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ON ω_*-CLOSED SETS AND THEIR TOPOLOGY

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Abstract. In [2], Arhangel'skiĭ introduced the notion of ω-closed sets in relation to countable tightness. This paper deals with a class of sets called ω_*-closed sets and its topology which is stronger than the class of ω-closed sets due to Arhangel'skiĭ.

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1.Introduction

The word tightness was first introduced in [1]. In [2], Arhangel'skiĭ introduced the notions of "countable tightness" and "ω-closedness". Spaces that are countably tight are essentially those in which the closure operator is determined by countable sets. A subset A of a space X is called ω-closed [2] if $Cl(B) \subset A$ for every countable subset $B \subset A$. It is shown in [2, 3] that the family of all ω-open subsets of a space forms a topology for it. A topological space X has countable tightness if every ω-closed subset is closed in X [2]. In the survey [6], Goreham discuss the use of sequences and countable sets in general topology. It is proved that every sequential space and every hereditarily separable space has countable tightness. In particular, every countable space has countable tightness. Also, every perfectly regular countable compact space has countable tightness [13]. On the other hand, countable tightness is concerned directly with Moore-Mrówka problem which is as follows: "Is every compact Hausdorff space of countable tightness a sequential space?". The problem was posed in [9]. Other results connected to the Moore-Mrówka problem can be found in [10]. The purpose of this paper is to introduce and study a class of sets called ω_*-closed sets and its topology which is stronger than the class of ω-closed sets due to Arhangel'skiĭ.

2. Preliminaries

Throughout this paper, by a space we will always mean a topological space. For a subset A of a space X, the closure and the interior of A will be denoted by $Cl(A)$ and $Int(A)$, respectively.

A subset A of a topological space X is said to be regular open [11] (resp. regular closed [11], preopen [8]) if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$, $A \subset Int(Cl(A))$). A point $x \in X$ is said to be in the θ-closure [12] of a subset A of X, denoted by θ-$Cl(U)$, if $Cl(U) \cap A \neq \varnothing$ for each open set U of X containing x. A subset A of a space X is called θ-closed if $A = \theta$-$Cl(A)$. The complement of a θ-closed set is called θ-open. The θ-interior of a subset A of X is the union of all open sets of X whose closures are contained in A and is denoted by θ-$Int(A)$. The family of all θ-open subsets of a space (X, τ) is denoted by τ_θ.

Definition 1. ([2]) *A subset A of a space (X, τ) is called*
(1) *ω-closed if $Cl(B) \subset A$ for every countable subset $B \subset A$.*
(2) *ω-open if its complement is an ω-closed set.*

The family of all ω-open subsets of a space (X, τ) is denoted by τ_ω. It was shown that τ_ω is a topology on X [2, 3]. A function $f : X \to Y$ is said to be θ-continuous if $f^{-1}(V)$ is θ-open in X for each open subset of Y. A function $f : X \to Y$ is called weakly continuous [7] if for each $x \in X$ and each open subset V in Y containing $f(x)$, there exists an open subset U in X containing x such that $f(U) \subset Cl(V)$. The graph of a function $f : X \to Y$, denoted by $G(f)$, is the subset $\{(x, f(x)) : x \in X\}$ of the product space $X \times Y$. For a function $f : X \to Y$, the graph function $g : X \to X \times Y$ of f is defined by $g(x) = (x, f(x))$ for each $x \in X$. A subset A of a space X is said to be N-closed relative to X [4] if for each cover $\{A_i : i \in I\}$ of A by open sets of X, there exists a finite subfamily $I_0 \subset I$ such that $A \subset \cup_{i \in I_0} Cl(A_i)$.

3. ω_*-closed sets and their topology

Definition 2. *A subset A of a space (X, τ) is called*
(1) *ω_*-closed if θ-$Cl(B) \subset A$ for every countable subset $B \subset A$.*
(2) *ω_*-open if its complement is an ω_*-closed set.*

The family of all ω_*-open subsets of a space (X, τ) is denoted by τ_{ω_*}.

Remark 3. *The following diagram holds for a subset A of a space (X, τ):*

$$\begin{array}{ccc} \omega_*\text{-open} & \longrightarrow & \omega\text{-open} \\ \uparrow & & \uparrow \\ \theta\text{-open} & \longrightarrow & \text{open} \end{array}$$

None of these implications is reversible as shown in the following examples:

Example 4. Let $(R, \tau_{\text{co-countable}})$ be the countable complement topological space with the real line. Then the set $(0, 1)$ is ω-open but it is not open.

Example 5. *(R, τ_u) be the usual topological space with the real line. Then the set $(0, 1)$ is ω-open but it is not ω_*-open.*
Question: Does there exist a set in a topological space (X, τ) which is ω_*-open and it is not θ-open?

Theorem 6. *Let (X, τ) be a regular space and $A \subset X$. The following are equivalent:*
(1) *A is ω-open,*
(2) *A is ω_*-open.*

Proof. It follows from the fact that $\tau = \tau_\theta$ in a regular space (X, τ).

Theorem 7. *Let A be a subset of a space (X, τ). The following are equivalent:*
(1) *A is ω_*-open,*
(2) *$A \subset \theta\text{-}Int(X \backslash B)$ for every countable subset B of X such that $A \subset X \backslash B$.*

Proof. (1) $\Rightarrow$ (2) : Let A be ω_*-open and B be a countable subset of X such that $A \subset X \backslash B$. We have $X \backslash A$ is ω_*-closed. Since $B \subset X \backslash A$ and B is countable, then $\theta\text{-}Cl(B) \subset X \backslash A$. Hence, $A \subset X \backslash \theta\text{-}Cl(B) = \theta\text{-}Int(X \backslash B)$.
(2) $\Rightarrow$ (1) : Let $A \subset \theta\text{-}Int(X \backslash B)$ for every countable subset B such that $A \subset X \backslash B$. Let $B \subset X \backslash A$ be a countable subset. Then $A \subset X \backslash B$ and $A \subset \theta\text{-}Int(X \backslash B) = X \backslash \theta\text{-}Cl(B)$. Thus, $\theta\text{-}Cl(B) \subset X \backslash A$ and $X \backslash A$ is ω_*-closed. Hence, A is ω_*-open.

Corollary 8. *Let A be a subset of a space (X, τ). The following are equivalent:*
(1) A is *ω_*-open,*
(2) *$A \subset \theta\text{-}Int(B)$ for every subset B having countable complement such that $A \subset B$,*
(3) *$A \subset \theta\text{-}Int(B)$ for every $B \in \tau_{co\text{-}countable}$ such that $A \subset B$, where $\tau_{co\text{-}countable}$ is the countable complement topology on X.*

Corollary 9. *Let A be a subset of a space (X, τ). The following are equivalent:*
(1) A is *ω_*-closed,*
(2) *$\theta\text{-}Cl(B) \subset A$ for every $B \in \tau^*_{co\text{-}countable}$ such that $B \subset A$, where $\tau^*_{co\text{-}countable}$ is the family of closed sets of $(X, \tau_{co\text{-}countable})$.*

Theorem 10. *For a topological space (X, τ), (X, τ_{ω_*}) is a topological space.*

Proof. It is obvious that $\varnothing, X \in \tau_{\omega_*}$.
Let A, $B \in \tau_{\omega_*}$. This implies that $X \backslash A$ and $X \backslash B$ is ω_*-closed. Let V be a countable subset such that $V \subset X \backslash (A \cap B) = (X \backslash A) \cup (X \backslash B)$. There exist the sets

V_1 and V_2 such that $V = V_1 \cup V_2$ and $V_1 \subset X \backslash A$ and $V_2 \subset X \backslash B$. Since $X \backslash A$ and $X \backslash B$ are ω_*-closed, then θ-$Cl(V_1) \subset X \backslash A$ and θ-$Cl(V_2) \subset X \backslash B$. Also, θ-$Cl(V) = \theta$-$Cl(V_1 \cup V_2) = \theta$-$Cl(V_1) \cup \theta$-$Cl(V_2) \subset (X \backslash A) \cup (X \backslash B)$. Thus, $(X \backslash A) \cup (X \backslash B) = X \backslash (A \cap B)$ is ω_*-closed and hence $A \cap B \in \tau_{\omega_*}$.

Let $\{A_i\}_{i \in I}$ be a family of ω_*-open subsets of X. Then $\{X \backslash A_i\}_{i \in I}$ is a family of ω_*-closed subsets of X. Suppose that $V \subset \cap_{i \in I}(X \backslash A_i)$ is a countable subset. Then $V \subset X \backslash A_i$ for each $i \in I$. Since $X \backslash A_i$ is an ω_*-closed subset of X for each $i \in I$, then θ-$Cl(V) \subset X \backslash A_i$ for each $i \in I$. Thus, θ-$Cl(V) \subset \cap_{i \in I}(X \backslash A_i)$ and hence, $\cap_{i \in I}(X \backslash A_i)$ is an ω_*-closed subset of X. Thus, $\cup_{i \in I} A_i$ is an ω_*-open subset of X.

Example 11. Let $(X, \tau_{\text{co-countable}})$ be the countable complement topological space. Then (X, τ_ω) is the discrete topological space. It is obvious that $\tau_\omega \subset P(X)$ where $P(X)$ is the power set of X. Let $A \subset X$ and $C \in \tau_{\text{co-countable}}$ such that $A \subset C$. Then $A \subset Int(C) = C$. This implies that $A \in \tau_\omega$. Thus, τ_ω is the discrete topology.

Definition 12. *Let X be a topological space. The intersection of all ω_*-closed (resp. ω-closed) sets of X containing a subset A is called the ω_*-closure (resp. ω-closure) of A and is denoted by ω_*-$Cl(A)$ (resp. ω-$Cl(A)$). The union of all ω_*-open (resp. ω-open) sets of a space X contained in a subset A is called the ω_*-interior (resp. ω-interior) of A and is denoted by ω_*-$Int(A)$ (resp. ω-$Int(A)$).*

Remark 13. For every open set A, we have ω-$Int(A) = Int(A)$ but the converse is not true as shown in the following example:

Example 14. Let R be the real line with the topology $\tau = \{\varnothing, R, Q'\}$ where Q' is the set of irrational numbers. Then ω-$Int(N) = Int(N)$ where N is the set of natural numbers but it is not open.

Theorem 15. *Let A be a subset of a space X. The following are equivalent:*
(1) *A is open,*
(2) *A is ω-open and ω-$Int(A) = Int(A)$.*

Proof. (1) $\Rightarrow$ (2) : Since every open set is ω-open and ω-$Int(A) = Int(A)$, it is obvious.
(2) $\Rightarrow$ (1) : Let A be an ω-open and ω-$Int(A) = Int(A)$. Then $A = \omega$-$Int(A) = Int(A)$. Thus, A is open.

Remark 16. *If A is an ω_*-open set, then ω_*-$int(A) = \omega$-$int(A)$. If A is a θ-open set, then ω_*-$int(A) = \theta$-$int(A)$. None of these implications is reversible as shown in the following examples.*

Example 17. Let R be the real line with the topology $\tau = \{\varnothing, R, (2, 3)\}$. Then for the set $A = (1, \frac{3}{2})$, ω_*-$int(A) = \theta$-$int(A)$ but it is not θ-open.

Example 18. Let R be the real line with the topology $\tau = \{\varnothing, R, Q'\}$ where Q' is the set of irrational numbers. Then for the natural number set N, ω_*-$int(A) = \omega$-$int(A)$ but it is not ω_*-open.

Theorem 19. *Let A be a subset of a space X. The following hold:*
(1) A is θ-open if and only if A is ω_*-open and ω_*-$int(A) = \theta$-$int(A)$.
(2) A is ω_*-open if and only if A is ω-open and ω_*-$int(A) = \omega$-$int(A)$.

Proof. (1) : It follows from the fact that every θ-open is ω_*-open and ω_*-$int(A) = \theta$-$int(A)$. Conversely, let A be an ω_*-open and ω_*-$int(A) = \theta$-$int(A)$. Then $A = \omega_*$-$Int(A) = \theta$-$Int(A)$. Thus, A is θ-open.

(2) : Since every ω_*-open is ω-open and ω_*-$int(A) = \omega$-$int(A)$, it is obvious. Conversely, let A be an ω-open and ω_*-$int(A) = \omega$-$int(A)$. Then $A = \omega$-$int(A) = \omega_*$-$int(A)$. Hence, A is ω_*-open.

4. The related functions via ω_*-closed sets

In this section, we introduce the related functions via ω_*-closed sets. Moreover, we investigate properties and characterizations of these classes of functions.

Definition 20. *A function $f : X \to Y$ is said to be ω_*-continuous if for every $x \in X$ and every open subset V in Y containing $f(x)$, there exists an ω_*-open subset U in X containing x such that $f(U) \subset V$.*

Theorem 21. *For a function $f : (X, \tau) \to (Y, \sigma)$, f is ω_*-continuous if and only if $f : (X, \tau_{\omega_*}) \to (Y, \sigma)$ is continuous.*

Definition 22. *A function $f : X \to Y$ is said to be ω-continuous if $f^{-1}(V)$ is ω-open in X for each open subset of Y.*

Remark 23. The following diagram holds for a function $f : X \to Y$:

$$\begin{array}{ccc} \omega_*\text{-continuous} & \longrightarrow & \omega\text{-continuous} \\ \uparrow & & \uparrow \\ \theta\text{-continuous} & \longrightarrow & \text{continuous} \end{array}$$

None of these implications is reversible as shown in the following examples:

Example 24. Let $(R, \tau_{\text{co-countable}})$ be the countable complement topological space and (R, τ_u) be the usual topological space with the real line. Then the identity function $i : (R, \tau_{\text{co-countable}}) \to (R, \tau_u)$ is ω-continuous but it is not continuous.

Example 25. Let (R, τ_u) be the usual topological space with the real line. Let $Y = \{a, b, c\}$ and $\sigma = \{Y, \varnothing, \{a\}, \{c\}, \{a, c\}\}$. Define a function $f : (R, \tau_u) \to (Y, \sigma)$

as follows: $f(x) = \begin{cases} a & , \text{if } x \in (0,1) \\ b & , \text{if } x \notin (0,1) \end{cases}$. Then f is ω-continuous but it is not ω_*-continuous.

Question: Does there exist a function $f : (X, \tau) \to (Y, \sigma)$ which is ω_*-continuous and it is not θ-continuous?

Definition 26. *A function $f : X \to Y$ is said to be weakly ω_*-continuous (resp. weakly ω-continuous) at $x \in X$ if for every open subset V in Y containing $f(x)$, there exists an ω_*-open (resp. ω-open) subset U in X containing x such that $f(U) \subset Cl(V)$. f is said to be weakly ω_*-continuous (resp. weakly ω-continuous) if f is weakly ω_*-continuous (resp. weakly ω-continuous) at every $x \in X$.*

Remark 27. The following diagram holds for a function $f : X \to Y$:

$$\begin{array}{ccc} \text{weakly } \omega_*\text{-continuous} & \longrightarrow & \text{weakly } \omega\text{-continuous} \\ \uparrow & & \uparrow \\ \omega_*\text{-continuous} & \longrightarrow & \omega\text{-continuous} \end{array}$$

None of these implications is reversible as shown in the following examples:

Example 28. Let (R, τ_u) be the usual topological space with the real line. Let $X = \{a, b, c, d\}$ and $\sigma = \{X, \varnothing, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$. Then the function $f : (R, \tau_u) \to (X, \sigma)$ defined by

$$f(x) = \begin{cases} a & , \text{if } x \in (-\infty, 0] \cup [1, \infty) \\ b & , \text{if } x \notin (-\infty, 0] \cup [1, \infty) \end{cases}$$

is weakly ω-continuous but it is not ω-continuous.

Example 29. Let (R, τ_u) be the usual topological space with the real line. Then the identity function $i : (R, \tau_u) \to (R, \tau_u)$ is weakly ω-continuous but it is not weakly ω_*-continuous.

Question: Does there exist a function $f : (X, \tau) \to (Y, \sigma)$ which is weakly ω_*-continuous and it is not ω_*-continuous?

Definition 30. *A function $f : X \to Y$ is coweakly ω_*-continuous if for every open subset U in Y, $f^{-1}(Fr(U))$ is ω_*-closed in X, where $Fr(U) = Cl(U) \backslash Int(U)$.*

Theorem 31. *Let $f : X \to Y$ be a function. The following are equivalent:*
(1) f is ω_*-continuous,
(2) f is weakly ω_*-continuous and coweakly ω_*-continuous.

Proof. (1) $\Rightarrow$ (2) : Obvious.
(2) $\Rightarrow$ (1) : Let f be weakly ω_*-continuous and coweakly ω_*-continuous. Suppose that $x \in X$ and A is an open subset of Y such that $f(x) \in A$. Since f

is weakly ω_*-continuous, then there exists an ω_*-open subset B of X containing x such that $f(B) \subset Cl(A)$. Also, $Fr(A) = Cl(A)\backslash A$ and $f(x) \notin Fr(A)$. Since f is coweakly ω_*-continuous, then $x \in B\backslash f^{-1}(Fr(A))$ is ω_*-open in X. For every $y \in f(B\backslash f^{-1}(Fr(A)))$, $y = f(x_*)$ for a point $x_* \in B\backslash f^{-1}(Fr(A))$. We have $f(x_*) = y \in f(B) \subset Cl(A)$ and $y \notin Fr(A)$. Also, $f(x_*) = y \notin Fr(A)$ and $f(x_*) \in A$. Thus, $f(B\backslash f^{-1}(Fr(A))) \subset A$ and hence f is ω_*-continuous.

Theorem 32. *Let $f : X \to Y$ be a function. The following are equivalent:*
(1) f is weakly ω_*-continuous,
(2) ω_*-$Cl(f^{-1}(Int(Cl(K)))) \subset f^{-1}(Cl(K))$ for every subset $K \subset Y$,
(3) ω_*-$Cl(f^{-1}(Int(U))) \subset f^{-1}(U)$ for every regular closed set $U \subset Y$,
(4) ω_*-$Cl(f^{-1}(U)) \subset f^{-1}(Cl(U))$ for every open set $U \subset Y$,
(5) $f^{-1}(U) \subset \omega_*$-$Int(f^{-1}(Cl(U)))$ for every open set $U \subset Y$,
(6) ω_*-$Cl(f^{-1}(U)) \subset f^{-1}(Cl(U))$ for each preopen set $U \subset Y$,
(7) $f^{-1}(U) \subset \omega_*$-$Int(f^{-1}(Cl(U)))$ for each preopen set $U \subset Y$.

Proof. (1) ⇒ (2) : Let $K \subset Y$ and $x \in X\backslash f^{-1}(Cl(K))$. Then $f(x) \in Y\backslash Cl(K)$. There exists an open set U containing $f(x)$ such that $U \cap K = \emptyset$. Then $Cl(U) \cap Int(Cl(K)) = \emptyset$. Since f is weakly ω_*-continuous, then there exists an ω_*-open set V containing x such that $f(V) \subset Cl(U)$. We have $V \cap f^{-1}(Int(Cl(K))) = \emptyset$. Hence, $x \in X\backslash\omega_*$-$Cl(f^{-1}(Int(Cl(K))))$ and ω_*-$Cl(f^{-1}(Int(Cl(K)))) \subset f^{-1}(Cl(K))$.

(2) ⇒ (3) : Let U be any regular closed set in Y. Hence, ω_*-$Cl(f^{-1}(Int(U))) = \omega_*$-$Cl(f^{-1}(Int(Cl(Int(U))))) \subset f^{-1}(Cl(Int(U))) = f^{-1}(U)$.

(3) ⇒ (4) : Let U be an open subset of Y. Since $Cl(U)$ is regular closed in Y, ω_*-$Cl(f^{-1}(U)) \subset \omega_*$-$Cl(f^{-1}(Int(Cl(U)))) \subset f^{-1}(Cl(U))$.

(4) ⇒ (5) : Let U be any open set of Y. Since $Y\backslash Cl(U)$ is open in Y, then $X\backslash\omega_*$-$Int(f^{-1}(Cl(U))) = \omega_*$-$Cl(f^{-1}(Y \setminus Cl(U))) \subset f^{-1}(Cl(Y \setminus Cl(U))) \subset X \setminus f^{-1}(U)$. Thus, $f^{-1}(U) \subset \omega_*$-$Int(f^{-1}(Cl(U)))$.

(5) ⇒ (1) : Let $x \in X$ and U be any open subset of Y containing $f(x)$. Then $x \in f^{-1}(U) \subset \omega_*$-$Int(f^{-1}(Cl(U)))$. Take $V = \omega_*$-$Int(f^{-1}(Cl(U)))$. Hence, $f(V) \subset Cl(U)$ and f is weakly ω_*-continuous at x in X.

(1) ⇒ (6) : Let U be any preopen set of Y and $x \in X\backslash f^{-1}(Cl(U))$. There exists an open set S containing $f(x)$ such that $S \cap U = \emptyset$. We have $Cl(S \cap U) = \emptyset$. Since U is preopen, then $U \cap Cl(S) \subset Int(Cl(U)) \cap Cl(S) \subset Cl(Int(Cl(U)) \cap S) \subset Cl(Int(Cl(U) \cap S)) \subset Cl(Int(Cl(U \cap S))) \subset Cl(U \cap S) = \emptyset$. Since f is weakly ω_*-continuous and S is an open set containing $f(x)$, there exists an ω_*-open set V in X containing x such that $f(V) \subset Cl(S)$. We have $f(V) \cap U = \emptyset$ and hence $V \cap f^{-1}(U) = \emptyset$. Hence, $x \in X\backslash\omega_*$-$Cl(f^{-1}(U))$ and ω_*-$Cl(f^{-1}(U)) \subset f^{-1}(Cl(U))$.

(6) ⇒ (7) : Let U be any preopen set of Y. Since $Y\backslash Cl(U)$ is open in Y, then $X \setminus \omega_*$-$Int(f^{-1}(Cl(U))) = \omega_*$-$Cl(f^{-1}(Y \setminus Cl(U))) \subset f^{-1}(Cl(Y \setminus Cl(U))) \subset X \setminus f^{-1}(U)$. Hence, $f^{-1}(U) \subset \omega_*$-$Int(f^{-1}(Cl(U)))$.

(7) $\Rightarrow$ (1) : Let $x \in X$ and U any open set of Y containing $f(x)$. Then $x \in f^{-1}(U) \subset \omega_*\text{-}Int(f^{-1}(Cl(U)))$. Take $V = \omega_*\text{-}Int(f^{-1}(Cl(U)))$. Then $f(V) \subset Cl(U)$. Thus, f is weakly ω_*-continuous at x in X.

Theorem 33. *For a function $f : X \to Y$, $f : X \to Y$ be weakly ω_*-continuous at $x \in X$ if and only if $x \in \omega_*\text{-}Int(f^{-1}(Cl(A)))$ for each neighborhood A of $f(x)$.*

Proof. Let A be any neighborhood of $f(x)$. There exists an ω_*-open set B containing x such that $f(B) \subset Cl(A)$. Since $B \subset f^{-1}(Cl(A))$ and B is ω_*-open, then $x \in B \subset \omega_*\text{-}Int(B) \subset \omega_*\text{-}Int(f^{-1}(Cl(A)))$.

Conversely, let $x \in \omega_*\text{-}Int(f^{-1}(Cl(A)))$ for each neighborhood A of $f(x)$. Take $U = \omega_*\text{-}Int(f^{-1}(Cl(A)))$. Thus, $f(U) \subset Cl(A)$ and U is ω_*-open. Hence, f is weakly ω_*-continuous at $x \in X$.

Theorem 34. *Let $f : X \to Y$ be a function. The following are equivalent:*
(1) *f is weakly ω_*-continuous,*
(2) *$f(\omega_*\text{-}Cl(G)) \subset \theta\text{-}Cl(f(G))$ for each subset $G \subset X$,*
(3) *$\omega_*\text{-}Cl(f^{-1}(A)) \subset f^{-1}(\theta\text{-}Cl(A))$ for each subset $A \subset Y$,*
(4) *$\omega_*\text{-}Cl(f^{-1}(Int(\theta\text{-}Cl(A)))) \subset f^{-1}(\theta\text{-}Cl(A))$ for every subset $A \subset Y$.*

Proof. (1) $\Rightarrow$ (2) : Let $G \subset X$ and $x \in \omega_*\text{-}Cl(G)$. Let U be any open set of Y containing $f(x)$. Then there exists an ω_*-open set B containing x such that $f(B) \subset Cl(U)$. Since $x \in \omega_*\text{-}Cl(G)$, then $B \cap G \neq \emptyset$. Thus, $\emptyset \neq f(B) \cap f(G) \subset Cl(U) \cap f(G)$ and $f(x) \in \theta\text{-}Cl(f(G))$. Hence, $f(\omega_*\text{-}Cl(G)) \subset \theta\text{-}Cl(f(G))$.

(2) $\Rightarrow$ (3) : Let $A \subset Y$. Then $f(\omega_*\text{-}Cl(f^{-1}(A))) \subset \theta\text{-}Cl(A)$. Hence, $\omega_*\text{-}Cl(f^{-1}(A)) \subset f^{-1}(\theta\text{-}Cl(A))$.

(3) $\Rightarrow$ (4) : Let $A \subset Y$. Since $\theta\text{-}Cl(A)$ is closed in Y, then $\omega_*\text{-}Cl(f^{-1}(Int(\theta\text{-}Cl(A)))) \subset f^{-1}(\theta\text{-}Cl(Int(\theta\text{-}Cl(A))))) = f^{-1}(Cl(Int(\theta\text{-}Cl(A))))) \subset f^{-1}(\theta\text{-}Cl(A))$.

(4) $\Rightarrow$ (1) : Let U be any open set of Y. Then $U \subset Int(Cl(U)) = Int(\theta\text{-}Cl(U))$. Hence, $\omega_*\text{-}Cl(f^{-1}(U)) \subset \omega_*\text{-}Cl(f^{-1}(Int(\theta\text{-}Cl(U)))) \subset f^{-1}(\theta\text{-}Cl(U)) = f^{-1}(Cl(U))$. By Theorem 32, f is weakly ω_*-continuous.

Definition 35. *If a space X can not be written as the union of two nonempty disjoint ω_*-open sets, then X is said to be ω_*-connected.*

Theorem 36. *If $f : X \to Y$ is a weakly ω_*-continuous surjection and X is ω_*-connected, then Y is connected.*

Proof. Suppose that Y is not connected. There exist nonempty open sets A and B of Y such that $Y = A \cup B$ and $A \cap B = \emptyset$. Then A and B are clopen in Y. By Theorem 32, $f^{-1}(A) \subset \omega_*\text{-}Int(f^{-1}(Cl(A))) = \omega_*\text{-}Int(f^{-1}(A))$. Hence $f^{-1}(A)$ is ω_*-open in X. Similarly, $f^{-1}(B)$ is ω_*-open in X. Hence, $f^{-1}(A) \cap f^{-1}(B) = \emptyset$, $X = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A)$ and $f^{-1}(B)$ are nonempty. Thus, X is not ω_*-connected.

Theorem 37. *Let $\{A_i : i \in I\}$ be an ω_*-open cover of a space X. Then a function $f : X \to Y$ is weakly ω_*-continuous if and only if for each $i \in I$, the restriction $f_{A_i} : A_i \to Y$ is weakly ω_*-continuous.*

Proof. Obvious.

Theorem 38. *Let $f : X \to Y$ be weakly ω_*-continuous and Y be Hausdorff. The following hold:*

(1) *for each $(x, y) \notin G(f)$, there exist an ω_*-open set $G \subset X$ and an open set $U \subset Y$ containing x and y, respectively, such that $f(G) \cap int(cl(U)) = \varnothing$.*

(2) *inverse image of each N-closed set of Y is ω_*-closed in X.*

Proof. (1) : Let $(x, y) \notin G(f)$. Then $y \neq f(x)$. Since Y is Hausdorff, there exist disjoint open sets U and V containing y and $f(x)$, respectively. Thus, $int(cl(U)) \cap cl(V) = \varnothing$. Since f is weakly ω_*-continuous, there exists an ω_*-open set G containing x such that $f(G) \subset cl(V)$. Hence, $f(G) \cap int(cl(U)) = \varnothing$.

(2) : Suppose that there exists a N-closed set $W \subset Y$ such that $f^{-1}(W)$ is not ω_*-closed in X. There exists a point $x \in \omega_*\text{-}cl(f^{-1}(W))\backslash f^{-1}(W)$. Since $f(x) \notin f^{-1}(W)$, then $(x, y) \notin G(f)$ for each $y \in W$. Then there exist ω_*-open sets $G_y(x) \subset X$ and an open set $B(y) \subset Y$ containing x and y, respectively, such that $f(G_y(x)) \cap int(cl(B(y))) = \varnothing$. The family $\{B(y) : y \in W\}$ is a cover of W by open sets of Y. Since W is N-closed, there exit a finite number of points y_1, y_2, ..., y_n in W such that $W \subset \cup_{j=1}^{n} int(cl(B(y_j)))$. Take $G = \cap_{j=1}^{n} G_{y_j}(x)$. Then $f(G) \cap W = \varnothing$. Since $x \in \omega_*\text{-}cl(f^{-1}(W))$, then $f(G) \cap W \neq \varnothing$. This is a contradiction.

Theorem 39. *For a function $f : X \to Y$, f is weakly ω_*-continuous if the graph function g is weakly ω_*-continuous.*

Proof. Let g be weakly ω_*-continuous and $x \in X$ and A be an open set of X containing $f(x)$. Then $X \times A$ is an open set containing $g(x)$. Then there exists an ω_*-open set B containing x such that $g(B) \subset Cl(X \times A) = X \times Cl(A)$. Thus, $f(B) \subset Cl(A)$ and f is weakly ω_*-continuous.

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