb{R}^n$. Every weight ω gives rise to a measure on the measurable subsets of \mathbb{R}^n through integration. This measure will also be denoted by ω . Thus $\omega(E) = \int_E \omega \, dx$ for measurable sets $E \subset \mathbb{R}^n$.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be open and let ω be a weight. For $1 < p < \infty$, we define $L^p(\Omega, \omega)$, the Banach space of all measurable functions f defined on Ω for which

$$\|f\|_{L^p(\Omega, \omega)} = \left(\int_{\Omega} |f(x)|^p \omega(x) \, dx \right)^{1/p} < \infty.$$

Definition 2.2. Let $1 \leq p < \infty$.

- (a) The weight ω belongs to the Muckenhoupt class A_p ($\omega \in A_p$) if there is a constant $\mathbf{C} = C_{p,\omega}$ (called A_p -constant) such that

$$\begin{aligned} \left(\frac{1}{|B|} \int_B \omega \, dx \right) \left(\frac{1}{|B|} \int_B \omega^{-1/(p-1)} \, dx \right)^{p-1} &\leq \mathbf{C}, & \text{when } 1 < p < \infty \\ \left(\frac{1}{|B|} \int_B \omega \, dx \right) \left(\operatorname{ess\,sup}_B \frac{1}{\omega} \right) &\leq \mathbf{C}, & \text{when } p = 1, \end{aligned}$$

for every ball $B \subset \mathbb{R}^n$, where $|B|$ is the n -dimensional Lebesgue measure of B .

- (b) Let ω and v be weights. We shall say that the pair of weights (v, ω) satisfies the condition A_p , $1 \leq p < \infty$, if there is a constant C such that

$$\left(\frac{1}{|B|} \int_B v(x) \, dx \right) \left(\frac{1}{|B|} \int_B \omega^{-1/(p-1)}(x) \, dx \right)^{p-1} \leq C, \quad \text{when } 1 < p < \infty,$$

$$\frac{1}{|B|} \int_B v(x) \, dx \leq C \operatorname{ess\,inf}_B \omega, \quad \text{when } p = 1,$$

for every ball B in \mathbb{R}^n . The smallest constant C will be called the A_p -constant for the pair (ω, v) .

Remark 2.3. If $(v, \omega) \in A_p$ and $\omega \leq v$ then $\omega \in A_p$ and $v \in A_p$.

Example 2.4. The function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is a weight A_p if and only if $-n < \alpha < n(p-1)$ (see [7, Chapter 15]).

Remark 2.5. If $\omega \in A_p$, $1 \leq p < \infty$, then since $\omega^{-1/(p-1)}$ is locally integrable, when $p > 1$, and $1/\omega$ is locally bounded, when $p = 1$, we have $L^p(\Omega, \omega) \subset L^1_{\text{loc}}(\Omega)$ for every open set Ω and such that convergence in $L^p(\Omega, \omega)$ implies local convergence in $L^1(\Omega)$. If Ω is bounded, in the same way one obtains $L^p(\Omega, \omega) \subset L^1(\Omega)$. It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega, \omega)$.

Definition 2.6. We shall say that the pair of weights (v, ω) satisfies the condition S_p ($1 < p < \infty$) if there is a constant C (called the S_p -constant) such that

$$\int_B \left| M(\mu\chi_B)(x) \right|^p v(x) \, dx \leq C\mu(B) < \infty,$$

for every ball B , where $[Mf](x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| dy$ is the Hardy-Littlewood maximal function, $\mu = \omega^{-1/(p-1)}$ and $\mu(B) = \int_B \mu(x) dx$.

Remark 2.7. If $(v, \omega) \in S_p$, $1 < p < \infty$, then $(v, \omega) \in A_p$.

Theorem 2.8 (Muckenhoupt generalized theorem). *Let $1 < p < \infty$ and let (v, ω) be a pair of weights in \mathbb{R}^n . Then $M : L^p(\mathbb{R}^n, \omega) \rightarrow L^p(\mathbb{R}^n, v)$ is bounded*

$$\|Mf\|_{L^p(\mathbb{R}^n, v)} \leq C_M \|f\|_{L^p(\mathbb{R}^n, \omega)},$$

if and only if $(v, \omega) \in S_p$. The constant C_M is called Muckenhoupt constant and C_M depends only on n , p and the S_p -constant of (v, ω) .

Proof. See Theorem 4.9, Chapter IV in [6]. □

Definition 2.9. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let v, ω be weights. We define the space

$$\begin{aligned} W^{k,2}(\Omega, \omega, v) \\ = \left\{ u \in L^2(\Omega, v) : \int_{\Omega} \langle A \nabla u, \nabla u \rangle dx < \infty \text{ and } D^{\alpha} u \in L^2(\Omega, \omega), 2 \leq |\alpha| \leq k \right\} \end{aligned}$$

with the norm

$$\|u\|_{W^{k,2}(\Omega, \omega, v)} = \left(\int_{\Omega} u^2 v dx + \int_{\Omega} \langle A \nabla u, \nabla u \rangle dx + \sum_{2 \leq |\alpha| \leq k} \int_{\Omega} |D^{\alpha} u|^2 \omega dx \right)^{1/2}$$

where $A = (a_{ij})_{i,j=1,\dots,n}$ is the coefficient matrix of the operator L .

Remark 2.10. If $(v, \omega) \in S_2$ and $\omega \leq v$ then $C^\infty(\Omega)$ is dense in $W^{k,2}(\Omega, \omega, v)$ (see [1, Theorem 4.7]). In this case, we define $W_0^{k,2}(\Omega, \omega, v)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{k,2}(\Omega, \omega, v)} = \left(\int_{\Omega} \langle A \nabla u, \nabla u \rangle dx + \sum_{2 \leq |\alpha| \leq k} \int_{\Omega} |D^\alpha u|^2 \omega dx \right)^{1/2}.$$

Note that, by (1.2), we have

$$\int_{\Omega} |\nabla u|^2 \omega dx \leq \int_{\Omega} \langle A \nabla u, \nabla u \rangle dx \leq \int_{\Omega} |\nabla u|^2 v dx.$$

Definition 2.11. We say that an element $u \in W^{1,2}(\Omega, \omega, v)$ is a weak solution of the equation $Lu = g$ if

$$\int_{\Omega} \sum_{i,j=1}^n a_{ij} D_i u D_j \varphi dx = \int_{\Omega} g \varphi dx$$

for every $\varphi \in W_0^{1,2}(\Omega, \omega, v)$.

Remark 2.12. The existence and uniqueness result for the Dirichlet problem

$$(P) \begin{cases} Lu = g, & \text{in } \Omega \\ u - \psi \in W_0^{1,2}(\Omega, \omega, v) \end{cases}$$

where $\psi \in W^{1,2}(\Omega, \omega, v)$, can be found in [1, Theorem 4.9].

3. DIFFERENTIABILITY OF WEAK SOLUTIONS

In this section we prove that weak solutions $u \in W^{1,2}(\Omega, \omega, v)$ of the equation $Lu = g$, with some hypotheses, are twice weakly differentiable and $D_{ij}u \in L^2(\Omega', \omega)$ (that is, $u \in W^{2,2}(\Omega', \omega, v)$, $\forall \Omega' \subset\subset \Omega$).

Definition 3.1. Let u be a function on a bounded open set $\Omega \subset \mathbb{R}^n$ and denote by e_i the unit coordinate vector in the x_i direction. We define the difference quotient of u at x in the direction e_i by

$$(3.1) \quad \Delta_k^h u(x) = \frac{u(x + he_k) - u(x)}{h}, \quad (0 < |h| < \text{dist}(x, \partial\Omega)).$$

Lemma 3.2. Let $\Omega' \subset\subset \Omega$ and $0 < |h| < \text{dist}(\Omega', \partial\Omega)$. If $u, \varphi \in L_{\text{loc}}^2(\Omega, \omega)$, $\text{supp}(\varphi) \subset \Omega'$ and g is a measurable function with $|g(x)| \leq C\omega(x)$, then

$$(a) \quad \Delta_k^h(u\varphi)(x) = u(x + he_k)\Delta_k^h\varphi(x) + \varphi(x)\Delta_k^h u(x), \quad \text{with } 1 \leq k \leq n.$$

$$(b) \quad \int_{\Omega} g(x)u(x)\Delta_k^{-h}\varphi(x) \, dx = - \int_{\Omega} \varphi(x)\Delta_k^h(gu)(x) \, dx.$$

$$(c) \quad \text{If } \varphi \in C^1(\Omega), \text{ then } \Delta_k^h(D_j\varphi)(x) = D_j(\Delta_k^h\varphi)(x).$$

Proof. The proof of this lemma follows trivially from the Definition 3.1. □

Definition 3.3. Let ω be a weight in \mathbb{R}^n . We say that ω is uniformly A_p in each coordinate if

$$(a) \quad \omega \in A_p(\mathbb{R}^n);$$

$$(b) \quad \omega_i(t) = \omega(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n) \text{ is in } A_p(\mathbb{R}), \text{ for } x_1, \dots, x_{i-1}, x_{i+1}, \dots, \dots, x_n \text{ a.e., } 1 \leq i \leq n, \text{ with } A_p \text{ constant of } \omega_i \text{ bounded independently of } x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$$

Example 3.4. Let $\omega(x, y) = \omega_1(x)\omega_2(y)$, with $\omega_1(x) = |x|^{1/2}$ and $\omega_2(y) = |y|^{1/2}$. We have ω is uniformly A_2 in each coordinate.

Definition 3.5. Let v, ω be weights in \mathbb{R}^n .



- (a) We say that (v, ω) is uniformly A_p in each coordinate if $(v, \omega) \in A_p(\mathbb{R}^n)$ and $(v_i, \omega_i) \in A_p(\mathbb{R})$ ($1 \leq i \leq n$) with constant A_p of (v_i, ω_i) bounded independently of $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$.
- (b) We say that (v, ω) is uniformly S_p in each coordinate if $(v, \omega) \in S_p(\mathbb{R}^n)$ and $(v_i, \omega_i) \in S_p(\mathbb{R})$ ($1 \leq i \leq n$) with S_p -constant of (v_i, ω_i) bounded independently of $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$.

Lemma 3.6. *Let $u \in W^{1,2}(\Omega, \omega, v)$ and let (ω, v) be uniformly S_2 in each coordinate. Then for any $\Omega' \subset\subset \Omega$ and $0 < |h| < \text{dist}(\Omega', \partial\Omega)$, we have*

$$(3.2) \quad \|\Delta_k^h u\|_{L^2(\Omega', v)} \leq C \|D_k u\|_{L^2(\Omega, \omega)}$$

where $C = 2C_M$, and C_M is the Muckenhoupt constant.

Proof. Case 1: Let us suppose initially that $u \in C^\infty(\Omega)$. We have,

$$\begin{aligned} \Delta_k^h u(x) &= \frac{u(x + he_k) - u(x)}{h} = \frac{1}{h} \int_0^h D_k(x + \zeta e_k) d\zeta \\ &= \frac{1}{h} \int_0^h D_k u(x_1, \dots, x_{k-1}, x_k + \zeta, x_{k+1}, \dots, x_n) d\zeta. \end{aligned}$$

For $1 \leq k \leq n$, we define the functions

$$G_k(x) = \begin{cases} D_k u(x), & \text{if } x \in \Omega \\ 0, & \text{if } x \notin \Omega. \end{cases}$$



We have for $x \in \Omega' \subset \subset \Omega$ and h satisfying $0 < |h| < \text{dist}(\Omega', \partial\Omega)$,

$$\begin{aligned}
 |\Delta_k^h u(x)| &\leq \frac{1}{|h|} \left| \int_0^h |D_k u(x_1, \dots, x_{k-1}, x_k + \zeta, x_{k+1}, \dots, x_n)| d\zeta \right| \\
 &= \frac{1}{|h|} \left| \int_{x_k}^{x_k+h} |G_k(x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_n)| dt \right| \\
 &\leq \frac{1}{|h|} \left| \int_{x_k-h}^{x_k+h} |G_k(x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_n)| dt \right| \\
 &\leq 2M(G_k^{x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n})(\mathbf{x}_k),
 \end{aligned}$$

where $G_k^{x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n}(\mathbf{x}_k) = G_k(x_1, \dots, \mathbf{x}_k, \dots, x_n)$. Consequently, using the notation $\widehat{dx}_k = dx_1 \dots dx_{k-1} dx_{k+1} \dots dx_n$ (where the hat indicates the term that must be omitted in the product) and by Theorem 2.8, we obtain

$$\begin{aligned}
 &\int_{\Omega'} |\Delta_k^h u(x)|^2 v(x) dx \\
 &\leq 2^2 \int_{\Omega'} [M(G_k^{x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n})]^2(\mathbf{x}_k) v(x_1, \dots, x_k, \dots, x_n) dx \\
 &\leq 4 \int_{\mathbb{R}^n} [M(G_k^{x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n})]^2(\mathbf{x}_k) v(x_1, \dots, x_k, \dots, x_n) dx_1 \dots dx_k \dots dx_n \\
 &\leq 4 \int_{\mathbb{R}^{n-1}} \left(C_M^2 \int_{\mathbb{R}} |G_k^{x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n}(\mathbf{x}_k)|^2 \omega(x_1, \dots, \mathbf{x}_k, \dots, x_n) dx_k \right) \widehat{dx}_k \\
 &= 4C_M^2 \int_{\mathbb{R}^n} |G_k(x)|^2 \omega(x) dx = 4C_M^2 \int_{\Omega} |D_k u(x)|^2 \omega(x) dx,
 \end{aligned}$$

where C_M is independent of $x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n$ because (v, ω) is uniformly S_2 in each coordinate. Therefore

$$\|\Delta_k^h u\|_{L^2(\Omega', v)} \leq C \|D_k u\|_{L^2(\Omega, \omega)}, \quad \text{where } C = 2C_M.$$

Case 2: If $u \in W^{1,2}(\Omega, \omega, v)$ then there exists a sequence $\{u_m\}$, $u_m \in C^\infty(\Omega)$, Cauchy sequence in the norm $\|\cdot\|_{W^{1,p}(\Omega, \omega, v)}$. By Remark 2.10, we have

$$u_m \rightarrow u \quad \text{in } L^2(\Omega, v), \quad \text{and} \quad D_k u_m \rightarrow D_k u \quad \text{in } L^2(\Omega, \omega).$$

Since $(v, \omega) \in S_2$ and $\omega \leq v$, we have $\omega \in A_2$ and $v \in A_2$. Consequently, by Remark 2.5, there exists a subsequence $\{u_{m_j}\}$ such that $u_{m_j} \rightarrow u$ a.e. and $D_k u_{m_j} \rightarrow D_k u$ a.e.. This implies, for $0 < |h| < \text{dist}(\Omega', \partial\Omega)$, that

$$\Delta_k^h u_{m_j} \rightarrow \Delta_k^h u \quad \text{a.e..}$$

We have $\{\Delta_k^h u_{m_j}\}$ is a Cauchy sequence in $L^2(\Omega', v)$, for any $\Omega' \subset\subset \Omega$. In fact, using the first case, we have

$$\begin{aligned} \|\Delta_k^h u_{m_r} - \Delta_k^h u_{m_s}\|_{L^2(\Omega', v)} &= \|\Delta_k^h (u_{m_r} - u_{m_s})\|_{L^2(\Omega', v)} \\ &\leq C \|D_k (u_{m_r} - u_{m_s})\|_{L^2(\Omega, \omega)} \\ &= C \|D_k u_{m_r} - D_k u_{m_s}\|_{L^2(\Omega, \omega)} \\ &\rightarrow 0, \quad \text{as } m_r, m_s \rightarrow \infty. \end{aligned}$$

Therefore, there exists $g \in L^2(\Omega', v)$ such that $\Delta_k^h u_{m_j} \rightarrow g$ in $L^2(\Omega', v)$. Consequently, there exists a subsequence $\Delta_k^h u_{m_{j_r}} \rightarrow g$ a.e.. We can conclude that $\Delta_k^h u = g$ a.e.. Hence

$$\Delta_k^h u_{m_j} \rightarrow \Delta_k^h u \quad \text{in } L^2(\Omega', v).$$

This implies that

$$\begin{aligned} \|\Delta_k^h u\|_{L^2(\Omega', v)} &= \lim_{m_j \rightarrow \infty} \|\Delta_k^h u_{m_j}\|_{L^2(\Omega', v)} \\ &\leq C \lim_{m_j \rightarrow \infty} \|D_k u_{m_j}\|_{L^2(\Omega, \omega)} \\ &= C \|D_k u\|_{L^2(\Omega, \omega)}, \end{aligned}$$

that is, $\|\Delta_k^h u\|_{L^2(\Omega', v)} \leq C \|D_k u\|_{L^2(\Omega, \omega)}$. □

Lemma 3.7. *Let $u \in L^p(\Omega, \omega)$, $1 < p < \infty$, $\omega \in A_p$ and suppose there exists a constant C such that*

$$(3.3) \quad \|\Delta_k^h u\|_{L^p(\Omega', \omega)} \leq C, \quad k = 1, 2, \dots, n$$

for any $\Omega' \subset \subset \Omega$ and $0 < |h| < \text{dist}(\Omega', \partial\Omega)$ (with C independent of h). Then there exists $\vartheta_k \in L^p(\Omega, \omega)$ such that $D_k u = \vartheta_k$ in the weak sense and $\|D_k u\|_{L^p(\Omega, \omega)} \leq C$.

Proof. Since $\|\Delta_k^h u\|_{L^p(\Omega', \omega)} \leq C$, using $L^p(\Omega, \omega)$ is reflexive ($1 < p < \infty$), there exists a sequence $\{h_m\}$, $h_m \rightarrow 0$, and a function $\vartheta_k \in L^p(\Omega, \omega)$, with $\|\vartheta_k\|_{L^p(\Omega, \omega)} \leq C$, such that

$$(3.4) \quad \int_{\Omega} \Delta_k^{h_m} u(x) \varphi(x) \omega(x) dx \rightarrow \int_{\Omega} \vartheta_k(x) \varphi(x) \omega(x) dx$$

for all $\varphi \in L^{p'}(\Omega, \omega)$. Since $\omega \in A_p$, we have $\varphi = \psi/\omega \in L^{p'}(\Omega, \omega)$ for any $\psi \in C_0^\infty(\Omega)$. In fact,

$$\begin{aligned} \int_{\Omega} |\varphi|^{p'} \omega dx &= \int_{\Omega} |\psi|^{p'} \omega^{-p'} \omega dx \\ &\leq C_\psi \int_{\Omega} \omega^{1-p'} dx < \infty \quad (\text{because } \omega \in A_p). \end{aligned}$$

Setting $\varphi = \psi/\omega$ in (3.4), we obtain

$$\int_{\Omega} \Delta_k^{h_m} u(x) \psi(x) \, dx \rightarrow \int_{\Omega} \vartheta_k(x) \psi(x) \, dx, \quad \forall \psi \in C_0^\infty(\Omega).$$

Now for $h_m < \text{dist}(\text{supp } \psi, \partial\Omega)$, we have

$$\begin{aligned} \int_{\Omega} \Delta_k^{h_m} u(x) \psi(x) \, dx &= - \int_{\Omega} u(x) \Delta_k^{-h_m} \psi(x) \, dx \\ &\rightarrow - \int_{\Omega} u(x) D_k \psi(x) \, dx, \quad \text{with } h_m \rightarrow 0. \end{aligned}$$

Hence

$$\int_{\Omega} \vartheta_k(x) \psi(x) \, dx = - \int_{\Omega} u(x) D_k \psi(x) \, dx, \quad \forall \psi \in C_0^\infty(\Omega).$$

Therefore $D_k u = \vartheta_k$ in the weak sense. □

Remark 3.8. If the assumptions of Lemma 3.6 are satisfied and $\omega \leq v$, then we have

$$\|\Delta_k^h u\|_{L^2(\Omega', \omega)} \leq \|\Delta_k^h u\|_{L^2(\Omega', v)} \leq C \|D_k u\|_{L^2(\Omega', \omega)}.$$

We are able now to prove the main result of this paper.

Theorem 3.9. *Let $u \in W^{1,2}(\Omega, \omega, v)$ be a weak solution of the equation $Lu = g$ in Ω , and assume that*

- (a) $g/v \in L^2(\Omega, v)$;
- (b) *The pair of weights (v, ω) is uniformly S_2 in each coordinate;*

(c) $|\Delta_k^h a_{ij}(x)| \leq C_1 v(x)$, $x \in \Omega' \subset \subset \Omega$ a.e., $0 < |h| < \text{dist}(\Omega', \partial\Omega)$, with constant C_1 is independent of Ω' and h .

Then for any subdomain $\Omega' \subset \subset \Omega$, we have $u \in W^{2,2}(\Omega', \omega, v)$ and

$$(3.5) \quad \|u\|_{W^{2,2}(\Omega', \omega, v)} \leq \mathbf{C} (\|u\|_{W^{1,2}(\Omega, \omega, v)} + \|g/v\|_{L^2(\Omega, v)})$$

for $\mathbf{C} = \mathbf{C}(n, C_M, C_1, d')$, and $d' = \text{dist}(\Omega', \partial\Omega)$.

Proof. Since $u \in W^{1,2}(\Omega, \omega, v)$ is a weak solution of the equation $Lu = g$, then by Definition 2.11 we have,

$$(3.6) \quad \int_{\Omega} a_{ij}(x) D_i u(x) D_j \varphi(x) \, dx = \int_{\Omega} g(x) \varphi(x) \, dx,$$

for all $\varphi \in W_0^{1,2}(\Omega, \omega, v)$ (in particular for all $\varphi \in C_0^\infty(\Omega)$).

In (3.6) let us replace φ by $\Delta_k^{-h} \varphi$ ($1 \leq k \leq n$), with $\varphi \in C_0^\infty(\Omega)$, $\text{supp}(\varphi) \subset \subset \Omega$ and let $|2h| < \text{dist}(\text{supp}(\varphi), \partial\Omega)$. Then, by Lemma 3.2, we obtain

$$(3.7) \quad \begin{aligned} - \int_{\Omega} g(x) \Delta_k^{-h} \varphi(x) \, dx &= - \int_{\Omega} a_{ij}(x) D_i u(x) D_j (\Delta_k^{-h} \varphi(x)) \, dx \\ &= - \int_{\Omega} a_{ij}(x) D_i u(x) \Delta_k^{-h} (D_j \varphi)(x) \, dx \\ &= \int_{\Omega} \Delta_k^h (a_{ij} D_i u)(x) D_j \varphi(x) \, dx \\ &= \int_{\Omega} (a(x + h e_k) \Delta_k^h D_i u(x) + D_i u(x) \Delta_k^h a_{ij}(x)) D_j \varphi(x) \, dx \\ &= \int_{\Omega} ([h \Delta_k^h a_{ij}(x) + a_{ij}(x)] \Delta_k^h D_i u(x) + D_i u(x) \Delta_k^h a_{ij}(x)) \cdot D_j \varphi(x) \, dx. \end{aligned}$$

By Lemma 3.6, if $u \in W^{1,2}(\Omega, \omega, v)$ we have

$$(3.8) \quad \|\Delta_k^h u\|_{L^2(\Omega', v)} \leq C \|D_k u\|_{L^2(\Omega, \omega)} = \tilde{C}, \quad \forall \Omega' \subset\subset \Omega.$$

Since $u \in L^2(\Omega, v)$ and $v \in A_2$ (see Remark 2.3 and Remark 2.7), by Lemma 3.7 we have that

$$(3.9) \quad \|D_k u\|_{L^2(\Omega', v)} \leq \|D_k u\|_{L^2(\Omega, v)} \leq \tilde{C} = C \|D_k u\|_{L^2(\Omega, \omega)}, \quad \forall \Omega' \subset\subset \Omega.$$

Consequently, in (3.7), we obtain

$$(3.10) \quad \begin{aligned} & \int_{\Omega} a_{ij}(x) D_i(\Delta_k^h u(x)) D_j \varphi(x) \, dx \\ &= - \int_{\Omega} g(x) \Delta_k^{-h} \varphi(x) \, dx - \int_{\Omega} \Delta_k^h a_{ij}(x) D_i u(x) D_j \varphi(x) \, dx \\ & \quad - \int_{\Omega} h \Delta_k^h a_{ij}(x) \Delta_k^h D_i u(x) D_j \varphi(x) \, dx \\ & \leq \int_{\Omega} |g(x)| |\Delta_k^{-h} \varphi(x)| \, dx + \int_{\Omega} |\Delta_k^h a_{ij}(x)| |D_i u(x)| |D_j \varphi(x)| \, dx \\ & \quad + |h| \int_{\Omega} |\Delta_k^h a_{ij}(x)| |\Delta_k^h D_i u(x)| |D_j \varphi(x)| \, dx \end{aligned}$$

We have, by Lemma 3.6,

$$\begin{aligned} \int_{\Omega} |g(x)| |\Delta_k^{-h} \varphi(x)| \, dx &= \int_{\Omega} \left(\frac{|g(x)|}{v(x)} \right) v^{1/2}(x) |\Delta_k^{-h} \varphi(x)| v^{1/2}(x) \, dx \\ &\leq \left\| \frac{g}{v} \right\|_{L^2(\Omega, v)} \|\Delta_k^{-h} \varphi\|_{L^2(\text{supp} \varphi, v)} \\ &\leq C \left\| \frac{g}{v} \right\|_{L^2(\Omega, v)} \|D_k \varphi\|_{L^2(\Omega, \omega)}. \end{aligned}$$

And, using (3.9), we obtain

$$\begin{aligned}
 & \int_{\Omega} |\Delta_k^h a_{ij}(x)| |D_i u(x)| |D_j \varphi(x)| \, dx \\
 & \leq C_1 \int_{\text{supp } \varphi} v(x) |D_i u(x)| |D_j \varphi(x)| \, dx \\
 & \leq C_1 \int_{\text{supp } \varphi} |D_i u(x)| v^{1/2}(x) |D_j \varphi(x)| v^{1/2}(x) \, dx \\
 & \leq C_1 C^2 \left(\int_{\Omega} |D_i u(x)|^2 \omega(x) \, dx \right)^{1/2} \left(\int_{\Omega} |D_j \varphi(x)|^2 \omega(x) \, dx \right)^{1/2}
 \end{aligned}$$

Hence, in (3.10), we obtain

$$\begin{aligned}
 & \int_{\Omega} a_{ij}(x) D_i(\Delta_k^h u(x)) D_j \varphi(x) \, dx \\
 (3.11) \quad & \leq C \left(\|u\|_{W^{1,2}(\Omega, \omega, v)} + \left\| \frac{g}{v} \right\|_{L^2(\Omega, \omega)} \right) \|D\varphi\|_{L^2(\Omega, \omega)} \\
 & \quad + C_1 |h| \int_{\Omega} v(x) |\Delta_k^h D_i u(x)| |D_j \varphi(x)| \, dx
 \end{aligned}$$

Let $\Omega' \subset\subset \Omega$. To proceed further let us take a function $\eta \in C_0^\infty(\Omega)$ satisfying $0 \leq \eta \leq 1$, $\eta \equiv 1$ in Ω' and with $\|D\eta\|_\infty \leq 2/d'$, where $d' = \text{dist}(\Omega', \partial\Omega)$ and set $\varphi = \eta^2 \Delta_k^h u$ (with $|2h| < \text{dist}(\text{supp}(\eta), \partial\Omega)$). We have

$$D_j \varphi = 2\eta D_j \eta \Delta_k^h u + \eta^2 D_j(\Delta_k^h u).$$

We denote by $a = \|u\|_{W^{1,2}(\Omega,\omega,v)} + \left\| \frac{g}{v} \right\|_{L^2(\Omega,v)}$. In (3.11) we obtain

$$\begin{aligned}
 & \int_{\Omega} \langle A\eta D(\Delta_k^h u), \eta D(\Delta_k^h u) \rangle dx \\
 &= \int_{\Omega} a_{ij}(x) [\eta(x) D_i(\Delta_k^h u(x))] [\eta(x) D_j(\Delta_k^h u(x))] dx \\
 &\leq C \left(\|u\|_{W^{1,2}(\Omega,\omega,v)} + \left\| \frac{g}{v} \right\|_{L^2(\Omega,v)} \right) \|2\eta D_j \eta \Delta_k^h u \\
 &\quad + \eta^2 \Delta_k^h (D_j u)\|_{L^2(\Omega,\omega)} + 2 \int_{\Omega} |a_{ij}(x)| |\eta D_i \Delta_k^h u| |D_j \eta \Delta_k^h u| dx \\
 &\quad + C_1 |h| \int_{\Omega} v |\Delta_k^h D_i u| |2\eta D_j \eta \Delta_k^h u + \eta^2 D_j \Delta_k^h u| dx \\
 &\leq Ca (2\|\eta D_j \eta \Delta_k^h u\|_{L^2(\text{supp}\eta,\omega)} + \|\eta D_j(\Delta_k^h u)\|_{L^2(\text{supp}\eta,\omega)}) \\
 &\quad + 2I_2 + C_1 |h| I_3 \\
 &\leq Ca (\|D_j \eta\|_{\infty} \|\Delta_k^h u\|_{L^2(\text{supp}\eta,\omega)} + \|\eta D_j \Delta_k^h u\|_{L^2(\text{supp}\eta,\omega)}) \\
 &\quad + 2I_2 + C_1 |h| I_3 \\
 &\leq Ca \|D_j \eta\|_{\infty} \|D_k u\|_{L^2(\Omega,\omega)} + Ca \|\eta D_j \Delta_k^h u\|_{L^2(\Omega,\omega)} + 2I_2 + C_1 |h| I_3,
 \end{aligned}$$

i. e.

$$\begin{aligned}
 (3.12) \quad & \int_{\Omega} \langle A\eta D(\Delta_k^h u), \eta D(\Delta_k^h u) \rangle dx \\
 & \leq C a^2 \|D_j \eta\|_{\infty} + Ca \|\eta D_j \Delta_k^h u\|_{L^2(\Omega,\omega)} + 2I_2 + C_1 |h| I_3.
 \end{aligned}$$

Let us estimate the integrals I_2 and I_3 . By (1.2), we have that $|a_{ij}(x)| \leq Cv(x)$ a.e. in Ω . Using (3.8), (3.9) and Remark 3.8, we obtain

$$\begin{aligned}
 I_2 &= \int_{\Omega} |a_{ij}| |\eta D_i \Delta_k^h u| |D_j \eta \Delta_k^h u| \, dx \\
 &\leq C \int_{\text{supp} \eta} v |\eta D_i \Delta_k^h u| |D_j \eta \Delta_k^h u| \, dx \\
 &\leq C \|\eta D_i \Delta_k^h u\|_{L^2(\text{supp} \eta, v)} \|D_j \eta \Delta_k^h u\|_{L^2(\text{supp} \eta, v)} \\
 &\leq C \|D_i(\eta \Delta_k^h u) - D_i \eta \Delta_k^h u\|_{L^2(\text{supp} \eta, v)} \|D_j \eta\|_{\infty} \|\Delta_k^h u\|_{L^2(\text{supp} \eta, v)} \\
 &\leq C (C_M \|D_i \eta \Delta_k^h u + \eta D_i \Delta_k^h u\|_{L^2(\text{supp} \eta, \omega)} + C \|u\|_{W^{1,2}(\Omega, \omega, v)}) \\
 &\quad \cdot \|u\|_{W^{1,2}(\Omega, \omega, v)} \\
 &\leq C (\|D_k u\|_{L^2(\Omega, \omega)} + \|\eta D_i \Delta_k^h u\|_{L^2(\Omega, \omega)} + \|u\|_{W^{1,2}(\Omega, \omega, v)}) \\
 &\quad \cdot \|u\|_{W^{1,2}(\Omega, \omega, v)} \\
 &\leq C (\|u\|_{W^{1,2}(\Omega, \omega, v)} + \|\eta D_i \Delta_k^h u\|_{L^2(\Omega, \omega)}) \|u\|_{W^{1,2}(\Omega, \omega, v)} \\
 &\leq C \|u\|_{W^{1,2}(\Omega, \omega, v)}^2 + C \|\eta D_i \Delta_k^h u\|_{L^2(\Omega, \omega)} \|u\|_{W^{1,2}(\Omega, \omega, v)} \\
 &\leq Ca^2 + Ca \|\eta D_j \Delta_k^h u\|_{L^2(\Omega, \omega)}.
 \end{aligned}
 \tag{3.13}$$

We also have,

$$\begin{aligned}
 I_3 &= \int_{\Omega} v |\Delta_k^h D_i u| |2\eta D_j \eta \Delta_k^h u + \eta^2 D_j \Delta_k^h u| \, dx \\
 &\leq 2 \int_{\text{supp} \eta} v |\Delta_k^h D_i u| |\eta D_j \eta \Delta_k^h u| \, dx + \int_{\text{supp} \eta} v |\Delta_k^h D_i u| |\eta^2 D_j \Delta_k^h u| \, dx
 \end{aligned}$$

$$\begin{aligned}
&= 2 \int_{\text{supp}\eta} v |\eta \Delta_k^h D_i u| |D_j \eta \Delta_k^h u| \, dx + \int_{\text{supp}\eta} v |\eta D_i \Delta_k^h u| |\eta D_j \Delta_k^h u| \, dx \\
&\leq 2 \|\eta D_i \Delta_k^h u\|_{L^2(\text{supp}\eta, v)} \|D_j \eta \Delta_k^h u\|_{L^2(\text{supp}\eta, v)} \\
&\quad + \|\eta D_i \Delta_k^h u\|_{L^2(\text{supp}\eta, v)} \|\eta D_j \Delta_k^h u\|_{L^2(\text{supp}\eta, v)}.
\end{aligned}
\tag{3.14}$$

Using (3.8) and (3.9), we obtain

$$\begin{aligned}
\|D_j \eta \Delta_k^h u\|_{L^2(\text{supp}\eta, v)} &\leq \|D_j \eta\|_\infty \|\Delta_k^h u\|_{L^2(\text{supp}\eta, v)} \leq C \|D_j \eta\|_\infty \|D_k u\|_{L^2(\Omega, \omega)} \\
\|D_j \eta\|_\infty \|u\|_{W^{1,2}(\Omega, \omega, v)} &\leq Ca \|D_j \eta\|_\infty.
\end{aligned}
\tag{3.15}$$

And we also have,

$$\begin{aligned}
\|\eta D_i \Delta_k^h u\|_{L^2(\text{supp}\eta, v)} &= \|D_i(\eta \Delta_k^h u) - \Delta_k^h u D_i \eta\|_{L^2(\text{supp}\eta, v)} \\
&\leq \|D_i(\eta \Delta_k^h u)\|_{L^2(\text{supp}\eta, v)} + \|\Delta_k^h u D_i \eta\|_{L^2(\text{supp}\eta, v)} \\
&\leq C_M \|D_i(\eta \Delta_k^h u)\|_{L^2(\text{supp}\eta, \omega)} + \|D_i \eta\|_\infty \|\Delta_k^h u\|_{L^2(\text{supp}\eta, v)} \\
&\leq C_M \|D_i \eta \Delta_k^h u + \eta D_i \Delta_k^h u\|_{L^2(\text{supp}\eta, \omega)} \\
&\quad + C_M \|D_i \eta\|_\infty \|D_k u\|_{L^2(\Omega, \omega)} \\
&\leq Ca \|D_j \eta\|_\infty + C_M \|\eta D_i \Delta_k^h u\|_{L^2(\text{supp}\eta, \omega)}.
\end{aligned}
\tag{3.16}$$

By condition (1.2) we have,

$$\begin{aligned}
\int_{\Omega} a_{ij}(x) [\eta(x) D_j(\Delta_k^h u(x))] [\eta(x) D_i(\Delta_k^h u(x))] \, dx &\geq \int_{\Omega} |\eta(x) D(\Delta_k^h u(x))|^2 \omega(x) \, dx \\
&= \int_{\Omega} |\eta(x) \Delta_k^h(Du(x))|^2 \omega(x) \, dx.
\end{aligned}
\tag{3.17}$$



We denote by $b = \|\eta D(\Delta_k^h u)\|_{L^2(\Omega, \omega)}$. By (3.12), (3.13), (3.14), (3.15), (3.16) and (3.17), and using Young's inequality, we obtain

$$\begin{aligned} b^2 &\leq C a^2 + C a b + C |h| a^2 + C a b |h| + C |h| b^2 \\ &\leq C a^2 + C \frac{\varepsilon^{-2}}{2} a^2 + C |h| a^2 + C \frac{\varepsilon^2}{2} b^2 + C \frac{\varepsilon^{-2}}{2} a^2 + C \frac{\varepsilon^2}{2} b^2 |h|^2 + C |h| b^2 \\ &= (C + C \varepsilon^{-2} + C |h|) a^2 + \left(C \frac{\varepsilon^2}{2} + C \frac{\varepsilon^2}{2} |h|^2 + C |h| \right) b^2. \end{aligned}$$

Choose $\varepsilon > 0$ and h such that

$$\frac{C \varepsilon^2}{2} + \frac{C \varepsilon^2}{2} |h|^2 + C |h| \leq \frac{1}{2},$$

we obtain,

$$\int_{\Omega} |\eta \Delta_k^h D u|^2 \omega \, dx \leq \mathbf{C} \left(\|u\|_{W^{1,2}(\Omega, \omega, v)} + \left\| \frac{g}{v} \right\|_{L^2(\Omega, v)} \right)^2.$$

Using $\eta \equiv 1$ in Ω' , we have

$$\int_{\Omega'} |\Delta_k^h D u|^2 \omega \, dx \leq \mathbf{C} \left(\|u\|_{W^{1,2}(\Omega, \omega, v)} + \left\| \frac{g}{v} \right\|_{L^2(\Omega, v)} \right)^2.$$

Then we conclude that

$$\begin{aligned} \|D_j(\Delta_k^h u)\|_{L^2(\Omega', \omega)}^2 &= \|\Delta_k^h D_j u\|_{L^2(\Omega', \omega)}^2 \leq \|\Delta_k^h D u\|_{L^2(\Omega', \omega)}^2 \\ &\leq \mathbf{C} \left(\|u\|_{W^{1,2}(\Omega, \omega, v)} + \left\| \frac{g}{v} \right\|_{L^2(\Omega, v)} \right)^2. \end{aligned}$$



By Lemma 3.7, we have there exists $D_{jk}u$ and

$$\|D_{jk}u\|_{L^2(\Omega',\omega)} \leq \mathbf{C} \left(\|u\|_{W^{1,2}(\Omega,\omega,v)} + \left\| \frac{g}{v} \right\|_{L^2(\Omega,v)} \right).$$

Therefore $u \in W^{2,2}(\Omega', \omega, v)$, $\forall \Omega' \subset \subset \Omega$. □

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