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SHARP BOUNDS OF FEKETE-SZEGŐ FUNCTIONAL FOR QUASI-SUBORDINATION CLASS INVOLVING Q-CALCULUS

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Abstract. In the present paper, we introduce a certain subclass $\mathcal{K}_Q(q,\lambda,\gamma,\varphi)$ of analytic functions by means of quasi-subordination and $q-$calculus. Sharp bounds of the Fekete-Szegő functional for functions belonging to the class $\mathcal{K}_Q(q,\lambda,\gamma,\varphi)$ are obtained. The results obtained here are extension of earlier known work of the certain subclasses involving the quasi-subordination and majorization. Several special cases of the main results are also mentioned briefly.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U}), \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ and satisfy the conditions $f(0) = 0$ and $f'(0) = 1$ for every $z \in \mathbb{U}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all univalent functions f in the unit disk $\mathbb{U}$.

For two analytic functions f and g, we say that f is subordinate to g, written as

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω in the unit disk $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$ for all $z \in \mathbb{U}$. In particular, if the function g is univalent in $\mathbb{U}$, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. Comprehensive details on subordination can be found in [2].

In 1970, Robertson [21] introduced the concept of quasi-subordination. An analytic function f is said to be quasi-subordinate to an analytic function g in the unit disk $\mathbb{U}$ if there exists an analytic function ϕ with $|\phi(z)| \leq 1$ such that $f(z)/\phi(z)$ is analytic in $\mathbb{U}$ and

$$\frac{f(z)}{\phi(z)} \prec g(z) \quad (z \in \mathbb{U}).$$

We denote the above espression by

$$f(z) \prec_Q g(z) \quad (z \in \mathbb{U}),$$

which is equivalent to $f(z) = \phi(z)g(\omega(z))$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$. It is observed that if $\phi(z) \equiv 1$, then $f(z) = g(\omega(z))$ so that $f(z) \prec g(z)$ in $\mathbb{U}$. Also, if $\omega(z) = z$, then $f(z) = \phi(z)g(z)$, and it is said to be that f is majorized by g denoted by $f(z) \ll g(z)$ in $\mathbb{U}$. The concept of majorization was introduced by MacGregor [16] in 1967. Therefore, it is obvious that quasi-subordination is a generalization of subordination and majorization. For works related to quasi-subordination, one may refer to [6, 11, 17, 20, 22] and references given therein.

A typical problem in geometric function theory is to study a functional made up of combinations of the coefficients of f. In 1933, Fekete and Szegő [4] obtained a sharp bound of the functional $a_3 - \nu a_2^2$ with real ν $(0 \leq \nu \leq 1)$ for a univalent function f. Since then, the problem of finding the sharp bounds for this functional of any compact family of functions $f \in \mathcal{A}$ with any complex ν is known as the classical Fekete-Szegő problem or inequality. Several authors have investigated the Fekete-Szegő functionals for various subclasses of univalent and multivalent functions (see [1, 13, 14, 18, 20, 23]).

Quantum calculus (or $q-$calculus) is a theory of calculus where smoothness is not required. In 1908 and 1910, Jackson initiated in-depth study of $q-$calculus and developed the $q-$derivative and $q-$integral in a systematic way (see [7, 8, 9]). Later, quantum calculus has been used in various branches of physics and mathematics, as for example, in the areas of ordinary fractional calculus, orthogonal polynomials, basic hypergeometric functions, combinatorics, the calculus of variations, the theory of relativity, optimal control problems, $q-$difference and $q-$integral equations and more recently in geometric function theory.

Let $q \in (0, 1)$. The $q-$derivative (or $q-$difference) operator, introduced by Jackson [7], is defined as

$$(D_q f)(z) = \frac{f(z) - f(qz)}{(1 - q)z} \quad (z \neq 0).$$

We note that $\lim_{q \to 1^-}(D_q f)(z) = f'(z)$ if f is differentiable at z. For a function f

of the form (1), we observe that

$$(D_q f)(z) = \sum_{n=1}^{\infty} [n]_q a_n z^{n-1},$$

where

$$[n]_q = \frac{1-q^n}{1-q}$$

is called $q-$number (or $q-$bracket) of n. Clearly, $\lim_{q\to 1^-}[n]_q = n$. For more details of $q-$calculus, one may refer to [5] and [10].

Ma and Minda [15] gave the Ma-Minda type convex functions

$$\mathcal{C}(\varphi) := \left\{ f \in \mathcal{A} : 1 + z\frac{f''(z)}{f'(z)} \prec \varphi(z) \right\},$$

where φ is an analytic function with positive real part in $\mathbb{U}$ with $\varphi(0) = 1$ and $\varphi'(0) > 0$, which maps the unit disk $\mathbb{U}$ onto the region of starlike with respect to 1 and $\varphi(\mathbb{U})$ is symmetric with respect to the real axis. The Taylor series expansion of such a function is

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots, \quad (B_1 \in \mathbb{R}^+). \tag{2}$$

Motivated by earlier works on quasi-subordination and using $q-$difference operator, we introduce the following new subclass of analytic functions.

Definition 1. *Let* $0 \le \lambda \le 1$ *and* $\gamma \in \mathbb{C}\backslash\{0\}$*. Then a function* $f \in \mathcal{A}$ *given by* (1) *is in the class* $\mathcal{K}_Q(q,\lambda,\gamma,\varphi)$ *if it satisfies the condition*

$$\frac{1}{\gamma}\Big(\frac{zD_q f(z) + qz^2 D_q(D_q f(z))}{(1-\lambda)z + \lambda z D_q f(z)} - 1\Big) \prec_Q (\varphi(z) - 1), \tag{3}$$

where φ *is given by* (2) *and* $z \in \mathbb{U}$.

It follows that a function f is in the class $\mathcal{K}_Q(q,\lambda,\gamma,\varphi)$ if and only if there exists an analytic function ϕ with $|\phi(z)| \le 1$ in $\mathbb{U}$ such that

$$\frac{\frac{1}{\gamma}\Big(\frac{zD_q f(z) + qz^2 D_q(D_q f(z))}{(1-\lambda)z + \lambda z D_q f(z)} - 1\Big)}{\phi(z)} \prec (\varphi(z) - 1),$$

where φ is given by (2) and $z \in \mathbb{U}$.

If we set $\phi(z) \equiv 1 \ (z \in \mathbb{U})$, then the class $\mathcal{K}_Q(q,\lambda,\gamma,\varphi)$ is denoted by $\mathcal{K}(q,\lambda,\gamma,\varphi)$ satisfying the condition that

$$1 + \frac{1}{\gamma}\Big(\frac{zD_q f(z) + qz^2 D_q(D_q f(z))}{(1-\lambda)z + \lambda z D_q f(z)} - 1\Big) \prec \varphi(z) \ \ (z \in \mathbb{U}).$$

For special values of parameters, we get the following new and known subclasses:

Example 1. *i) For $\lambda = 0$, the class $\mathcal{K}_Q(q, \lambda, \gamma, \varphi)$ reduces to the class $\mathcal{K}_Q(q, \gamma, \varphi)$ defined by*

$$\frac{1}{\gamma}\big(D_q f(z) + qzD_q(D_q f(z)) - 1\big) \prec_Q (\varphi(z) - 1).$$

ii) For $\lambda = 1$, the class $\mathcal{K}_Q(q, \lambda, \gamma, \varphi)$ reduces to the class $\mathcal{C}_Q(q, \gamma, \varphi)$ defined by

$$\frac{1}{\gamma}\Big(\frac{qzD_q(D_q f(z))}{D_q f(z)}\Big) \prec_Q (\varphi(z) - 1).$$

Remark 1. *For $\lambda = 1$ and $q \to 1^-$, we get the class $\mathcal{K}_Q(q, \lambda, \gamma, \varphi) =: \mathcal{C}_Q(\gamma, \varphi)$ defined in [3]. For $\gamma = 1$, $\lambda = 1$ and $q \to 1^-$, we get the class $\mathcal{K}_Q(q, \lambda, \gamma, \varphi) =: \mathcal{C}_Q(\varphi)$ defined in [17]. We also note that when $\phi(z) \equiv 1$, $\gamma = 1$, $\lambda = 1$ and $q \to 1^-$, we get the class $\mathcal{K}_Q(q, \lambda, \gamma, \varphi) =: \mathcal{C}(\varphi)$ of Ma-Minda type convex functions [15].*

In order to derive our main results, we recall here the following well-known lemmas:

Lemma 1. *[12, p.10] Let the Schwarz function ω be given by*

$$\omega(z) = \omega_1 z + \omega_2 z^2 + \cdots, \quad (z \in \mathbb{U}) \tag{4}$$

then

$$|\omega_1| \le 1, \quad |\omega_2 - \nu\omega_1^2| \le 1 + (|\nu| - 1)|\omega_1^2| \le \max\{1, |\nu|\},$$

where $\nu \in \mathbb{C}$. The result is sharp for the functions $\omega(z) = z$ or $\omega(z) = z^2$.

Lemma 2. *[19, p.172] Let ϕ be an analytic function in $\mathbb{U}$ with $|\phi(z)| \le 1$ and let*

$$\phi(z) = A_0 + A_1 z + A_2 z^2 + \cdots, \quad (z \in \mathbb{U}). \tag{5}$$

Then $|A_0| \le 1$ and $|A_n| \le 1 - |A_0|^2 \le 1$ for $n > 0$.

In the present paper, we introduce upper bounds on the Fekete-Szegő functional for functions belonging to the class $\mathcal{K}_Q(q, \lambda, \gamma, \varphi)$. Several new and known consequences of these results are also pointed out.

2. Main Results

Theorem 3. *Let $0 \le \lambda \le 1$ and $\gamma \in \mathbb{C}\backslash\{0\}$. If a function $f \in \mathcal{A}$ of the form (1) belongs to the class $\mathcal{K}_Q(q, \lambda, \gamma, \varphi)$, then*

$$|a_2| \le \frac{|\gamma| B_1}{[2]_q([2]_q - \lambda)}, \tag{6}$$

and for any $\nu \in \mathbb{C}$

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{[3]_q([3]_q - \lambda)} \max\left\{1, \left|\frac{B_2}{B_1} - J B_1\right|\right\}, \tag{7}$$

where

$$J = \gamma\Big(\frac{[3]_q([3]_q - \lambda)}{[2]_q^2([2]_q - \lambda)^2}\nu - \frac{\lambda}{[2]_q - \lambda}\Big). \tag{8}$$

The results are sharp.

Proof. Let $f \in \mathcal{K}_Q(q, \lambda, \gamma, \varphi)$. In view of (3), there exist a Schwarz function ω and an analytic function ϕ such that

$$\frac{1}{\gamma}\Big(\frac{zD_q f(z) + qz^2 D_q(D_q f(z))}{(1-\lambda)z + \lambda z D_q f(z)} - 1\Big) = \phi(z)(\varphi(\omega(z)) - 1) \quad (z \in \mathbb{U}).$$

Series expansions for f and its successive derivatives from (1) gives us

$$\begin{aligned}
&\frac{1}{\gamma}\Big(\frac{zD_q f(z) + qz^2 D_q(D_q f(z))}{(1-\lambda)z + \lambda z D_q f(z)} - 1\Big) \\
&\quad = \frac{1}{\gamma}\Big[[2]_q([2]_q - \lambda)a_2 z + \big([3]_q([3]_q - \lambda)a_3 - [2]_q^2\lambda([2]_q - \lambda)a_2^2\big)z^2 + ...\Big].
\end{aligned} \tag{9}$$

By utilizing (2), (4) and (5), we obtain

$$\varphi(\omega(z)) - 1 = B_1\omega_1 z + (B_1\omega_2 + B_2\omega_1^2)z^2 + \cdots,$$

and

$$\phi(z)\big(\varphi(\omega(z)) - 1\big) = A_0 B_1 \omega_1 z + [A_1 B_1 \omega_1 + A_0(B_1\omega_2 + B_2\omega_1^2)]z^2 + \cdots. \tag{10}$$

Comparing the coefficients of the expansions (9) and (10), we get

$$a_2 = \frac{\gamma A_0 B_1 \omega_1}{[2]_q([2]_q - \lambda)} \tag{11}$$

and

$$a_3 = \frac{\gamma B_1}{[3]_q([3]_q - \lambda)}\Big[A_1\omega_1 + A_0\left\{\omega_2 + \Big(\frac{\gamma\lambda A_0 B_1}{[2]_q - \lambda} + \frac{B_2}{B_1}\Big)\omega_1^2\right\}\Big].$$

Thus for any $\nu \in \mathbb{C}$, we have

$$\begin{aligned}
a_3 - \nu a_2^2 &= \frac{\gamma B_1}{[3]_q([3]_q - \lambda)} \\
&\quad \times \Big[A_1\omega_1 + \Big(\omega_2 + \frac{B_2}{B_1}\omega_1^2\Big)A_0 - \Big(\frac{[3]_q([3]_q - \lambda)\gamma}{[2]_q^2([2]_q - \lambda)^2}\nu - \frac{\gamma\lambda}{[2]_q - \lambda}\Big)B_1 A_0^2 \omega_1^2\Big] \\
&= \frac{\gamma B_1}{[3]_q([3]_q - \lambda)}\Big[A_1\omega_1 + \Big(\omega_2 + \frac{B_2}{B_1}\omega_1^2\Big)A_0 - J B_1 A_0^2 \omega_1^2\Big],
\end{aligned} \tag{12}$$

where J is given by (8).

Since $\phi(z) = A_0 + A_1 z + A_2 z^2 + \cdots$ is analytic and bounded by one in $\mathbb{U}$, therefore we have (see [19], p. 172)

$$|A_0| \leq 1 \;\; and \;\; A_1 = (1 - A_0^2)y \qquad (y \leq 1). \tag{13}$$

Using $|A_0| \leq 1$ and $|\omega(z)| \leq 1$ in (11), we easily get (6). Also, substituting (13) into (12), we obtain

$$a_3 - \nu a_2^2 = \frac{\gamma B_1}{[3]_q([3]_q - \lambda)}\Big[y\omega_1 + \Big(\omega_2 + \frac{B_2}{B_1}\omega_1^2\Big)A_0 - \Big(B_1 J\omega_1^2 + y\omega_1\Big)A_0^2\Big]. \tag{14}$$

We consider two cases of (14). Firstly, if A_0=0 in (14), we arrive at

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{[3]_q([3]_q - \lambda)}. \tag{15}$$

But if $A_0 \neq 0$, let us then suppose that

$$G(A_0) = y\omega_1 + \Big(\omega_2 + \frac{B_2}{B_1}\omega_1^2\Big)A_0 - \Big(B_1 J\omega_1^2 + y\omega_1\Big)A_0^2,$$

which is a quadratic polynomial in A_0, and hence analytic in $|A_0| \leq 1$. Maximum value of $|G(A_0)|$ is attained at $A_0 = e^{\iota\theta}$ $(0 \leq \theta < 2\pi)$, we find that

$$\begin{aligned} \max|G(A_0)| &= \max_{0\leq\theta<2\pi}|G(e^{\iota\theta})| = |G(1)| \\ &= \Big|\omega_2 - \Big(JB_1 - \frac{B_2}{B_1}\Big)\omega_1^2\Big|. \end{aligned}$$

Therefore, it follows from (14) that

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{[3]_q([3]_q - \lambda)}\Big|\omega_2 - \Big(JB_1 - \frac{B_2}{B_1}\Big)\omega_1^2\Big|, \tag{16}$$

which on using Lemma 1, shows that

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{[3]_q([3]_q - \lambda)} \max\left\{1, \left|\frac{B_2}{B_1} - JB_1\right|\right\},$$

and this last above inequality together with (15) establish the result given by (7). These results are sharp for the function f given by

$$1 + \frac{1}{\gamma}\Big(\frac{zD_q f(z) + qz^2 D_q(D_q f(z))}{(1-\lambda)z + \lambda z D_q f(z)} - 1\Big) = \varphi(z),$$

$$1+\frac{1}{\gamma}\Big(\frac{zD_qf(z)+qz^2D_q(D_qf(z))}{(1-\lambda)z+\lambda zD_qf(z)}-1\Big)=\varphi(z^2),$$

and

$$1+\frac{1}{\gamma}\Big(\frac{zD_qf(z)+qz^2D_q(D_qf(z))}{(1-\lambda)z+\lambda zD_qf(z)}-1\Big)=z(\varphi(z)-1).$$

This completes the proof of Theorem 3.

For $\lambda=0$ and $\lambda=1$, Theorem 3 reduces to the following corollaries, respectively.

Corollary 4. *If $f\in\mathcal{A}$ is in the class $\mathcal{K}_Q(q,\gamma,\varphi)$, then we have*

$$|a_2|\leq\frac{|\gamma|B_1}{[2]_q^2},$$

and for some $\nu\in\mathbb{C}$

$$|a_3-\nu a_2^2|\leq\frac{|\gamma|B_1}{[3]_q^2}\max\Big\{1,\Big|\frac{B_2}{B_1}-\frac{[3]_q^2\gamma B_1}{[2]_q^4}\nu\Big|\Big\}.$$

The results are sharp.

Corollary 5. *If $f\in\mathcal{A}$ is in the class $\mathcal{C}_Q(q,\gamma,\varphi)$, then we have*

$$|a_2|\leq\frac{|\gamma|B_1}{[2]_q([2]_q-1)},$$

and for some $\nu\in\mathbb{C}$

$$|a_3-\nu a_2^2|\leq\frac{|\gamma|B_1}{[3]_q([3]_q-1)}\max\left\{1,\left|\frac{B_2}{B_1}-\Big(\frac{[3]_q([3]_q-1)}{[2]_q^2([2]_q-1)^2}\nu-\frac{1}{[2]_q-1}\Big)\gamma B_1\right|\right\}.$$

The results are sharp.

Remark 2. *Letting $\gamma=\lambda=1$ and $q\to1^-$, Corollary 5 reduces to an improved result of given in [17, Theorem 2.4].*

The next theorem gives a result based on majorization.

Theorem 6. *Let $0\leq\lambda\leq1$ and $\gamma\in\mathbb{C}\backslash\{0\}$. If a function $f\in\mathcal{A}$ of the form* (1) *satisfies*

$$\frac{1}{\gamma}\Big(\frac{zD_qf(z)+qz^2D_q(D_qf(z))}{(1-\lambda)z+\lambda zD_qf(z)}-1\Big)\ll(\varphi(z)-1)\quad(z\in\mathbb{U}),\tag{17}$$

then

$$|a_2| \leq \frac{|\gamma| B_1}{[2]_q([2]_q-\lambda)}, \tag{18}$$

and for any $\nu \in \mathbb{C}$

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{[3]_q([3]_q-\lambda)} \max\left\{1, \left|\frac{B_2}{B_1} - JB_1\right|\right\}, \tag{19}$$

where J *is given by* (8). *The results are sharp.*

Proof. Assume that (17) holds. From the definition of majorization, there exists an analytic function ϕ such that

$$\frac{1}{\gamma}\Big(\frac{zD_qf(z)+qz^2D_q(D_qf(z))}{(1-\lambda)z+\lambda zD_qf(z)} - 1\Big) = \phi(z)(\varphi(z)-1) \quad (z \in \mathbb{U}).$$

Following similar steps as in the proof of Theorem 3, and by setting $\omega(z) \equiv z$ so that $\omega_1 = 1$ and $\omega_n = 0$ $(n \geq 2)$, we obtain

$$a_2 = \frac{\gamma A_0 B_1}{[2]_q([2]_q-\lambda)} \tag{20}$$

and

$$a_3 - \nu a_2^2 = \frac{\gamma B_1}{[3]_q([3]_q-\lambda)}\Big[A_1 + \frac{B_2}{B_1}A_0 - JB_1A_0^2\Big]. \tag{21}$$

Because $|A_0| \leq 1$, from (20) we easily get (18). Also, substituting (13) into (21), we obtain

$$a_3 - \nu a_2^2 = \frac{\gamma B_1}{[3]_q([3]_q-\lambda)}\Big[y + \frac{B_2}{B_1}A_0 - \Big(JB_1 + y\Big)A_0^2\Big]. \tag{22}$$

If we consider the case A_0=0 in (22), we at once get

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{[3]_q([3]_q-\lambda)}. \tag{23}$$

But if $A_0 \neq 0$, let us then suppose that

$$T(A_0) = y + \frac{B_2}{B_1}A_0 - \Big(JB_1 + y\Big)A_0^2,$$

which is a quadratic polynomial in A_0 and hence analytic in $|A_0| \leq 1$, and maximum value of $|T(A_0)|$ is attained at $A_0 = e^{\iota\theta}$ $(0 \leq \theta < 2\pi)$, we find that

$$\max|T(A_0)| = \max_{0\leq\theta<2\pi}|T(e^{\iota\theta})| = |T(1)|.$$

Hence from (22), we obtain

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{[3]_q([3]_q - \lambda)} \Big| J B_1 - \frac{B_2}{B_1} \Big|.$$

Thus, the assertion (19) follows from this last above inequality together with (23). The results are sharp for the function given by

$$1 + \frac{1}{\gamma}\Big(\frac{zD_qf(z) + qz^2D_q(D_qf(z))}{(1-\lambda)z + \lambda z D_q f(z)} - 1\Big) = \varphi(z),$$

which completes the proof of Theorem 6.

Remark 3. *Letting $\gamma = \lambda = 1$ and $q \to 1^-$, Theorem 6 reduces to an improved result of given in [17, Theorem 2.5].*

Theorem 7. *Let $0 \leq \lambda \leq 1$ and $\gamma \in \mathbb{C}\backslash\{0\}$. If a function $f \in \mathcal{A}$ of the form (1) belongs to the class $\mathcal{K}(q, \lambda, \gamma, \varphi)$, then*

$$|a_2| \leq \frac{|\gamma| B_1}{[2]_q([2]_q - \lambda)}, \tag{24}$$

and for any $\nu \in \mathbb{C}$

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{[3]_q([3]_q - \lambda)} \max\left\{1, \Big|\frac{B_2}{B_1} - J B_1\Big|\right\}, \tag{25}$$

where J is given by (8). The results are sharp.

Proof. Let $f \in \mathcal{K}(q, \lambda, \gamma, \varphi)$. If $\phi(z) \equiv 1$, then $A_0 = 1$ and $A_n = 0$ $(n \in \mathbb{N})$. Therefore, in view of (11) and (12), and by application of Lemma 1, we obtain the assertions (24) and (25). Because the proof is similar to the proof of Theorem 3, therefore it is omitted.

The results are sharp for the function f given by

$$1 + \frac{1}{\gamma}\Big(\frac{zD_qf(z) + qz^2D_q(D_qf(z))}{(1-\lambda)z + \lambda z D_q f(z)} - 1\Big) = \varphi(z),$$

or

$$1 + \frac{1}{\gamma}\Big(\frac{zD_qf(z) + qz^2D_q(D_qf(z))}{(1-\lambda)z + \lambda z D_q f(z)} - 1\Big) = \varphi(z^2).$$

Now, we determine the bounds on the functional $|a_3 - \nu a_2^2|$ for real ν.

Theorem 8. *Let* $0 \le \lambda \le 1$ *and let a function* $f \in \mathcal{A}$ *of the form* (1) *belongs to the class* $\mathcal{K}_Q(q, \lambda, \gamma, \varphi)$. *Then for real* ν *and* γ, *we have*

$$|a_3 - \nu a_2^2| \le \begin{cases} \frac{|\gamma| B_1}{[3]_q([3]_q-\lambda)}\Big[B_1\gamma\big(\frac{\lambda}{[2]_q-\lambda} - \frac{[3]_q([3]_q-\lambda)}{[2]_q^2([2]_q-\lambda)^2}\nu\big) + \frac{B_2}{B_1}\Big], & (\nu \le \sigma_1), \\ \frac{|\gamma| B_1}{[3]_q([3]_q-\lambda)}, & (\sigma_1 \le \nu \le \sigma_1 + 2\rho), \\ -\frac{|\gamma| B_1}{[3]_q([3]_q-\lambda)}\Big[B_1\gamma\big(\frac{\lambda}{[2]_q-\lambda} - \frac{[3]_q([3]_q-\lambda)}{[2]_q^2([2]_q-\lambda)^2}\nu\big) + \frac{B_2}{B_1}\Big], & (\nu \ge \sigma_1 + 2\rho), \end{cases} \tag{26}$$

where

$$\sigma_1 = \frac{[2]_q^2([2]_q-\lambda)\lambda}{[3]_q([3]_q-\lambda)} - \frac{[2]_q^2([2]_q-\lambda)^2}{[3]_q([3]_q-\lambda)\gamma}\Big(\frac{1}{B_1} - \frac{B_2}{B_1^2}\Big), \tag{27}$$

and

$$\rho = \frac{[2]_q^2([2]_q-\lambda)^2}{[3]_q([3]_q-\lambda)\gamma B_1}. \tag{28}$$

Each of the estimates in (26) *is sharp.*

Proof. For real values of ν and γ, the above bounds can be obtained from (7), respectively, under the following cases:

$$B_1 J - \frac{B_2}{B_1} \le -1, \; -1 \le B_1 J - \frac{B_2}{B_1} \le 1 \text{ and } B_1 J - \frac{B_2}{B_1} \ge 1,$$

where J is given by (8). We also note that

(i) When $\nu < \sigma_1$ or $\nu > \sigma_1 + 2\rho$, then the equality holds if and only if $\phi(z) \equiv 1$ and $\omega(z) = z$ or one of its rotations.

(ii) When $\sigma_1 < \nu < \sigma_1 + 2\rho$, then the equality holds if and only if $\phi(z) \equiv 1$ and $\omega(z) = z^2$ or one of its rotations.

(iii) Equality holds for $\nu = \sigma_1$ if and only if $\phi(z) \equiv 1$ and $\omega(z) = \frac{z(z+\epsilon)}{1+\epsilon z}$ $(0 \le \epsilon \le 1)$, or one of its rotations, while for $\nu = \sigma_1 + 2\rho$, the equality holds if and only if $\phi(z) \equiv 1$ and $\omega(z) = -\frac{z(z+\epsilon)}{1+\epsilon z}$ $(0 \le \epsilon \le 1)$, or one of its rotations.

The bounds of the functional $a_3 - \nu a_2^2$ for real values of ν and γ for the middle range of the parameter ν can be improved further as follows.

Theorem 9. *Let* $0 \le \lambda \le 1$ *and let a function* $f \in \mathcal{A}$ *of the form* (1) *belongs to the class* $\mathcal{K}_Q(q, \lambda, \gamma, \varphi)$. *Then for real* ν *and* γ, *we have*

$$|a_3 - \nu a_2^2| + (\nu - \sigma_1)|a_2|^2 \le \frac{|\gamma| B_1}{[3]_q([3]_q-\lambda)}, \quad (\sigma_1 \le \nu \le \sigma_1 + \rho) \tag{29}$$

and

$$|a_3 - \nu a_2^2| + (\sigma_1 + 2\rho - \nu)|a_2|^2 \leq \frac{|\gamma|B_1}{[3]_q([3]_q - \lambda)}, \quad (\sigma_1 + \rho \leq \nu \leq \sigma_1 + 2\rho) \tag{30}$$

where σ_1 *and* ρ *are given by* (27) *and* (28), *respectively.*

Proof. Let $f \in \mathcal{K}_Q(q, \lambda, \gamma, \varphi)$. For real ν satisfying $\sigma_1 + \rho \leq \nu \leq \sigma_1 + 2\rho$ and using (11) and (16), we get

$$|a_3 - \nu a_2^2| + (\nu - \sigma_1)|a_2|^2 \leq \frac{|\gamma|B_1}{[3]_q([3]_q - \lambda)} \Big[|w_2| - \frac{|\gamma|B_1[3]_q([3]_q - \lambda)}{[2]_q^2([2]_q - \lambda)^2}$$

$$(\nu - \sigma_1 - \rho)|w_1|^2 + \frac{|\gamma|B_1[3]_q([3]_q - \lambda)}{[2]_q^2([2]_q - \lambda)^2}(\nu - \sigma_1)|\omega_1|^2 \Big].$$

Therefore by virtue of Lemma 1, we get

$$|a_3 - \nu a_2^2| + (\nu - \sigma_1)|a_2|^2 \leq \frac{|\gamma|B_1}{[3]_q([3]_q - \lambda)}[1 - |\omega_1|^2 + |\omega_1|^2],$$

which yields the assertion (29).

If $\sigma_1 + \rho \leq \nu \leq \sigma_1 + 2\rho$, then from (11), (16) and an application of Lemma 1, we have

$$|a_3 - \nu a_2^2| + (\sigma_1 + 2\rho - \nu)|a_2|^2 \leq \frac{|\gamma|B_1}{[3]_q([3]_q - \lambda)} \Big[|\omega_2| + \frac{\gamma|B_1[3]_q|([3]_q - \lambda)}{[2]_q^2(2 - \lambda)^2}$$

$$(\nu - \sigma_1 - \rho)|\omega_1|^2 + \frac{\gamma|B_1[3]_q|([3]_q - \lambda)}{[2]_q^2(2 - \lambda)^2}(\sigma_1 + 2\rho - \nu)|\omega_1|^2 \Big],$$

$$\leq \frac{|\gamma|B_1}{[3]_q([3]_q - \lambda)}[1 - |\omega_1|^2 + |\omega_1|^2],$$

which estimates (30).

Conflicts of Interest

The authors declare that there are no conflict of interest regarding the publication of this paper.

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