RECENT ADVANCES IN THE THEORY OF MINIMAL MODELS

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Dedicated to the memory of Professor Gheorghe Galbură (1916-2007)

1.Introduction

In this note we survey recent developments in the classification theory of complex function fields. The subject dates back to the italian school of algebraic geometry at the beginning of the 20th century[1]. In each complex function field of transcendence degree 2, they constructed a so called *minimal surface*, which is unique for most fields, and they investigated its geometry. By the 1960's, these results were reconsidered and provided with a solid algebraic and analytic basis by the schools of Kodaira, Shafarevich and Zariski.

Minimal models are the higher dimensional analog of minimal surfaces. They were constructed in dimension 3 by Mori and his collaborators in the 1980's, and in dimension 4 by Shokurov in 2000. In 2006, Birkar, Cascini, Hacon and McKernan announced the existence of minimal models for fields of general type of arbitrary dimension. We will survey these developments.

1. Geometric classification of fields

Let $Q/\mathbb{C}$ be a field extension of finite type. A *model of* Q is a complex projective manifold X endowed with an isomorphism over $\mathbb{C}$ between Q and the field $\mathbb{C}(X)$ of rational functions on X. The geometric classification of fields aims to understand a field through its models, that is:

I) Find a *simple* model X in Q.

II) Describe the geometry of X.

[1]Gheorghe Galbură received his doctorate in Roma in 1942, under the guidance of Francesco Severi, the leader of the italian school at the time.

Informally, a model is simple if it is uniformly curved. A given field may contain several non-isomorphic simple models. For a given manifold X, the geometric classification of $\mathbb{C}(X)/\mathbb{C}$ is called the *birational classification* of X.

A simple model is constructed by choosing a model X in Q and analyzing its global differential forms. The local sections of the vector bundle $T_X^\vee$ form the sheaf of Kähler differentials Ω^1_X. The sheaf $\omega_X = \wedge^{\dim X}\Omega^1_X$ is called the *canonical sheaf*. The divisor $K_X = (\omega)$ of zeros and poles of a top rational differential form $\omega \in \wedge^{\dim X}\Omega^1_X \otimes \mathbb{C}(X)$ is called the *canonical divisor*. The *canonical ring* is the graded ring $R(X) = \oplus_{i\in\mathbb{N}}\Gamma(X, \omega_X^{\otimes i})$. The dimension of X coincides with the transcendence degree of Q over $\mathbb{C}$.

1.1. Curves

Let $Q/\mathbb{C}$ be a field extension of transcendence degree 1. It is known that there is a unique isomorphism type of nonsingular projective curve C with $\mathbb{C}(C) \simeq Q$. Therefore Q has a unique model C, up to isomorphism.

The main invariant of C is its genus

$$g = \frac{b_1(C)}{2} = \dim\Gamma(C, \omega_C).$$

The geometry of C is of 3 types

genus	$g = 0$	$g = 1$	$g \geq 2$
description	$\mathbb{P}^1$	$\mathbb{C}/\Lambda$	H/Γ
curvature	$c_1(T_C) > 0$	$c_1(T_C) = 0$	$c_1(T_C) < 0$

Here Λ is a lattice in $\mathbb{C}$, $H = \{z \in \mathbb{C}; \operatorname{Im} z > 0\}$ and Γ is a subgroup of $\mathrm{SL}_2(\mathbb{R})$. For $g \geq 2$ and $m \geq 3$, the linear system $|\omega_C^{\otimes m}|$ defines an embedding $X \hookrightarrow \mathbb{P}^{m(2g-2)-g}$. This embedding is Chow stable for $m \geq 5$, and the theory of invariants defines a structure of algebraic variety on M_g, the set of isomorphism classes of curves of genus g. M_g is a quasi-projective variety of dimension $3g - 3$, with normal quotient singularities.

1.2. Surfaces

Let $Q/\mathbb{C}$ be a field extension of transcendence degree 2. We define a partial order on the set of models of Q by setting $S_1 > S_2$ if the induced birational map $S_1 \dashrightarrow S_2$ is regular everywhere. A minimal element of this partial order is called a *minimal surface*.

Castelnuovo showed that a minimal surface exists, and he provided an algorithm to construct it. We choose an arbitrary model S in Q. If S is not minimal, it is the blow-up of another model S' at a closed point. We obtain a birational morphism $S \to S'$. If S' is not minimal, we repeat the process. The Picard number drops at each step, so the algorithm stops in finitely many steps with a minimal surface:

$$S \to S' \to S'' \to \cdots \to T.$$

The minimal surface T satisfies exactly one of the following properties:

a) $T \simeq \mathbb{P}^2$, or $T \simeq \mathbb{P}_C(\mathcal{E})$ is a geometrically ruled surface over a curve; or

b) The linear system $|\omega_T^{\otimes m}|$ defines a morphism to the projective space for some $m \geq 1$.

In case b), the minimal surface is unique up to isomorphism. In case a), non-isomorphic minimal surfaces may exist. Moreover, case b) may be decomposed into 3 types:

b1) $\omega_T^{\otimes m}$ is trivial for some $m \in \{1, 2, 3, 4, 6\}$. Examples are Abelian or K3 surfaces.

b2) The linear system $|\omega_T^{\otimes m}|$, for some $m \leq 42$, defines a fibration $T \to C$ with general fiber an elliptic curve. T is called an elliptic surface.

b3) The linear system $|\omega_T^{\otimes m}|$ induces a birational morphism $T \to V$ for every $m \geq 6$. T is called a surface of general type.

In case b3), V is isomorphic to $\operatorname{Proj} R(S)$ for any model S of Q. It is called the *canonical model* of Q. V may have singularities (Du Val), but some power of its canonical sheaf defines an embedding.

An analogue of M_g is the moduli space of surfaces of general type $M_{c_1^2, c_2}$.

2.The Iitaka program

In the 1970's, it was not known if minimal surfaces have an analog in higher dimension. Instead, manifolds were studied according to natural fibrations, such as the Albanese or Iitaka fibration. If non-empty, the linear systems $|\omega_X^{\otimes m}|$ define dominant rational maps $\varphi_m \colon X \dashrightarrow Y_m$. For m sufficiently large and divisible, these maps are birational to a given fibration $\varphi \colon X \dashrightarrow Y$, called the *Iitaka fibration* of

X. The dimension of Y is called the *Kodaira dimension* of X, denoted $\kappa(X)$. If $|\omega_X^{\otimes m}| = \emptyset$ for every $m \geq 1$, we set $\kappa(X) = -\infty$. Thus

$$\kappa(X) \in \{-\infty\} \cup \{0, \dots, \dim X\}.$$

The generic fibre F of the Iitaka fibration $\varphi\colon X \dashrightarrow Y$ satisfies $\kappa(F) = 0$. Therefore manifolds with $\kappa(X) \in \{-\infty, 0, \dim X\}$ are the building blocks of an arbitrary manifold. Manifolds with $\kappa(X) \in \{-\infty, 0\}$ were studied via the Albanese map.

Much of the work of this time was stimulated by two problems posed by Iitaka:

- (addition) if $f\colon X \to Y$ is a fibration, then $\kappa(X) \geq \kappa(F) + \kappa(Y)$.
- (invariance of plurigenera) if $f\colon X \to Y$ is a smooth morphism and $m \geq 1$, then $\dim \Gamma(X_y, \omega_{X_y}^{\otimes m})$ is independent of $y \in Y$.

The addition problem was solved in several important cases, by using moduli spaces of curves and surfaces first, and Hodge theory later. This led to the discovery that the sheaves $f_*(\omega_{X/Y}^{\otimes m})$ ($m \geq 1$) are positive in a certain sense, which in turn was used by Viehweg to construct quasi-projective moduli spaces of canonically polarized manifolds of arbitrary dimension.

Another influential proposal of Iitaka was to classify open and compact manifolds on the same footing. Given a quasi-projective manifold X, one can compactify it to a manifold $\bar{X}$ such that $\Sigma = \bar{X} \setminus X$ is a divisor with simple normal crossings. Then global differentials on X with logarithmic poles along Σ play the same role as differentials in the compact case.

The reader may consult [4] for more details.

3. Birational classification of 3-folds

The simple models in dimension 2 are minimal surfaces. In dimension 3, the partial order on the set of models of a given field may have no minimal elements, so the definition of a simple model had to be reconsidered. Mori took the first step in this direction, by recasting the construction of a minimal surface as the process of contracting negative extremal rays.

A surface S is not minimal only if it is the total space of a blow up $\sigma\colon S \to T$. The exceptional locus of σ is a rational curve C with $(\omega_S \cdot C) < 0$. Let $N_1(S)$ be the group of 1-cycles modulo numerical equivalence, and $NE(S) \subset N_1(S) \otimes_{\mathbb{Z}} \mathbb{R}$ the cone spanned by effective 1-cycles. The numerical class $[C]$ spans an extremal ray of $NE(S)$ included in the ω_S-negative halfspace. Such a ray is called *negative* *extremal ray*. Mori observed that any negative extremal ray defines a morphism which is either a blow-up (viewed from above) or a $\mathbb{P}^1$-fibration. This suggests to construct simple models by contracting negative extremal rays.

For every negative extremal ray R on a smooth projective 3-fold X, Mori constructed a morphism with connected fibers

$$f_R \colon X \to Y,$$

which contracts exactly the curves C with $[C] \in R$. Compared with surfaces, two new phenomena appear.

First, Y may have *singular points*. Thus, in order to construct simple models by contracting negative extremal rays, one has to enlarge the category of models to allow certain singularities. The smallest category is that of normal projective varieties with *terminal singularities*. In this category, the canonical divisor K_X is only a Weil divisor such that rK_X is Cartier for some $r \geq 1$.

Next, f_R is either a fibration, or birational. If birational, f_R is either a *divisorial contraction* or a *flipping contraction*, depending whether its exceptional locus is a surface or a curve. The existence of flipping contractions is the second new phenomena. For such a contraction, no multiple of K_Y is Cartier, so the notion of negative extremal ray is not well defined on Y. In this case, X is replaced by another model defined as $X^+ = \mathrm{Proj}(\oplus_{i\in\mathbb{N}} f_{R*}\mathcal{O}_X(iK_X))$. The induced rational map

$$\begin{array}{ccc} X & \overset{\phi_R}{\dashrightarrow} & X^+ \\ & {\scriptstyle f_R}\searrow \quad \swarrow & \\ & Y & \end{array}$$

is called the *flip of R*. It exists if the graded $\mathcal{O}_Y$-algebra $\oplus_{i\in\mathbb{N}} f_{R*}\mathcal{O}_X(iK_X)$ is finitely generated.

Taking all these into account, a simple 3-fold in a given field Q is constructed by the following algorithm: choose an arbitary model X in Q (with at most terminal singularities). If X has no negative extremal ray, stop. Otherwise, choose a negative extremal ray R and construct $f_R \colon X \to Y$. If f_R is a fibration, stop. If f_R is a divisorial contraction, replace X by Y and repeat. If f_R is a flipping contraction, replace X by X^+ and repeat. We obtain a sequence of models

$$X = X_0 \to X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_n = Y$$

ending up with a model satisfying exactly one of the following properties:

a) Y contains a negative extremal ray R such that f_R is a fibration, called a *Mori-Fano fiberspace structure* on X.

b) Y contains no negative extremal rays. Equivalently, $(K_Y \cdot C) \geq 0$ for every curve C. We call Y a *minimal model*.

A model of Q is called simple if it satisfies one of the properties a) or b). The termination of the algorithm in finitely many steps requires a new invariant. Divisorial contractions decrease the Picard number strictly, while flips preserve it. Shokurov introduced an invariant of the singularities of a model, which is a non-negative integer decreasing strictly after a flip.

As for the geometry of a simple model Y, we first consider case a). There are three cases:

a1) Y admits a $\mathbb{P}^1$-fibration ($\dim f_R(Y) = 2$);

a2) Y admits a del Pezzo fibration ($\dim f_R(Y) = 1$);

a3) $\varrho(Y) = 1$ and $|-mK_Y|$ defines an embedding for some $m \geq 1$ ($\dim f_R(Y) = 0$). We say that Y is a *Fano* 3-fold.

Fano 3-folds form a bounded family. Smooth Fano 3-folds are classified.

In case b), $|mK_Y|$ defines a morphism to the projective space for some $m \geq 1$. There are three main cases:

b1) $mK_Y \sim 0$ for some m. We call Y a *Calabi-Yau*.

b2) $|mK_Y|$ defines a fibration with connected fibers $f\colon Y \to V$ for m large.

b3) $|mK_Y|$ defines a birational morphism $f\colon Y \to W$ for m large.

In case b3), W is isomorphic to $\operatorname{Proj} R(X)$ for any model X in $\mathbb{C}(Y)$. It is called the *canonical model* of Y. It has so called *canonical singularities*, and $|mK_W|$ defines an embedding into the projective space for some $m \geq 1$.

These results were obtained around 1990's through the joint effort of many mathematicians, especially Benveniste, Iskovkikh, Kawamata, Kollár, Miyaoka, Mori, Reid and Shokurov. We refer the reader to [3] for more details.

4. Birational classification in higher dimension

In dimension 4 or more, it is expected that a simple model can be constructed by contracting or flipping negative extremal rays, exactly as in dimension 3. The contraction f_R associated to an extremal ray R exists in arbitrary dimension. Therefore simple models can be constructed modulo two open problems:

- Flips exist.
- The construction algorithm stops in finitely many steps.

As for the geometry of simple models, qualitative results are desirable, such as effective embeddings into the projective space or the existence of canonical hermitian metrics. For varieties of type a), a first problem is

- Fano varieties of given dimension form a bounded family.

Fano varieties without singular points are known to be bounded. An Einstein-Kähler metric may exist or not, and this is expected to be related to the stability of the embeddings defined by $|-mK_Y|$ for large m. For varieties of type b), a first problem is the so called abundance problem:

- If Y is a minimal model, then $|mK_Y|$ defines a morphism to the projective space for some $m \geq 1$.

For a Calabi-Yau model Y, it is expected that $mK_Y \sim 0$ for some $m \geq 1$ depending only on $\dim Y$. If Y has no singular points, it has a unique Einstein-Kähler metric in each Kähler class. For a canonical model W (with canonical singularities), it is expected that $|mK_W|$ defines an embedding for some $m \geq 1$ depending only on $\dim W$ and $(K_W^{\dim W})$. If W has no singular points, it has a unique Einstein-Kähler metric.

Important corollaries of the existence of simple models and the abundance problem are:

i) The canonical ring $R(X)$ is finitely generated.

ii) Addition and Invariance of Plurigenera hold.

iii) The moduli space of canonically polarized varieties has a geometric compactification.

4.1. Recent progress

Significant progress was made recently. Siu [6, 7] proved the invariance of plurigenera. For a one parameter family $X \to C$, this is equivalent to the surjectivity of the adjunction map

$$\Gamma(X, m(K_X + Y)) \to \Gamma(Y, mK_Y)$$

for any fiber Y and $m \geq 1$. For this lifting problem, Siu introduced a new technique, based on singular hermitian metrics and L^2-estimates for the solution of the $\bar{\partial}$-equation. An algebraic equivalent of this technique is not known yet.

Shokurov [5] proved the existence of flips in dimension 4. In fact, he showed that flips exist in dimension d if 1) minimal models exist in dimension $d-1$, and 2) certain algebras defined on Fano varieties of dimension $d-1$ are finitely generated. Hacon and McKernan [2] removed the assumption 2), based on a generalization of Siu's lifting technique. Therefore flips exist in dimension d if minimal models exist in dimension $d - 1$. Together with Birkar and Cascini [1], they improved this to obtain the unconditional existence of flips. They claim much more, namely the existence of minimal models of general type and the finite generation of the canonical ring. The finite generation of the canonical ring of a variety of general type was also announced by Siu [8].

Acknowledgement: This work was partially supported by the research contract CEx05-D11-11/04.10.05.

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