DESCH-SCHAPPACHER PERTURBATION THEOREM FOR C_0-SEMIGROUPS ON THE DUAL OF A BANACH SPACE

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Abstract.Let $(\mathcal{X}, \|\,.\,\|)$ be a Banach space. Consider on $\mathcal{X}^*$ the topology of uniform convergence on compact subsets of $(\mathcal{X}, \|\,.\,\|)$ denoted by $\mathcal{C}(\mathcal{X}^*, \mathcal{X})$, for which the usual semigroups in literature becomes C_0-semigroups.
The main purpose of this paper is to prove a Desch-Schappacher perturbation theorem for C_0-semigroups on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$.

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1. Preliminary

Perturbation theory has long been a very useful tool in the hand of the analyst and physicist. A very elegant brief introduction to one-parameter semigroups is given in the treatise of Kato [5] where on can find all results on perturbation theory. A complete information on the general theory of semigroups of linear operators can be obtained by consulting the books of Yosida [11], Davies [2], Pazy [7] or Goldstein [4]. The perturbation by bounded operators is due to Phillips [8] who also investigate permanence of smoothness properties by this kind of perturbation. The perturbation by continuous operators on the graph norm of the generator is due to Desch and Schappacher [3].

In general, for a C_0-semigroup $\{T(t)_{t\geq 0}\}$ on a Banach space $(\mathcal{X}, \|\,.\,\|)$, it is well known that its adjoint semigroup $\{T^*(t)_{t\geq 0}\}$ is no longer strongly continuous on the dual space $\mathcal{X}^*$ with respect to the strong topology of $\mathcal{X}^*$. In [10] Wu and Zhang introduce on $\mathcal{X}^*$ a topology for which the usual semigroups in literature becomes C_0-semigroups. That is *the topology of uniform convergence on compact subsets of* $(\mathcal{X}, \|\,.\,\|)$, denoted by $\mathcal{C}(\mathcal{X}^*, \mathcal{X})$. If $\{T(t)_{t\geq 0}\}$ is a C_0-semigroup on $(\mathcal{X}, \|\,.\,\|)$ with generator $\mathcal{L}$, by [10, Theorem 1.4, p.564] it follows that $\{T^*(t)_{t\geq 0}\}$ is a C_0-semigroup on $\mathcal{X}^*_{\mathcal{C}} := (\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$ with generator $\mathcal{L}^*$.

2. The main results

The main result of this paper is a Desch-Schappacher perturbation theorem for C_0-semigroups on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$. We begin with a perturbation by bounded operators for C_0-semigroups on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$.

Theorem 1. *Let $(\mathcal{X}, \| \,.\, \|)$ be a Banach space, $\mathcal{L}$ the generator of a C_0-semigroup $\{T(t)_{t\geq 0}\}$ on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$ and C a linear operator on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$ with domain $\mathcal{D}(C) \supset \mathcal{D}(\mathcal{L})$. If C is $\mathcal{C}(\mathcal{X}^*, \mathcal{X})$-continuous, then $\mathcal{L} + C$ with domain $\mathcal{D}(\mathcal{L} + C) = \mathcal{D}(\mathcal{L})$ is the generator of some C_0-semigroup on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$.*

Proof. By the [10, Theorem 1.4, p.564] and using [10, Lemma 1.10, p.567], $\mathcal{L}^*$ is the generator of the C_0-semigroup $\{T^*(t)_{t\geq 0}\}$ on $(\mathcal{X}, \mathcal{C}(\mathcal{X}, \mathcal{X}^*_{\mathcal{C}})) = (\mathcal{X}, \| \,.\, \|)$. Under the condition on C, by [10, Lemma 1.12, p.568] it follows that the operator C^* is bounded on $(\mathcal{X}, \| \,.\, \|)$. By a well known perturbation result (see [10, Theorem 1, p.68]), we find that $\mathcal{L}^* + C^* = (\mathcal{L} + C)^*$ is the generator of some C_0-semigroup on $(\mathcal{X}, \| \,.\, \|)$. By using again [10, Theorem 1.4, p.564], we obtain that $(\mathcal{L} + C)^{**}$ is the generator of some C_0-semigroup on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$. Moreover, $\mathcal{D}((\mathcal{L} + C)^*)$ is dense in $(\mathcal{X}, \| \,.\, \|)$. Hence $\mathcal{D}((\mathcal{L} + C)^*)$ is dense in $(\mathcal{X}, \sigma(\mathcal{X}, \mathcal{X}^*))$. Then by [9, Theorem 7.1, p.155] it follows that

$$(\mathcal{L} + C)^{**} = \overline{(\mathcal{L} + C)}^{\sigma(\mathcal{X}^*, \mathcal{X})} \tag{1}$$

Since C is $\mathcal{C}(\mathcal{X}^*, \mathcal{X})$-continuous, by [10, Lemma 1.5, p.564] it follows that C is $\sigma(\mathcal{X}^*, \mathcal{X})$-continuous hence $\sigma(\mathcal{X}^*, \mathcal{X})$-closed. Consequently

$$\mathcal{L} + C = \overline{(\mathcal{L} + C)}^{\sigma(\mathcal{X}^*, \mathcal{X})} \tag{2}$$

from where it follows that $(\mathcal{L} + C)^{**} = \mathcal{L} + C$. Hence $\mathcal{L} + C$ is the generator of some C_0-semigroup on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$.
Finaly we present the Desch-Schappacher perturbation theorem for C_0-semigroups on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$.

Theorem 2. *Let $(\mathcal{X}, \| \,.\, \|)$ be a Banach space, $\mathcal{L}$ the generator of a C_0-semigroup $\{T(t)_{t\geq 0}\}$ on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$ and C a linear operator on $(\mathcal{X}^*, \mathcal{C}(\mathcal{X}^*, \mathcal{X}))$ with domain $\mathcal{D}(C) \supset \mathcal{D}(\mathcal{L})$. If $C : \mathcal{D}(\mathcal{L}) \to \mathcal{D}(\mathcal{L})$ is continuous with respect to the graph topology of $\mathcal{L}$ induced by the topology $\mathcal{C}(\mathcal{X}^*, \mathcal{X})$,* *then* $\mathcal{L}+C$ *with domain* $\mathcal{D}(\mathcal{L}+C)=\mathcal{D}(\mathcal{L})$ *is the generator of some* C_0*-semigroup on* $(\mathcal{X}^*,\mathcal{C}(\mathcal{X}^*,\mathcal{X}))$.

Proof. We will follows closely the proof of Arendt [1, Theorem 1.31, p.45]. Remark that $C:\mathcal{D}(\mathcal{L})\to\mathcal{D}(\mathcal{L})$ is continuous with respect to the graph topology of $\mathcal{L}$ induced by the topology $\mathcal{C}(\mathcal{X}^*,\mathcal{X})$ if and only if for all $\lambda>\lambda_0$ the operator

$$\tilde{C}:=(\lambda I-\mathcal{L})CR(\lambda;\mathcal{L}) \tag{3}$$

is continuous on $\mathcal{X}^*$ with respect to the topology $\mathcal{C}(\mathcal{X}^*,\mathcal{X})$. Consequently, by Theorem 1 we find that $\mathcal{L}+\tilde{C}$ is the generator of some C_0-semigroup on $(\mathcal{X}^*,\mathcal{C}(\mathcal{X}^*,\mathcal{X}))$. We shall prove that $\mathcal{L}+\tilde{C}$ is similar to $\mathcal{L}+C$. Remark that C is continuous with respect to the graph norm $\|\,.\,\|^*+\|\mathcal{L}.\,\|^*$. By the prove of [1, Theorem 1.31, p.45], there exists some $\lambda>\lambda_0$ such that the operators

$$U:=I-CR(\lambda;\mathcal{L})\quad,\quad U^{-1} \tag{4}$$

are bounded on $(\mathcal{X}^*,\|\,.\,\|^*)$. Moreover

$$\begin{aligned}
&U(\mathcal{L}+\tilde{C})U^{-1}=U(\mathcal{L}-\lambda I+\tilde{C})U^{-1}+\lambda I=\\
=\;&U[\mathcal{L}-\lambda I+(\lambda I-\mathcal{L})CR(\lambda;\mathcal{L})]U^{-1}+\lambda I=\\
=\;&U(\mathcal{L}-\lambda I)[I-CR(\lambda;\mathcal{L})]U^{-1}+\lambda I=\\
=\;&U(\mathcal{L}-\lambda I)+\lambda I=[I-CR(\lambda;\mathcal{L})](\mathcal{L}-\lambda I)+\lambda I=\\
=\;&\mathcal{L}-\lambda I+C+\lambda I=\mathcal{L}+C
\end{aligned}$$

Now we have only to prove that U and U^{-1} are continuous with respect to the topology $\mathcal{C}(\mathcal{X}^*,\mathcal{X})$. Since $CR(\lambda;\mathcal{L})=R(\lambda;\mathcal{L})\tilde{C}$ is continuous with respect to the topology $\mathcal{C}(\mathcal{X}^*,\mathcal{X})$, hence $U=I-CR(\lambda;\mathcal{L})$ is continuous with respect to the topology $\mathcal{C}(\mathcal{X}^*,\mathcal{X})$. On the other hand, by [10, Lemma 1.5, p.564], U^* and $[CR(\lambda;\mathcal{L})]^*$ are continuous on $(\mathcal{X},\|\,.\,\|)$. By Phillips theorem [6, Proposition 5.9, p.246], $1\in\rho([CR(\lambda;\mathcal{L})]^*)$ if and only if $1\in[CR(\lambda;\mathcal{L})]^{**}$ and

$$[I-([CR(\lambda;\mathcal{L})]^*)^{-1}]^*=(I-[CR(\lambda;\mathcal{L})]^{**})^{-1} \tag{5}$$

But by [9, Theorem 1.1, p.155] we have $[CR(\lambda;\mathcal{L})]^{**}=CR(\lambda;\mathcal{L})$ and the right hand side above becomes U^{-1}. Hence U^{-1}, being the dual of some bounded operator on $(\mathcal{X},\|.\|)$, is continuous on $(\mathcal{X}^*,\mathcal{C}(\mathcal{X}^*,\mathcal{X}))$ by [10, Lemma 1.5, p.564] and the proof is completed.

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