COMMON FIXED POINT THEOREMS FOR SUBCOMPATIBLE SINGLE AND SET-VALUED D-MAPS

Hakima Bouhadjera, Ahcène Djoudi

Abstract. The aim of this paper is to establish and prove a unique common fixed point theorem for two pairs of subcompatible single and set-valued D-maps satisfying an implicit relation. This result improves and extends especially the result of [1] and references therein.

2000 *Mathematics Subject Classification*: 47H10, 54H25.

Key words and phrases: Weakly compatible maps, compatible and compatible maps of type (A), (B) and (C), subcompatible maps, D-maps, implicit relations, single and set-valued maps, common fixed point theorems.

1. Introduction and preliminaries

Throughout this paper, (X, d) denotes a metric space and $B(X)$ is the set of all nonempty bounded subsets of X. For all A, B in $B(X)$, we define

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}.$$

If $A = \{a\}$, we write $\delta(A, B) = \delta(a, B)$. Also, if $B = \{b\}$, it yields that $\delta(A, B) = d(a, b)$.

From the definition of $\delta(A, B)$, for all A, B, C in $B(X)$ it follows that

$$\delta(A, B) = \delta(B, A) \geq 0,$$

$$\delta(A, B) \leq \delta(A, C) + \delta(C, B),$$

$$\delta(A, A) = diam A,$$

$$\delta(A, B) = 0 \Leftrightarrow A = B = \{a\}.$$

Definition 1.1. [3] *A sequence $\{A_n\}$ of nonempty subsets of X is said to be convergent to a subset A of X if:*
(i) *each point a in A is the limit of a convergent sequence $\{a_n\}$, where a_n is in A_n for $n \in N$,*
(ii) *for arbitrary $\epsilon > 0$, there exists an integer m such that $A_n \subseteq A_\epsilon$ for $n > m$, where A_ϵ denotes the set of all points x in X for which there exists a point a in A, depending on x, such that $d(x, a) < \epsilon$. A is then said to be the limit of the sequence $\{A_n\}$.*

Lemma 1.1. [3] *If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$ converging to A and B in $B(X)$, respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

Lemma 1.2. [4] *Let $\{A_n\}$ be a sequence in $B(X)$ and y be a point in X such that $\delta(A_n, y) \to 0$. Then the sequence $\{A_n\}$ converges to the set $\{y\}$ in $B(X)$.*

To generalize commuting and weakly commuting maps, Jungck introduced the concept of compatible maps as follows:

Definition 1.2. [5] *Self-maps f and g of a metric space X are said to be compatible if*

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t.$$

for some $t \in X$.

Further, Jungck et al. gave another generalization of weakly commuting maps by introducing the concept of compatible maps of type (A).

Definition 1.3. [7] *Self-maps f and g of a metric space X are said to be compatible of type (A) if*

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) = 0$$

and

$$\lim_{n\to\infty} d(gfx_n, f^2x_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t.$$

for some $t \in X$.

Extending type (A) maps, Pathak et al. introduced the notion of compatible maps of type (B).

Definition 1.4. [10] *Self-maps* f *and* g *of a metric space* X *are said to be compatible of type* (B) *if*

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, f^2x_n)\right]$$

and

$$\lim_{n\to\infty} d(gfx_n, f^2x_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, g^2x_n)\right]$$

whenever $\{x_n\}$ *is a sequence in* X *such that*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t.$$

for some $t \in X$.

In 1998, Pathak et al. added a new extension of compatibility of type (A) by introducing the concept of compatibility of type (C).

Definition 1.5. [9] *Self-maps* f *and* g *of a metric space* X *are said to be compatible of type* (C) *if*

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) \leq \frac{1}{3}\left[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, f^2x_n) + \lim_{n\to\infty} d(ft, g^2x_n)\right]$$

and

$$\lim_{n\to\infty} d(gfx_n, f^2x_n) \leq \frac{1}{3}\left[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, g^2x_n) + \lim_{n\to\infty} d(gt, f^2x_n)\right]$$

whenever $\{x_n\}$ *is a sequence in* X *such that*

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t.$$

for some $t \in X$.

Afterwards, Jungck generalized all of the above notions by giving the concept of weak compatibility.

Definition 1.6. [6] *Self-maps* f *and* g *of a metric space* X *are called weakly compatible if* $ft = gt$, $t \in X$ *implies* $fgt = gft$.

Obviously, compatible maps and compatible maps of type (A) (resp. (B), (C)) are weakly compatible, however, there exist weakly compatible maps which are neither compatible nor compatible of type (A) (resp. type (B), (C)) (see [1]).

To extend the weak compatibility of single valued maps to the setting of single and set-valued maps, the same author with Rhoades gave the next generalization:

Definition 1.7. [8] *Maps $f : X \to X$ and $F : X \to B(X)$ are subcompatible if*

$$\{t \in X/Ft = \{ft\}\} \subseteq \{t \in X/Fft = fFt\}.$$

Recently in 2003, Djoudi and Khemis introduced the following definition:

Definition 1.8. [2] *Maps $f : X \to X$ and $F : X \to B(X)$ are said to be D-maps if there exists a sequence $\{x_n\}$ in X such that for some $t \in X$*

$$\lim_{n\to\infty} fx_n = t$$

and

$$\lim_{n\to\infty} Fx_n = \{t\}.$$

Example 1.1.
(1) Let $X = [0, \infty)$ with the usual metric d. Define $f : X \to X$ and $F : X \to B(X)$ as follows:

$$fx = x$$

and

$$Fx = [0, 2x]$$

for $x \in X$.
Let $x_n = \frac{1}{n}$ for $n \in N^* = \{1, 2, \ldots\}$. Then,

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} x_n = 0$$

and

$$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} [0, 2x_n] = \{0\}.$$

Therefore f and F are D-maps.

(2) Endow $X = [0, \infty)$ with the usual metric d and define

$$fx = x + 2$$

and

$$Fx = [0, x]$$

for every $x \in X$. Suppose there exists a sequence $\{x_n\}$ in X such that $fx_n \to t$ and $y_n \to t$ for some $t \in X$, with $y_n \in Fx_n = [0, x_n]$. Then,

$$\lim_{n\to\infty} x_n = t - 2$$

and $0 \leq t \leq t - 2$ which is impossible.

Let R_+ be the set of all non-negative real numbers and $\mathcal{G}$ be the set of all continuous functions $G : R_+^6 \to R$ satisfying the conditions:
(G_1) : G is nondecreasing in variables t_5 and t_6,
(G_2) : there exists $\theta \in (1, \infty)$, such that for every u, $v \geq 0$ with
(G_a) : $G(u, v, u, v, u + v, 0) \geq 0$ or
(G_b) : $G(u, v, v, u, 0, u + v) \geq 0$
we have $u \geq \theta v$.
(G_3) : $G(u, u, 0, 0, u, u) < 0 \ \forall \ u > 0$.

In his paper [1], Djoudi established the following result:

Theorem 1.1. *Let f, g, h and k be maps from a complete metric space X into itself having the following conditions*:
(i) f, g are surjective,
(ii) the pairs of maps f, h as well as g, k are weakly compatible,
(iii) the inequality

$$G(d(fx, gy), d(hx, ky), d(fx, hx), d(gy, ky), d(fx, ky), d(gy, hx)) \geq 0$$

for all x, $y \in X$, where $G \in \mathcal{G}$. Then f, g, h and k have a unique common fixed point.

Our aim here is to extend the above result to the setting of single and set-valued maps in a metric space by deleting some conditions required on G. Also, we give a generalization of our result.

2. Implicit relations

Let R_+ and let Φ be the set of all continuous functions $\varphi : R_+^6 \to R$ satisfying the conditions
(φ_1) : for every u, $v \geq 0$ with
$(\varphi_a) : \varphi(u, v, u, v, 0, u+v) \geq 0$ or
$(\varphi_b) : \varphi(u, v, v, u, u+v, 0) \geq 0$
we have $u \leq v$.
$(\varphi_2) : \varphi(u, u, 0, 0, u, u) < 0 \ \forall \ u > 0$.

Example 2.1.
$\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - k \max\{t_2, t_3, t_4, \frac{1}{2}(t_5 + t_6)\}$, where $k > 1$.
(φ_1) : Let $u > 0$ and $v \geq 0$, suppose that $u > v$, then $\varphi(u, v, u, v, 0, u+v) = \varphi(u, v, v, u, u+v, 0) = u - ku \geq 0$, which implies that $u \geq ku > u$ which is a contradiction, then $u \leq v$. For $u = 0$, we have $u \leq v$.
$(\varphi_2) : \varphi(u, u, 0, 0, u, u) = u(1-k) < 0 \ \forall \ u > 0$.

Example 2.2.
$\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1^p - c_1 \max\{t_2^p, t_3^p, t_4^p\} - c_2 t_5^{p-1} t_6 - c_3 t_5 t_6^{p-1}$, where $c_1 > 1$, c_2, $c_3 \geq 0$ and p is an integer such that $p \geq 2$.
(φ_1) : Let $u > 0$ and $v \geq 0$, suppose that $u > v$, then $\varphi(u, v, u, v, 0, u+v) = \varphi(u, v, v, u, u+v, 0) = u^p - c_1 u^p \geq 0$, which implies that $u^p \geq c_1 u^p > u^p$ which is a contradiction, then $u \leq v$. For $u = 0$, we have $u \leq v$.
$(\varphi_2) : \varphi(u, u, 0, 0, u, u) = u^p(1 - c_1 - c_2 - c_3) < 0 \ \forall \ u > 0$.

Example 2.3.
$\varphi(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - a \max\{t_2, t_3, t_4, b\sqrt{t_3 t_4}\}$, where $a > 1$ and $0 \leq b \leq 1$.
(φ_1) : Let $u > 0$ and $v \geq 0$, suppose that $u > v$, then $\varphi(u, v, u, v, 0, u+v) = \varphi(u, v, v, u, u+v, 0) = u(1-a) \geq 0$ impossible, then $u \leq v$. For $u = 0$, we have $u \leq v$.
$(\varphi_2) : \varphi(u, u, 0, 0, u, u) = u(1-a) < 0 \ \forall \ u > 0$.

3. Main results

Theorem 3.1. *Let f, g be self-maps of a metric space (X, d) and let F, $G : X \to B(X)$ be two set-valued maps satisfying the conditions*
(1) $FX \subseteq gX$ *and* $GX \subseteq fX$,

$$(2)\varphi(\delta(Fx, Gy), d(fx, gy), \delta(fx, Fx), \delta(gy, Gy), \delta(fx, Gy), \delta(gy, Fx)) \geq 0$$

for all x, y in X, where $\varphi \in \Phi$. If either
(3) f and F are subcompatible D-maps; g and G are subcompatible and FX is closed, or
(3′) g and G are subcompatible D-maps; f and F are subcompatible and GX is closed.
Then, f, g, F and G have a unique common fixed point $t \in X$ such that $Ft = Gt = \{t\} = \{ft\} = \{gt\}$.

Proof. Suppose that F and f are D-maps, then, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = t$$

and

$$\lim_{n\to\infty} Fx_n = \{t\}$$

for some $t \in X$. Since FX is closed and $FX \subseteq gX$, there is a point u in X such that $gu = t$. First, we show that $Gu = \{gu\} = \{t\}$. Using inequality (2) we get

$$\varphi(\delta(Fx_n, Gu), d(fx_n, gu), \delta(fx_n, Fx_n), \delta(gu, Gu), \delta(fx_n, Gu), \delta(gu, Fx_n)) \geq 0.$$

Since φ is continuous, using lemma 1.1 we obtain at infinity

$$\varphi(\delta(t, Gu), 0, 0, \delta(t, Gu), \delta(t, Gu), 0) \geq 0$$

thus, by (φ_b) we have $\delta(t, Gu) = 0$ hence $Gu = \{t\} = \{gu\}$. Since the pair (g, G) is subcompatible then, $Ggu = gGu$ and hence $GGu = Ggu = gGu = \{ggu\}$. Now, we show that $Ggu = \{ggu\} = \{gu\}$. If $Ggu \neq \{gu\}$ then $\delta(Ggu, t) > 0$. Using inequality (2) we obtain

$$\varphi(\delta(Fx_n, Ggu), d(fx_n, g^2u), \delta(fx_n, Fx_n),$$

$$\delta(g^2u, Ggu), \delta(fx_n, Ggu), \delta(g^2u, Fx_n)) \geq 0.$$

Since φ is continuous, using lemma 1.1 we get at infinity

$$\varphi(\delta(t, Ggu), d(t, g^2u), 0, 0, \delta(t, Ggu), \delta(g^2u, t))$$

$$= \varphi(\delta(t, Ggu), \delta(t, Ggu), 0, 0, \delta(t, Ggu), \delta(Ggu, t)) \geq 0$$

which contradicts (φ_2). Hence $\delta(t, Ggu) = 0$; that is, $Ggu = \{ggu\} = \{gu\} = \{t\}$. Since $GX \subseteq fX$, there exists an element $v \in X$ such that $\{fv\} = Gu = \{gu\} = \{t\}$. By inequality (2) we have

$$\varphi(\delta(Fv, Gu), d(fv, gu), \delta(fv, Fv), \delta(gu, Gu), \delta(fv, Gu), \delta(gu, Fv))$$

$$= \varphi(\delta(Fv, fv), 0, \delta(fv, Fv), 0, 0, \delta(fv, Fv)) \geq 0$$

thus, by (φ_a) we have $Fv = \{fv\}$. Since f and F are subcompatible then, $Ffv = fFv$ and hence $FFv = Ffv = fFv = \{ffv\}$. Suppose that $\delta(Ffv, fv) > 0$ then by (2) it yields

$$\varphi(\delta(Ffv, Gu), d(f^2v, gu), \delta(f^2v, Ffv), \delta(gu, Gu), \delta(f^2v, Gu), \delta(gu, Ffv))$$

$$= \varphi(\delta(Ffv, fv), \delta(Ffv, fv), 0, 0, \delta(Ffv, fv), \delta(fv, Ffv)) \geq 0$$

which contradicts (φ_2). Hence $Ffv = \{fv\} = \{ffv\} = \{t\}$. Therefore $t = gu = fv$ is a common fixed point of both f, g, F and G.
Similarly, we can obtain this conclusion by using $(3')$ in lieu of (3).
Now, suppose that f, g, F and G have two common fixed points t and t' such that $t' \neq t$. Then inequality (2) gives

$$\varphi(\delta(Ft, Gt'), d(ft, gt'), \delta(ft, Ft), \delta(gt', Gt'), \delta(ft, Gt'), \delta(gt', Ft))$$

$$= \varphi(d(t, t'), d(t, t'), 0, 0, d(t, t'), d(t', t)) \geq 0$$

contradicts (φ_2). Therefore $t' = t$.

If we let in the above theorem, $F = G$ and $f = g$ then we get the following result:

Corollary 3.1. *Let (X, d) be a metric space and let $f : X \to X$, $F : X \to B(X)$ be a single and a set-valued map respectively such that*
(i) $FX \subseteq fX$,

$$(ii)\varphi(\delta(Fx, Fy), d(fx, fy), \delta(fx, Fx), \delta(fy, Fy), \delta(fx, Fy), \delta(fy, Fx)) \geq 0$$

for all x, y in X, where $\varphi \in \Phi$. If f and F are subcompatible D-maps and FX is closed, then, f and F have a unique common fixed point $t \in X$ such that $Ft = \{t\} = \{ft\}$.

Now, if we put $f = g$ then we get the next corollary:

Corollary 3.2. *Let f be a self-map of a metric space (X, d) and let F, $G : X \to B(X)$ be two set-valued maps satisfying the conditions*
(i) $FX \subseteq fX$ and $GX \subseteq fX$,

$$(ii)\varphi(\delta(Fx, Gy), d(fx, fy), \delta(fx, Fx), \delta(fy, Gy), \delta(fx, Gy), \delta(fy, Fx)) \geq 0$$

for all x, y in X, where $\varphi \in \Phi$. If either
(iii) f and F are subcompatible D-maps; f and G are subcompatible and FX is closed, or
$(iii)'$ f and G are subcompatible D-maps; f and F are subcompatible and GX is closed.
Then, f, F and G have a unique common fixed point $t \in X$ such that $Ft = Gt = \{ft\} = \{t\}$.

Corollary 3.3. If in Theorem 3.1 we have instead of (2) the inequality

$$\delta(Fx, Gy) \geq k \max\{d(fx, gy), \delta(fx, Fx), \delta(gy, Gy), \frac{1}{2}(\delta(fx, Gy) + \delta(gy, Fx))\}$$

for all x, y in X, where $k > 1$. Then, f, g, F and G have a unique common fixed point $t \in X$.

Proof. Take a function φ as in Example 2.1 then

$$\varphi(\delta(Fx, Gy), d(fx, gy), \delta(fx, Fx), \delta(gy, Gy), \delta(fx, Gy), \delta(gy, Fx)) = \delta(Fx, Gy)$$

$$-k \max\{d(fx, gy), \delta(fx, Fx), \delta(gy, Gy), \frac{1}{2}(\delta(fx, Gy) + \delta(gy, Fx))\} \geq 0$$

which implies that

$$\delta(Fx, Gy) \geq k \max\{d(fx, gy), \delta(fx, Fx), \delta(gy, Gy), \frac{1}{2}(\delta(fx, Gy) + \delta(gy, Fx))\}$$

for all x, y in X, where $k > 1$. Conclude by using Theorem 3.1.

Remark. As in Corollary 3.3 we can get two other corollaries using Examples 2.2 and 2.3 above.

Corollary 3.4. *Let f, g, F and G be maps satisfying* (1), (3) *and* (3′) *of Theorem 3.1. Suppose that for all x, $y \in X$ we have the inequality*

$$\delta^p(Fx, Gy) \geq k \max\{d^p(fx, gy), \delta^p(fx, Fx), \delta^p(gy, Gy)\}$$

with $k > 1$ and p is an integer such that $p \geq 1$. Then, f, g, F and G have a unique common fixed point $t \in X$.

Proof. Take a function φ as in Example 2.2 with $c_1 = k$, $c_2 = c_3 = 0$. Observe by condition (2)

$$\varphi(\delta(Fx, Gy), d(fx, gy), \delta(fx, Fx), \delta(gy, Gy), \delta(fx, Gy), \delta(gy, Fx))$$

$$= \delta^p(Fx, Gy) - k \max\{d^p(fx, gy), \delta^p(fx, Fx), \delta^p(gy, Gy)\} \geq 0.$$

Conclude by using Theorem 3.1.

Remark. We can get other results if we let in the corollaries $f = g$ and also $f = g$ and $F = G$.

Now, we give a generalization of Theorem 3.1.

Theorem 3.2. *Let f, g be self-maps of a metric space (X, d) and $F_n : X \to B(X)$, $n \in N^* = \{1, 2, \ldots\}$ be set-valued maps with*
(i) $F_nX \subseteq gX$ and $F_{n+1}X \subseteq fX$,
(ii) the inequality

$$\varphi(\delta(F_nx, F_{n+1}y), d(fx, gy), \delta(fx, F_nx),$$

$$\delta(gy, F_{n+1}y), \delta(fx, F_{n+1}y), \delta(gy, F_nx)) \geq 0$$

holds for all x, y in X, where $\varphi \in \Phi$. If either
(iii) f and $\{F_n\}_{n \in N^*}$ are subcompatible D-maps; g and $\{F_{n+1}\}_{n \in N^*}$ are subcompatible and F_nX is closed, or
$(iii)'$ g and $\{F_{n+1}\}_{n \in N^*}$ are subcompatible D-maps; f and $\{F_n\}_{n \in N^*}$ are subcompatible and $F_{n+1}X$ is closed.
Then, there is a unique common fixed point $t \in X$ such that $F_nt = \{t\} = \{ft\} = \{gt\}$, $n \in N^*$.

Proof. Letting $n = 1$, we get the hypotheses of Theorem 3.1 for maps f, g, F_1 and F_2 with the unique common fixed point t. Now, t is a unique common fixed point of f, g, F_1 and of f, g, F_2. Otherwise, if t' is a second distinct fixed point of f, g and F_1, then by inequality (ii), we get

$$\varphi(\delta(F_1t', F_2t), d(ft', gt), \delta(ft', F_1t'), \delta(gt, F_2t), \delta(ft', F_2t), \delta(gt, F_1t'))$$

$$= \varphi(d(t', t), d(t', t), 0, 0, d(t', t), d(t, t')) \geq 0$$

which contradicts (φ_2) hence $t' = t$.
By the same method, we prove that t is the unique common fixed point of maps f, g and F_2.
Now, by letting $n = 2$, we get the hypotheses of Theorem 3.1 for maps f, g, F_2 and F_3 and consequently they have a unique common fixed point t'. Analogously, t' is the unique common fixed point of f, g, F_2 and of f, g, F_3. Thus $t' = t$. Continuing in this way, we clearly see that t is the required point.

References

[1] Djoudi, A. *General fixed point theorems for weakly compatible maps.* Demonstratio Math. 38 (2005), no. 1, 197-205.

[2] Djoudi, A.; Khemis, R. *Fixed points for set and single valued maps without continuity.* Demonstratio Math. 38 (2005), no. 3, 739-751.

[3] Fisher, B. *Common fixed points of mappings and set-valued mappings.* Rostock. Math. Kolloq. No. 18 (1981), 69-77.

[4] Fisher, B.; Sessa, S. *Two common fixed point theorems for weakly commuting mappings.* Period. Math. Hungar. 20 (1989), no. 3, 207-218.

[5] Jungck, G. *Compatible mappings and common fixed points.* Internat. J. Math. Math. Sci. 9 (1986), no. 4, 771-779.

[6] Jungck, G. *Common fixed points for noncontinuous nonself maps on nonmetric spaces.* Far East J. Math. Sci. 4 (1996), no. 2, 199-215.

[7] Jungck, G.; Murthy, P. P.; Cho, Y. J. *Compatible mappings of type (A) and common fixed points.* Math. Japon. 38 (1993), no. 2, 381-390.

[8] Jungck, G.; Rhoades, B. E. *Fixed points for set valued functions without continuity.* Indian J. Pure Appl. Math. 29 (1998), no. 3, 227-238.

[9] Pathak, H. K.; Cho, Y. J.; Kang, S. M.; Madharia, B. *Compatible mappings of type (C) and common fixed point theorems of Greguš type.* Demonstratio Math. 31 (1998), no. 3, 499-518.

[10] Pathak, H. K.; Khan, M. S. *Compatible mappings of type (B) and common fixed point theorems of Greguš type.* Czechoslovak Math. J. 45(120) (1995), no. 4, 685-698.

Authors:

Hakima Bouhadjera
Laboratoire de Mathématiques Appliquées

Université Badji Mokhtar, B. P. 12, 23000 Annaba, Algérie
email: *b_hakima2000@yahoo.fr*

Ahcène Djoudi
Laboratoire de Mathématiques Appliquées
Université Badji Mokhtar, B. P. 12, 23000 Annaba, Algérie
email: *adjoudi@yahoo.com*