NEW ITERATIVE METHODS FOR SOLVING NONLINEAR EQUATIONS BY USING MODIFIED HOMOTOPY PERTURBATION METHOD

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Abstract. In this paper, we establish some iterative methods for solving real and complex zeroes of nonlinear equations by using the modified homotopy perturbation method which is mainly due to Golbabai and Javidi [5] . The proposed methods are then applied to solve test problems in order to assess there validity and accuracy.

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1. Introduction

During the last few years, the numerical techniques for solving nonlinear equations has been successfully applied. There are many papers that deal with nonlinear equations (see for example [1-6, 9, 17] and the references therein).

In this paper, some new numerical methods based on modified homotopy perturbation method (mainly due to Golbabai and Javidi [5]) are introduced for solving real and complex zeroes of nonlinear equations. The proposed methods are then applied to solve test problems in order to assess there validity and accuracy.

2. Homotopy perturbation method

The homotopy perturbation method was established by He in 1999 [7] and systematical description in 2000 [8] which is, in fact, a coupling of the traditional perturbation method and homotopy in topology [8]. This method was further developed and improved by He and applied to nonlinear oscillators with discontinuities [11], nonlinear wave equations [12], asymptotology [10], boundary value problem [14], limit cycle and bifurcation of nonlinear problems [13] and many other subjects. Thus He's method is a universal one which can solve various kinds of nonlinear equations. After that many researchers applied the method to various linear and nonlinear problems (see for example [1, 5-6, 17]).

To illustrate basic ideas of modified homotopy perturbation method [5], we consider the following nonlinear equation.

$$f(x) = 0, \ x \in \mathbb{R}. \tag{2.1}$$

We construct a homotopy $(\mathbb{R} \times [0,1]) \times \mathbb{R} \to \mathbb{R}$ which satisfies

$$H(\bar{x}, p, \alpha) = pf(\bar{x}) + (1-p)[f(\bar{x}) - f(x_0)] + p(1-p)\alpha = 0, \ \alpha, \bar{x} \in \mathbb{R}, \ p \in [0,1], \tag{2.2}$$

or

$$H(\bar{x}, p, \alpha) = f(\bar{x}) - f(x_0) + pf(x_o) + p(1-p)\alpha = 0, \ \alpha, \bar{x} \in \mathbb{R}, \ p \in [0,1], \tag{2.3}$$

where α is an unknown real number and p is the embedding parameter, x_o is an initial approximation of (2.1). It is obvious that

$$H(\bar{x}, 0) = f(\bar{x}) - f(x_0) = 0, \tag{2.4}$$

$$H(\bar{x}, 1) = f(\bar{x}) = 0. \tag{2.5}$$

The embedding parameter p monotonically increases from zero to unit as trivial problem $H(\bar{x}, 0) = f(\bar{x}) - f(x_0) = 0$ is continuously deformed to original problem $H(\bar{x}, 1) = f(\bar{x}) = 0$. The modified HPM uses the homotopy parameter p as an expanding parameter to obtain [7]:

$$\bar{x} = x_0 + px_1 + p^2x_2 + p^3x_3 + p^4x_4 + ... \tag{2.6}$$

The approximate solution of (2.1), therefore, can be readily obtained:

$$x = \lim_{p \to 1} \bar{x} = x_0 + x_1 + x_2 + x_3 + x_4 + ... \tag{2.7}$$

The convergence of the series (2.7) has been proved by He in [7].

For the application of modified HPM to (2.1) we can write (2.3) as follows by expanding $f(\bar{x})$ into Taylor series around x_0 :

$$[f(x_0)+(x-x_0)f'(x_0)+(x-x_0)^2\frac{f''(x_0)}{2!}+(x-x_0)^3\frac{f'''(x_0)}{3!}+(x-x_0)^4\frac{f^{(iv)}(x_0)}{4!}+...]$$

$$-f(x_0)+pf(x_o)+p(1-p)\alpha=0. \tag{2.8}$$

Substitution of (2.6) into (2.8) yields

$$\begin{aligned}&[f(x_0)+(x_0+px_1+p^2x_2+p^3x_3+p^4x_4+...-x_0)f'(x_0)\\&+(x_0+px_1+p^2x_2+p^3x_3+p^4x_4+...-x_0)^2\frac{f''(x_0)}{2!}\\&+(x_0+px_1+p^2x_2+p^3x_3+p^4x_4+...-x_0)^3\frac{f'''(x_0)}{3!}\\&+(x_0+px_1+p^2x_2+p^3x_3+p^4x_4+...-x_0)^4\frac{f^{(iv)}(x_0)}{4!}+...]\\&-f(x_0)+pf(x_o)+p(1-p)\alpha=0.\end{aligned} \tag{2.9}$$

By equating the terms with identical powers of p, we have

$$p^0: f(x_0)-f(x_0)=0, \tag{2.10}$$

$$p^1: x_1f'(x_0)+f(x_0)+\alpha=0, \tag{2.11}$$

$$p^2: x_2f'(x_0)+x_1^2\frac{f''(x_0)}{2}-\alpha=0, \tag{2.12}$$

$$p^3: x_3f'(x_0)+x_1x_2f''(x_0)+x_1^3\frac{f'''(x_0)}{6}=0, \tag{2.13}$$

$$p^4: x_4f'(x_0)+(\frac{1}{2}x_2^2+x_1x_3)f''(x_0)+x_1^2x_2\frac{f'''(x_0)}{2}+x_1^4\frac{f^{(iv)}(x_0)}{24}=0, \tag{2.14}$$

$$\begin{aligned}&p^5: x_5f'(x_0)+(x_2x_3+x_1x_4)f''(x_0)\\&+x_1(x_1x_3+x_2^2)\frac{f'''(x_0)}{2}+x_1^3x_2\frac{f^{(iv)}(x_0)}{6}+x_1^5\frac{f^{(v)}(x_0)}{120}=0,\end{aligned} \tag{2.15}$$

$$\begin{aligned}
&p^6 : x_6 f'(x_0) + (x_1x_5 + x_2x_4 + \frac{1}{2}x_3^2)f''(x_0) \\
&+ (x_1x_2x_3 + \frac{1}{6}x_2^3 + \frac{1}{2}x_1^2x_4)f'''(x_0) \\
&+ \frac{1}{2}x_1^2(\frac{1}{360}x_1^4 + \frac{1}{3}x_1x_3f^{(iv)}(x_0) + \frac{1}{2}x_2^2)f^{(iv)}(x_0) \\
&+ x_1^4x_2\frac{f^{(v)}(x_0)}{24} = 0, \qquad (2.16)
\end{aligned}$$

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We try to find parameter α, such that

$$x_1 = 0. \qquad (2.17)$$

By putting $x_1 = 0$, in (2.11), we get the value of α, i.e.,

$$\alpha = -f(x_0). \qquad (2.18)$$

Similarly in this way, after putting $x_1 = 0$ and using the obtained results (2.18) in (2.13), (2.14), (2.15) and (2.16), we get

$$x_2 = -\frac{f(x_0)}{f'(x_0)}, \qquad (2.19)$$

$$x_3 = 0, \qquad (2.20)$$

$$x_4 = -x_2^2\frac{f''(x_0)}{2f'(x_0)}, \qquad (2.21)$$

$$x_5 = 0, \qquad (2.22)$$

$$x_6 = -x_2x_4\frac{f''(x_0)}{f'(x_0)} - x_2^3\frac{f'''(x_0)}{6f'(x_0)}. \qquad (2.23)$$

From (2.19), (2.21) and (2.23) we obtain

$$x_4 = -\frac{f^2(x_0)f''(x_0)}{2f'^3(x_0)}, \tag{2.24}$$

and

$$\begin{aligned} x_6 &= \frac{f^3(x_0)[-3f''^2(x_0) + f'(x_0)f'''(x_0)]}{6f'^5(x_0)}, \\ &\cdot \\ &\cdot \\ &\cdot \end{aligned} \tag{2.25}$$

By substituting (2.17), (2.19), (2.20), (2.22), (2.24) and (2.25) in (2.7), we can obtain the zero of (2.1) as follows:

$$\begin{aligned} x &= x_0 + x_2 + x_4 + x_6 + ... \\ &= x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{f^2(x_0)f''(x_0)}{2f'^3(x_0)} \\ &\quad + \frac{f^3(x_0)[-3f''^2(x_0) + f'(x_0)f'''(x_0)]}{6f'^5(x_0)} + ... \end{aligned} \tag{2.26}$$

This formulation allows us to suggest the following iterative methods for solving nonlinear equation (2.1).

1. For a given x_0, calculate the approximation solution x_{n+1} by the iterative scheme
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \ f'(x_n) \neq 0,$$
which is the classical Newton-Raphson method.

2. For a given x_0, calculate the approximation solution x_{n+1} by the iterative scheme
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)f''(x_n)}{2f'^3(x_n)}, \ f'(x_n) \neq 0,$$
which is the famous householder method.

3. For a given x_0, calculate the approximation solution x_{n+1} by the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)f''(x_n)}{2f'^3(x_n)} + \frac{f^3(x_n)[-3f''^2(x_n) + f'(x_n)f'''(x_n)]}{6f'^5(x_n)},$$

$$f'(x_n) \neq 0.$$

3. Analysis of convergence

Theorem. *Let $w \in I$ be a simple zero of a sufficiently differentiable function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ on an open interval I. If x_0 is close to w, then the Algorithm 3 has fourth order of convergence.*

Proof. The iterative technique is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)f''(x_n)}{2f'^3(x_n)} + \frac{f^3(x_n)[-3f''^2(x_n) + f'(x_n)f'''(x_n)]}{6f'^5(x_n)}. \tag{3.1}$$

Let w be a simple zero of f . By Taylor's expansion, we have,

$$f(x_n) = f'(w)\left[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O\left(e_n^7\right)\right], \tag{3.2}$$

$$f'(x_n) = f'(w)\left[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + O(e_n^6)\right], \tag{3.3}$$

$$f''(x_n) = f'(w)[2c_2 + 6c_3e_n + 12c_4e_n^2 + O(e_n^3)], \tag{3.4}$$

$$f'''(x_n) = f'(w)[6c_3 + 24c_4e_n + O(e_n^2)], \tag{3.5}$$

where

$$c_k = \left(\frac{1}{k!}\right)\frac{f^{(k)}(w)}{f'(w)},\ k = 2, 3,\ ...,\ \text{and } e_n = x_n - w.$$

From (3.1), (3.2), (3.3), (3.4) and (3.5) we have

$$x_{n+1} = w + (c_4 - 5c_2c_3 + 5c_2^3)e_n^4 + O(e_n^5),$$

implies

$$e_{n+1} = (c_4 - 5c_2c_3 + 5c_2^3)e_n^4 + O(e_n^5).$$

This completes the proof.

4. Applications

We present some examples to illustrate the efficiency of the developed methods. In Table-A, we apply the Algorithm 2 (A1) and Algorithm 3 (A2) for the following functions and compare them with the methods J1 (Algorithm 2.1, [5]) and J2 (Algorithm 2.2, [5]) developed by Javidi in [5].

$f_1(x) = x^3 + 4x^2 + 8x + 8,$
$f_2(x) = x - 2 - e^{-x},$
$f_3(x) = x^2 - (1 - x)^5,$
$f_4(x) = e^x - 3x^2,$
$f_5(x) = \sin^2(x) - x^2 + 1.$

Table-A:

	IT	Root	$f(x_n)$
$f_1, x_0 = -1$			
J1	8	2.0813-2.733i	-2.4535e-111-1.60923e-113i
J2	10	2.0813-2.733i	2.1309e-19+8.206e-19i
A1	5	-1.2076478271309	5.81586e-115
A2	4	-1.2076478271309	-2.06436e-120

	IT	Root	$f(x_n)$
$f_2, x_0 = 0$			
J1	7	-1.2076478271-1.49e-107i	2.901108e-106-3.0276e-106i
J2	4	0.257530	9.4420e-100
A1	4	0.257530	3.27433755e-114
A2	3	0.257530	-4.413912e-70

	IT	Root	$f(x_n)$
$f_3, x_0 = 1$			
J1	5	0.11785097732	3.5354109e-38
J2	6	-2.058925+2.0606i	-4.66e-57+9.484e-57i
A1	5	0.11785097732	1.237889811e-98
A2	3	0.11785097732	5.62000755291066013

	IT	Root	$f(x_n)$
$f_4, x_0 = -10$			
J1	5	-0.4589622675369	3.0643975e-86
J2	5	0.910007522488	2.493934020e-48
A1	4	0.910007522488	-2.7259720e-91
A2	4	-0.4589622675369	-1.29618555762e-95

	IT	Root	$f(x_n)$
$f_5, x_0 = -1$			
J1	4	-0.473465807729126	-9.77549922e-47
J2	7	0.236732-2.041i	9.0452e-72-2.2502e-72i
A1	3	-0.473465807729126	-3.0457465e-47
A2	4	-0.473465807729126	0

5. Conclusions

Modified homotopy perturbation method is applied to numerical solution for solving real and complex zeroes of nonlinear equations. Comparison of the results obtained by the presented methods with the existing methods reveal that the presented methods are very effective, convenient and easy to use.

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