ABOUT AN INTERMEDIATE POINT PROPERTY IN SOME QUADRATURE FORMULAS

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Abstract. In this paper we study a property of the intermediate point from the quadrature formula of the Gauss-Jacobi type, the quadrature formula obtained in the paper [3] by using connection between the monospline function and the numerical integration formula, and the generalized mean-value formula of N. Ciorănescu.

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1. Introduction

In the specialized literature there are a lot of mean-value theorems. In [1] and [8] the authors studied the property above for the intermediate point from the quadrature formula of Gauss type.

The generalized quadrature formula of Gauss-Jacobi type has the form ([9])

$$\int_a^b (b-x)^\alpha (x-a)^\beta f(x)dx = \sum_{k=0}^m B_{m,k} f(\gamma_k) + \mathcal{R}_m[f], \tag{1}$$

the nodes γ_k , $k=\overline{0,m}$, with appears in (1) are given by

$$\gamma_k = \frac{b-a}{2}a_k + \frac{b+a}{2}$$

where a_k , $k=\overline{0,m}$ are the zeros of the Jacobi polynomial, $J_{m+1}^{(\alpha,\beta)}$ and

$$B_{m,k} = \frac{1}{2}\frac{(b-a)^{\alpha+\beta+1}(2m+\alpha+\beta+2)\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{(m+1)!\Gamma(m+\alpha+\beta+2)J_m^{(\alpha,\beta)}(a_k)\dfrac{d}{dx}\left[J_{m+1}^{(\alpha,\beta)}(x)\right]_{x=a_k}}.$$

For $f\in C^{2m+2}[a,b]$ the rest term is given by

$$\mathcal{R}_m[f] = (b-a)^{2m+\alpha+\beta+3}\frac{f^{(2m+2)}(\xi)}{(2m+2)!}\cdot\frac{(m+1)!\Gamma(m+\alpha+2)\Gamma(m+\beta+2)\Gamma(m+\alpha+\beta+2)}{\Gamma(2m+\alpha+\beta+3)\Gamma(2m+\alpha+\beta+4)},$$

$$a < \xi < b\,.$$

For $\alpha = \beta = 0$ we have so-called Gauss-Legendre quadrature formula. If $f \in C^{2m+2}[a,b]$, then for any $x \in (a\;, b]$ there is $c_x \in (a,x)$ such that

$$\int_a^x f(t)dt = \frac{(x-a)}{m+1}\sum_{k=0}^{m}\frac{1}{J_m^{(0,0)}(a_k)\frac{d}{dx}\left[J_{m+1}^{(0,0)}(x)\right]_{x=a_k}}\cdot f\left(\frac{x-a}{2}a_k+\frac{x+a}{2}\right) + \frac{(x-a)^{2m+3}[(m+1)!]^4}{(2m+3)[(2m+2)!]^3}f^{(2m+2)}(c_x). \qquad (2)$$

Theorem 1. *[1] If $f \in C^{2m+4}[a,b]$ and $f^{(2m+3)}(a) \neq 0$, then for the intermediate point c_x which appears in formula (2) we have* $\lim\limits_{x\to a}\dfrac{c_x - a}{x-a} = \dfrac{1}{2}$.

In this paper we want to study the property of the intermediate point from the quadrature formula of Gauss type with weight function $w(x) = (b-x)(x-a)$.

In recent years a number of authors have considered generalization of some known and some new quadrature rules. For example, P. Cerone and S.S. Dragomir in [5] give a generalization of the midpoint quadrature rule:

$$\int_a^b f(t)dt = \sum_{k=0}^{m-1}[1+(-1)^k]\frac{(b-a)^{k+1}}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)+(-1)^m\int_a^b K_m(t)f^{(m)}(t)dt \qquad (3)$$

where

$$K_m(t) = \begin{cases} \dfrac{(t-a)^m}{m!}\,, & t \in \left[a, \dfrac{a+b}{2}\right] \\ \dfrac{(t-b)^m}{m!}\,, & t \in \left(\dfrac{a+b}{2}, b\right] \end{cases}$$

If $f \in C^m[a,b]$ and m is even , then for any $x \in (a\;, b\,]$ there is $c_x \in (a,x)$ such that

$$\int_a^x f(t)dt = \sum_{k=0}^{m-2}[1+(-1)^k]\frac{(x-a)^{k+1}}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+x}{2}\right)+\frac{(x-a)^{m+1}}{2^m(m+1)!}f^{(m)}(c_x) \qquad (4)$$

In [2] was studied the following property of intermediate point from the quadrature formula (4)

Theorem 2. *[2] If $f \in C^{2m}[a,b]$, m is even and $f^{(m+1)}(a) \neq 0$, then for the intermediate point c_x that appears in formula (4), it follows:*

$$\lim_{x \to a} \frac{c_x - a}{x - a} = \frac{1}{2}.$$

In this paper we want to give a property of intermediate point from a quadrature formula with the weight function $w : [a,b] \to (0, \infty)$, $w(x) = (b-x)(x-a)$.

N. Ciorănescu demonstrated in [6] that the following formula is valid

$$\int_a^b f(x) p_m(x) w(x) dx = \frac{f^{(m)}(c_b)}{m!} \int_a^b x^m p_m(x) w(x) dx, \tag{5}$$

where $f \in C^m[a,b]$ and $(p_n)_{n \geq 0}$ is a sequence of orthogonal polynomials on $[a,b]$, in respect to a weight function $w : [a,b] \to (0, \infty)$.

In [4], the authors study a property of the intermediate point from the mean-value formula of N. Ciorănescu.

Theorem 3. *[4] If $f \in C^{m+1}[a,b]$ and $f^{(m+1)}(a) \neq 0$, then the intermediate point of the mean-value formula (5) satisfies the relation:*

$$\lim_{b \to a} \frac{c_b - a}{b - a} = \frac{1}{m+2}$$

In this paper we give a property of the intermediate point from the generalized mean-value formula of N. Ciorănescu.

2. An intermediate point property in the quadrature formulas of Gauss-Jacobi type

Let

$$\begin{aligned} \int_a^b (b-x)(x-a) f(x) dx &= \frac{(b-a)^3}{m+3} \sum_{k=0}^{m} \frac{1}{J_m^{(1,1)}(a_k) \frac{d}{dx}\left[J_{m+1}^{(1,1)}(x)\right]_{x=a_k}} f\left(\frac{b-a}{2} a_k + \frac{b+a}{2}\right) \\ &+ (b-a)^{2m+5} \frac{(m+1)!^4 (m+2)(m+3)}{4(2m+2)!^3 (2m+3)^2 (2m+5)} f^{(2m+2)}(\xi) \end{aligned} \tag{6}$$

be the quadrature formula of Gauss-Jacobi type (1), which $\alpha=\beta=1$.

If $f\in C^{2m+2}[a,b]$, then for any $x\in(a\;,b]$ there is $c_x\in(a,x)$ such that

$$\begin{aligned}\int_a^x (x-t)(t-a)f(t)dt &= \frac{(x-a)^3}{m+3}\sum_{k=0}^{m}\frac{1}{J_m^{(1,1)}(a_k)\frac{d}{dx}\left[J_{m+1}^{(1,1)}(x)\right]_{x=a_k}}f\left(\frac{x-a}{2}a_k+\frac{x+a}{2}\right)\\ &+ (x-a)^{2m+5}\frac{(m+1)!^4(m+2)(m+3)}{4(2m+2)!^3(2m+3)^2(2m+5)}f^{(2m+2)}(c_x). \quad (7)\end{aligned}$$

In this section we give a property of the intermediate point, c_x, from the quadrature formula of Gauss-Jacobi type (7). Here we prove a lemma which help us in proving our theorem.

Lemma 1. *If a_k, $k=\overline{0,m}$ are the zeroes of the Jacobi polynomials, $J_{m+1}^{(1,1)}$, then the following relations hold:*

$$\sum_{k=0}^{m}\frac{\left(\frac{a_k+1}{2}\right)^{i-3}}{J_m^{(1,1)}(a_k)\frac{d}{dx}\left[J_{m+1}^{(1,1)}(x)\right]_{x=a_k}} = \frac{m+3}{i(i-1)},\ \textit{for } i=\overline{3,2m+4}, \quad (8)$$

$$\begin{aligned}\sum_{k=0}^{m}\frac{\left(\frac{a_k+1}{2}\right)^{2m+2}}{J_m^{(1,1)}(a_k)\frac{d}{dx}\left[J_{m+1}^{(1,1)}(x)\right]_{x=a_k}} &= \frac{m+3}{(2m+3)(2m+4)(2m+5)} \quad (9)\\ &\cdot \left[(2m+3)-\frac{(m+1)!^4(m+2)^2(m+3)}{2(2m+2)!^2(2m+3)}\right],\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{m}\frac{\left(\frac{a_k+1}{2}\right)^{2m+3}}{J_m^{(1,1)}(a_k)\frac{d}{dx}\left[J_{m+1}^{(1,1)}(x)\right]_{x=a_k}} &= \frac{1}{2^{2m+3}}\left\{\frac{2^{2m+2}}{2m+5}\right.\\ &- \left.\frac{2^{2m}(m+1)!^4(m+2)(m+3)^2}{(2m+2)!^2(2m+3)(2m+5)}\right\}. \quad (10)\end{aligned}$$

Proof. If we choose $a=0$, $b=1$ and $f(t)=t^{i-3}$, $i=\overline{3,2m+5}$ in the quadrature formula (6), then we obtain the relations (8) and (9).

If we choose $a=-1$, $b=1$ and $f(t)=t^i$, $i=\overline{0,2m+2}$ in the quadrature formula (6), then we obtain the following relations:

$$\sum_{k=0}^{m}\frac{a_k^i}{J_m^{(1,1)}(a_k)\frac{d}{dx}\left[J_{m+1}^{(1,1)}(x)\right]_{x=a_k}} = \frac{m+3}{4(i+1)(i+3)}\left[1+(-1)^i\right],\ i=\overline{0,2m+1} \quad (11)$$

$$\sum_{k=0}^{m}\frac{a_k^{2m+2}}{J_m^{(1,1)}(a_k)\frac{d}{dx}\left[J_{m+1}^{(1,1)}(x)\right]_{x=a_k}} = \frac{m+3}{2(2m+3)(2m+5)} \cdot \left[1-2^{2m+1}\frac{(m+1)!^4(m+2)(m+3)}{(2m+2)!^2(2m+3)}\right]. \quad (12)$$

By using the following formulas (see [10]):

$$J_{2m}^{(\alpha,\alpha)}(x)=\frac{\Gamma(2m+\alpha+1)\Gamma(m+1)}{\Gamma(m+\alpha+1)\Gamma(2m+1)}J_m^{(\alpha,-\frac{1}{2})}(2x^2-1),$$
$$J_{2m+1}^{(\alpha,\alpha)}(x)=\frac{\Gamma(2m+\alpha+2)\Gamma(m+1)}{\Gamma(m+\alpha+1)\Gamma(2m+2)}xJ_m^{(\alpha,\frac{1}{2})}(2x^2-1),$$
$$\frac{d}{dx}\left\{J_m^{(\alpha,\beta)}(x)\right\}=\frac{1}{2}(m+\alpha+\beta+1)J_{m-1}^{(\alpha+1,\beta+1)}(x),$$

we obtain

$$J_{2m}^{(1,1)}(a_k)\frac{d}{dx}\left[J_{2m+1}^{(1,1)}(x)\right]_{x=a_k}=\frac{2(2m+1)^2}{m+1}J_m^{\left(1,-\frac{1}{2}\right)}(2a_k^2-1)J_m^{\left(2,-\frac{1}{2}\right)}(2a_k^2-1), \quad (13)$$

$$J_{2m+1}^{(1,1)}(a_k)\frac{d}{dx}\left[J_{2m+2}^{(1,1)}(x)\right]_{x=a_k}=\frac{2(2m+5)(2m+3)}{(m+2)}\cdot a_k^2 \cdot J_m^{\left(1,\frac{1}{2}\right)}(2a_k^2-1)J_m^{\left(2,\frac{1}{2}\right)}(2a_k^2-1). \quad (14)$$

From the identity

$$J_m^{(\alpha,\beta)}(x)=(-1)^m J_m^{(\beta,\alpha)}(-x)$$

it follows that

$$a_k+a_{m-k}=0, \quad (15)$$

where a_k, $k=\overline{0,m}$ are the zeroes of Jacobi polynomial of degree $m+1$, $J_{m+1}^{(1,1)}$.

From (13), (14) and (15) we obtain

$$\sum_{k=0}^{m} \frac{a_k^{2m+3}}{J_m^{(1,1)}(a_k)\frac{d}{dx}\left[J_{m+1}^{(1,1)}(x)\right]_{x=a_k}} = 0,$$

therefore

$$\sum_{k=0}^{m} \frac{\left(\frac{a_k+1}{2}\right)^{2m+3}}{J_m^{(1,1)}(a_k)\frac{d}{dx}\left[J_{m+1}^{(1,1)}(x)\right]_{x=a_k}} = \frac{1}{2^{2m+3}}\left[\sum_{k=0}^{m} \frac{a_k^{2m+3}}{J_m^{(1,1)}(a_k)\frac{d}{dx}\left[J_{m+1}^{(1,1)}(x)\right]_{x=a_k}}\right.$$

$$\left.+\sum_{k=0}^{m}\sum_{i=0}^{2m+2}\binom{2m+3}{i}\frac{a_k^i}{J_m^{(1,1)}(a_k)\frac{d}{dx}\left[J_{m+1}^{(1,1)}(x)\right]_{x=a_k}}\right]$$

$$= \frac{1}{2^{2m+3}}\sum_{i=0}^{2m+2}\binom{2m+3}{i}\sum_{k=0}^{m}\frac{a_k^i}{J_m^{(1,1)}(a_k)\frac{d}{dx}\left[J_{m+1}^{(1,1)}(x)\right]_{x=a_k}}$$

and by using (11) and (12) it follows the relation (10).

Theorem 4. *If $f \in C^{2m+6}[a,b]$ and $f^{(2m+3)} \neq 0$, then for the intermediate point c_x which appears in formula (7) we have*

$$\lim_{x\to a}\frac{c_x-a}{x-a} = \frac{1}{2}. \tag{16}$$

Proof. Let us consider $F, G : [a,b] \to \mathrm{R}$ defined by

$$\begin{aligned} F(x) &= \int_a^x (x-t)(t-a)f(t)dt \\ &- \frac{(x-a)^3}{m+3}\sum_{k=0}^{m}\frac{1}{J_m^{(1,1)}(a_k)\frac{d}{dx}\left[J_{m+1}^{(1,1)}(x)\right]_{x=a_k}} f\left(\frac{x-a}{2}a_k+\frac{x+a}{2}\right) \\ &- (x-a)^{2m+5}\frac{(m+1)!^4(m+2)(m+3)}{4(2m+2)!^3(2m+3)^2(2m+5)}f^{(2m+2)}(a), \qquad (17) \\ G(x) &= (x-a)^{2m+6}. \end{aligned}$$

We have that F and G are $(2m+6)$ times derivable on $[a,b]$,

$$G^{(i)}(x) \neq 0,\ i=\overline{1,2m+5} \quad \text{any} \quad x \in (a,\ b],$$
$$G^{(i)}(a) = 0,\ i=\overline{1,2m+5}.$$

We observe that $F(a)=F'(a)=F''(a)=0$.
For $i=\overline{3,2m+4}$ we have

$$F^{(i)}(a)=(i-2)f^{(i-3)}(a)-\frac{i(i-1)(i-2)}{m+3}\sum_{k=0}^{m}\frac{\left(\frac{a_k+1}{2}\right)^{i-3}}{J_m^{(1,1)}(a_k)\frac{d}{dx}\left[J_{m+1}^{(1,1)}(x)\right]_{x=a_k}}\cdot f^{(i-3)}(a)$$

and by using relation (8) we obtain $F^{(i)}(a)=0$.
From relations (9) and (17) we obtain

$$\begin{aligned}F^{(2m+5)}(a) &= (2m+3)f^{(2m+2)}(a)-\frac{(2m+3)(2m+4)(2m+5)}{m+3}\\ &\cdot \sum_{k=0}^{m}\frac{\left(\frac{a_k+1}{2}\right)^{2m+2}}{J_m^{(1,1)}(a_k)\frac{d}{dx}\left[J_{m+1}^{(1,1)}(x)\right]_{x=a_k}}\cdot f^{(2m+2)}(a)\\ &- \frac{(m+1)!^4(m+2)^2(m+3)}{2(2m+2)!^2(2m+3)}f^{(2m+2)}(a)=0.\end{aligned}$$

By using successive l′Hospital rule and

$$\begin{aligned}F^{(2m+6)}(a) &= (2m+4)f^{(2m+3)}(a)-2(2m+4)(2m+5)\\ &\cdot \sum_{k=0}^{m}\frac{\left(\frac{a_k+1}{2}\right)^{2m+3}}{J_m^{(1,1)}(a_k)\frac{d}{dx}\left[J_{m+1}^{(1,1)}(x)\right]_{x=a_k}}\cdot f^{(2m+3)}(a),\\ G^{(2m+6)}(a) &= (2m+6)!\end{aligned}$$

we obtain

$$\lim_{x\to a}\frac{F(x)}{G(x)}=\lim_{x\to a}\frac{F^{(2m+6)}(x)}{G^{(2m+6)}(x)}=\frac{f^{(2m+3)}(a)}{(2m+6)!} \tag{18}$$

$$\cdot\left[(2m+4)-2(2m+4)(2m+5)\cdot\sum_{k=0}^{m}\frac{\left(\frac{a_k+1}{2}\right)^{2m+3}}{J_m^{(1,1)}(a_k)\frac{d}{dx}\left[J_{m+1}^{(1,1)}(x)\right]_{x=a_k}}\right],$$

but

$$\begin{aligned}\lim_{x\to a}\frac{F(x)}{G(x)} &= \lim_{x\to a}(x-a)^{2m+5}\frac{(m+1)!^4(m+2)(m+3)}{4(2m+2)!^3(2m+3)^2(2m+5)}\frac{f^{(2m+2)}(c_x)-f^{(2m+2)}(a)}{(x-a)^{2m+6}}\\ &= \lim_{x\to a}\frac{(m+1)!^4(m+2)(m+3)}{4(2m+2)!^3(2m+3)^2(2m+5)}\cdot\frac{f^{(2m+2)}(c_x)-f^{(2m+2)}(a)}{c_x-a}\cdot\frac{c_x-a}{x-a},\end{aligned}$$

namely

$$\lim_{x\to a}\frac{F(x)}{G(x)} = \frac{(m+1)!^4(m+2)(m+3)}{4(2m+2)!^3(2m+3)^2(2m+5)}\cdot f^{(2m+3)}(a)\cdot\lim_{x\to a}\frac{c_x-a}{x-a}. \tag{19}$$

From (10), (18) and (19) it follows that the intermediate point c_x which appears in formula (7) verifies the property (16).

3. An intermediate point property in a quadrature formula with weight function $w(x)=(b-x)(x-a)$

In [3] was studied the following quadrature formula

$$\int_a^b w(t)f(t)dt=\sum_{k=0}^{m-1}\left[(-1)^k+1\right]\cdot\frac{(b-a)^{k+3}}{2^{k+2}(k+1)!(k+3)}f^{(k)}\left(\frac{a+b}{2}\right)+\mathcal{R}[f]\,,$$

where $f\in C^m[a,b]$, $w(t)=(b-t)(t-a)$,

$$\mathcal{R}[f]=(-1)^m\int_a^b M_n(t)f^{(m)}(t)dt$$

and

$$M_m(t)=\begin{cases}(b-a)\dfrac{(t-a)^{m+1}}{(m+1)!}-2\dfrac{(t-a)^{m+2}}{(m+2)!}\,, & t\in\left[a,\ \dfrac{a+b}{2}\right)\\ (a-b)\dfrac{(t-b)^{m+1}}{(m+1)!}-2\dfrac{(t-b)^{m+2}}{(m+2)!}\,, & t\in\left[\dfrac{a+b}{2},b\right]\end{cases}.$$

If $f\in C^m[a,b]$ and m is even , then for any $x\in(a\,,b\,]$ there is $c_x\in(a,x)$ such that

$$\begin{aligned}\int_a^x(x-t)(t-a)f(t)dt &= \sum_{k=0}^{m-2}\left[(-1)^k+1\right]\cdot\frac{(x-a)^{k+3}}{2^{k+2}(k+1)!(k+3)}f^{(k)}\left(\frac{a+x}{2}\right)\\ &+ \frac{(x-a)^{m+3}}{2^{m+1}(m+1)!(m+3)}f^{(m)}(c_x).\end{aligned} \tag{20}$$

In the above condition we have the following theorem

Theorem 5. *If $f \in C^{2m+2}[a,b]$, m is even and $f^{(m+1)}(a) \neq 0$, then for the intermediate point c_x that appears in formula (20), it follows:*

$$\lim_{x\to a} \frac{c_x - a}{x-a} = \frac{1}{2}.$$

Proof. Let $F, G : [a,b] \to \mathrm{R}$ definite as follows

$$\begin{aligned} F(x) &= \int_a^x (x-t)(t-a)f(t)dt - \sum_{k=0}^{m-2}\left[(-1)^k+1\right]\cdot\frac{(x-a)^{k+3}}{2^{k+2}(k+1)!(k+3)}f^{(k)}\left(\frac{a+x}{2}\right) \\ &- \frac{(x-a)^{m+3}}{2^{m+1}(m+1)!(m+3)}f^{(m)}(a), \\ G(x) &= (x-a)^{m+4}. \end{aligned}$$

We observe that $F(a) = F'(a) = F''(a) = 0$. For $i = \overline{3, m+2}$ we have

$$\begin{aligned} F^{(i)}(a) &= f^{(i-3)}(a)\left\{(i-2) - \frac{i(i-1)(i-2)}{2^{i-1}}\right. \\ &\cdot \left.\sum_{k=0}^{i-3}\left[(-1)^k+1\right]\binom{i-3}{k}\frac{1}{(k+1)(k+3)}\right\} = 0. \end{aligned}$$

We find

$$\begin{aligned} F^{(m+3)}(a) &= f^{(m)}(a)\left\{(m+1) - \frac{(m+1)(m+2)(m+3)}{2^{m+2}}\right. \\ &\cdot \left.\sum_{k=0}^{m-2}\left[(-1)^k+1\right]\binom{m}{k}\frac{1}{(k+1)(k+3)} - \frac{m+2}{2^{m+1}}\right\} = 0 \end{aligned}$$

and

$$\begin{aligned} F^{(m+4)}(a) &= f^{(m+1)}(a)\left\{(m+2) - \frac{(m+2)(m+3)(m+4)}{2^{m+3}}\right. \\ &\cdot \left.\sum_{k=0}^{m-2}\left[(-1)^k+1\right]\binom{m+1}{k}\frac{1}{(k+1)(k+3)}\right\} \\ &= \frac{(m+2)(m+4)}{2^{m+2}}f^{(m+1)}(a). \end{aligned}$$

We have that F and G are $(m+4)$ times derivable on $[a,b]$, $F^{(i)}(a)=G^{(i)}(a)=0$, for $i=\overline{0,m+3}$ and $G^{(i)}(x)\neq 0$, $i=\overline{1,m+3}$ any $x\in(a,\ b]$. By using successive l′ Hospital rule we obtain

$$\lim_{x\to a}\frac{F(x)}{G(x)}=\lim_{x\to a}\frac{F^{m+4}(x)}{G^{m+4}(x)}=\frac{1}{2^{m+2}(m+1)!(m+3)}f^{(m+1)}(a),\qquad (21)$$

but

$$\lim_{x\to a}\frac{F(x)}{G(x)}=\frac{1}{2^{m+1}(m+1)!(m+3)}f^{(m+1)}(a)\lim_{x\to a}\frac{c_x-a}{x-a}.\qquad (22)$$

From relation (21) and (22) we have

$$\lim_{x\to a}\frac{c_x-a}{x-a}=\frac{1}{2}.$$

4. An intermediate point property from the mean-value formula of N. Ciorănescu

The polynomial

$$P_{s,m}(x)=x^m+a_{m-1}x^{m-1}+\cdots+a_1x+a_0,$$

which satisfies the orthogonality conditions

$$\int_a^b\left[P_{s,m}(x)\right]^{2s+1}x^kw(x)dx=0,\quad k=0,1,\cdots,m-1$$

is called s-orthogonal polynomial with respect to the weight function $w:[a,b]\to(0,\infty)$.

The following equality is the generalized mean-value formula of N. Ciorănescu (see [9])

$$\int_a^b f(x)P_{s,m}^{2s+1}(x)w(x)dx=\frac{f^{(m)}(c_b)}{m!}\int_a^b x^mP_{s,m}^{2s+1}(x)w(x)dx.\qquad (23)$$

We observe that for $s=0$ we obtain the mean-value formula of N. Ciorănescu (5).

We construct the functions $(V_k)_{k=0,m}$ as follows

$$V_0(x) = w(x)P_{s,m}^{2s+1}(x),$$

$$V_j(x) = \int_a^x V_{j-1}(x)dx, j = \overline{1,m}.$$

Here we prove a lemma which help us in proving our theorem.

Lemma 2. *We have the following equalities*

$$V_j(a) = 0, V_j(b) = 0, \quad \textit{for any} \quad j = \overline{1,m}, \tag{24}$$

$$\int_a^b f(x)P_{s,m}^{2s+1}(x)w(x)dx = (-1)^m \int_a^b f^{(m)}(x)V_m(x)dx. \tag{25}$$

Proof. We have $V_j(a) = 0$, for any $j = \overline{1,m}$.

$$V_1(b) = \int_a^b V_0(x)dx = \int P_{s,m}^{2s+1}(x)w(x)dx = 0.$$

For every $k \in \{2, 3, \ldots, m\}$ we have

$$\begin{aligned}
V_k(b) &= \int_a^b V_{k-1}(x)dx = \int_a^b \left[\frac{x^{k-1}}{(k-1)!}\right]^{(k-1)} V_{k-1}(x)dx \\
&= \sum_{\nu=0}^{k-2}(-1)^{k-\nu-2}\left[V_{k-1}(x)\right]^{(k-\nu-2)}\left[\frac{x^{k-1}}{(k-1)!}\right]^{(\nu)}\Bigg|_a^b \\
&+ (-1)^{k-1}\int_a^b \left[V_{k-1}(x)\right]^{(k-1)}\frac{x^{k-1}}{(k-1)!}dx \\
&= \sum_{\nu=0}^{k-2}(-1)^{k-\nu-2}\, V_{\nu+1}(x)\frac{x^{k-\nu-1}}{(k-\nu-1)!}\Bigg|_a^b + \frac{(-1)^{k-1}}{(k-1)!}\int_a^b V_0(x)x^{k-1}dx \\
&= \frac{(-1)^{k-1}}{(k-1)!}\int_a^b x^{k-1}P_{s,m}^{2s+1}(x)w(x)dx = 0.
\end{aligned}$$

We have

$$\begin{aligned}
\int_a^b f(x)P_{s,m}^{2s+1}(x)w(x)dx &= \sum_{\nu=0}^{m-1}(-1)^{m-\nu-1}\, f^{(m-\nu-1)}(x)V_{m-\nu}(x)\Big|_a^b \\
&+ (-1)^m\int_a^b f^{(m)}(x)V_m(x)dx,
\end{aligned}$$

and by using relation (24) we obtain the equality (25).

Theorem 6. *If $f \in C^{m+1}[a,b]$ and $f^{(m+1)}(a) \neq 0$, then the intermediate point of the mean-value formula (23) satisfies the relation:*

$$\lim_{b \to a} \frac{c_b - a}{b - a} = \frac{1}{m+2} \tag{26}$$

Proof. From (23) and (25) we can written

$$\begin{aligned} \int_a^b f(x) P_{s,m}^{2s+1}(x) w(x) dx &= \frac{f^{(m)}(c_b)}{m!} \int_a^b x^m P_{s,m}^{2s+1}(x) w(x) dx \\ &= (-1)^m f^{(m)}(c_b) \int_a^b V_m(x) dx. \end{aligned} \tag{27}$$

From relations (25) and (27) we obtain

$$\int_a^b f^{(m)}(x) V_m(x) dx = f^{(m)}(c_b) \int_a^b V_m(x) dx.$$

We consider the functions

$$\begin{aligned} &F(b) = \int_a^b f^{(m)}(x) V_m(x) dx - f^{(m)}(a) \int_a^b V_m(x) dx, \\ &G(b) = (b-a)^{m+2}. \end{aligned}$$

Since

$$\begin{aligned} F^{(k)}(b) &= \sum_{\nu=0}^{k-1} \binom{k-1}{\nu} f^{(m+k-1-\nu)}(b) V_{m-\nu}(b) - f^{(m)}(a) V_{m-k+1}(b), \\ F^{(m+1)}(b) &= \sum_{\nu=0}^{m-1} \binom{m}{\nu} f^{(2m-\nu)}(b) V_{m-\nu}(b) + \left[f^{(m)}(b) - f^{(m)}(a) \right] V_0(b), \end{aligned}$$

by using successive l'Hospital rule, we have

$$\lim_{b \to a} \frac{F(b)}{G(b)} = \frac{f^{(m+1)}(a)}{(m+2)!} V_0(a), \tag{28}$$

but

$$\begin{aligned}\lim_{b\to a}\frac{F(b)}{G(b)} &= \lim_{b\to a}\frac{f^{(m)}(c_b)\displaystyle\int_a^b V_m(x)dx - f^{(m)}(a)\int_a^b V_m(x)dx}{(b-a)^{m+2}}\\ &= \frac{V_0(a)}{(m+1)!}f^{(m+1)}(a)\cdot\lim_{b\to a}\frac{c_b-a}{b-a}. \qquad (29)\end{aligned}$$

From (28) and (29) it follows that the intermediate point c_b from the generalized mean-value formula of N. Ciorănescu verifies the property (26).

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