Acta Universitatis Apulensis No. 23/2010
ISSN: 1582-5329 pp. 21-38

NEW EXACT TRAVELING WAVE SOLUTIONS FOR THE KAWAHARA AND MODIFIED KAWAHARA EQUATIONS BY USING MODIFIED TANH-COTH METHOD

A. Jabbari, H. Kheiri

Abstract. In this paper we use the modified tanh-coth method to solve the Kawahara and the modified Kawahara equations. New multiple traveling wave solutions are obtained for the Kawahara and the modified Kawahara equations.

2000 *Mathematics Subject Classification*: 35K01; 35J05.

1. Introduction

The world around us is inherently nonlinear. Nonlinear evolution equations (NEEs) are widely used as models to describe complex physical phenomena in various fields of sciences, especially in fluid mechanics, solid state physics, plasma physics, plasma wave and chemical physics. Particularly, various methods have been utilized to explore different kinds of solutions of physical models described by nonlinear PDEs. One of the basic physical problems for those models is to obtain their traveling wave solutions. Concepts like solitons, peakons, kinks, breathers, cusps and compactons are now being thoroughly investigated in the scientific literature [1-3]. During the past decades, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have proposed a variety of powerful methods, such as, Painleve expansion method [4], Jacobi elliptic function method [5], Hirota's bilinear method [6], the Sine-Cosine function method [7], the Exp-function method [8], the tanh method [9, 10] and so on. Among those, the tanh method, established by Malfliet [9], uses a particularly straightforward and effective algorithm to obtain solutions for a large numbers of nonlinear PDEs. In recent years, much research work has been concentrated on the various extensions and applications of the tanh method. Fan [11, 12] has proposed an extended tanh method and obtained new traveling wave solutions that cannot be obtained by the tanh method. Recently, Wazwaz extended the tanh method and call it first the extended tanh method [13-15] and later as the tanh-coth method [16]. Most recently, El-Wakil [17, 18] and Soliman [19] modified the extended tanh method (the tanh-coth method) and obtained new solutions for some nonlinear PDEs. The goal of this work is to implement the tanh-coth method and the Riccati equation in [20] that named modified tanh-coth method, to obtain more new exact travelling wave solutions of the Kawahara and the modefied Kawahara equations. The Kawahara equation occurs in the theory of magneto-acoustic waves in a plasmas and in theory of shallow water waves with surface tension. This equation was proposed by Kawahara in 1972, as a model equation describing solitary-wave propagation in media [21]. In the literature this equation is also referred as fifth-order KdV equation or singularly perturbed KdV equation [22]. The modified Kawahara equation was proposed first by Kawahara [21] as an important dispersive equation. This equation is also called the singularly perturbed KdV equation. This equation arises in the theory of shallow water waves.

2.Description of modified tanh-coth method

Consider the general nonlinear wave PDE

$$u_t = G(u, u_x, u_{xx}, ...) = 0. \tag{1}$$

In order to apply the tanh-coth method, the independent variables, x and t, are combined into a new variable $\xi = \mu(x - ct)$, where μ and c are undetermined parameters which represent the wave number and velocity of the traveling wave, respectively. Therefore, $u(x, t)$ is replaced by $u(\xi)$, which defines the traveling wave solutions of (1). Equations such as (1) are then transformed into

$$-\mu c \frac{du}{d\xi} = G(u, k\frac{du}{d\xi}, \mu^2 \frac{d^2u}{d\xi^2}, ...). \tag{2}$$

Hence, under the transformation $\xi = \mu(x - ct)$, the PDE in (1) has been reduced to an ordinary differential equation (ODE) given by (2). The resulting ODE is then solved by the modified tanh-coth method, which admits the use of a finite series of functions of the form

$$u(x, t) = u(\xi) = a_0 + \sum_{j=1}^{N} [a_j Y^j(\xi) + b_j Y^{-j}(\xi)], \tag{3}$$

and the Riccati equation

$$Y' = \alpha + \beta Y + \gamma Y^2, \tag{4}$$

where α, β and γ are constants to be prescribed later. The parameter N in (3) is a positive constant that can be determined by balancing the linear term of highest order with the nonlinear term in (2). substituting (3) into the ODE in (2) and using (4), we obtain an algebraic equation in powers of Y. Since all coefficients of Y^j must vanish. This will give a system of algebraic equations with respect to parameters a_i, b_i, μ and c. With the aid of Maple, we can determine a_i, b_i, μ and c. We will consider the following special solutions of the Riccati equation (4) are given in [23] by

- $\alpha = \beta = 1, \gamma = 0, \quad Y(\xi) = e^{\xi} - 1,$
- $\alpha = \frac{1}{2}, \gamma = -\frac{1}{2}, \beta = 0, \quad Y(\xi) = \coth(\xi) \pm csch(\xi) \ or \ Y(\xi) = \tanh(\xi) \pm isech(\xi),$
- $\alpha = \gamma = \pm\frac{1}{2}, \beta = 0, \quad Y(\xi) = \sec(\xi) \pm \tan(\xi) \ or \ Y(\xi) = \csc(\xi) \mp \cot(\xi),$
- $\alpha = 1, \gamma = -1, \beta = 0, \quad Y(\xi) = \tanh(\xi) \ or \ Y(\xi) = \coth(\xi),$
- $\alpha = \gamma = \pm 1, \beta = 0, \quad Y(\xi) = \tan(\xi) \ or \ Y(\xi) = \cot(\xi),$
- $\alpha = 1, \gamma = -4, \beta = 0, \quad Y(\xi) = \dfrac{\tanh(\xi)}{1 + \tanh^2(\xi)},$
- $\alpha = 1, \gamma = 4, \beta = 0, \quad Y(\xi) = \dfrac{\tan(\xi)}{1 - \tan^2(\xi)},$ (5)
- $\alpha = -1, \gamma = -4, \beta = 0, \quad Y(\xi) = \dfrac{\cot(\xi)}{1 - \cot^2(\xi)},$
- $\alpha = 1, \beta = \pm 2, \gamma = 2, \quad Y(\xi) = \dfrac{\tan(\xi)}{1 \mp \tan(\xi)},$
- $\alpha = -1, \beta = \pm 2, \gamma = -2, \quad Y(\xi) = \dfrac{\cot(\xi)}{1 \pm \cot(\xi)}.$

Other values for Y can be derived for other arbitrary values for α, β and γ.

3.The Kawahara equation

Let us first consider the Kawahara equation which has the form

$$u_t + auu_x + bu_{xxx} - ku_{xxxxx} = 0, \tag{6}$$

where a, b and k are nonzero real constants. In order to obtain travelling wave solutions for Eq. (6), we use

$$u(x,t) = u(\xi), \quad \xi = \mu(x - ct). \tag{7}$$

Substituting (7) into (6), we obtain

$$-cu' + auu' + b\mu^2 u''' - k\mu^4 u^{(5)} = 0, \tag{8}$$

and by once time integrating we find

$$-cu + \frac{a}{2}u^2 + b\mu^2 u'' - k\mu^4 u^{(4)} = 0. \tag{9}$$

Balancing the order of the nonlinear term u^2 with the highes order linear term $u^{(4)}$ in (9), we obtain N=4.
Thus, the solution of (3) has the form

$$u(\xi) = a_0 + a_1 Y + a_2 Y^2 + a_3 Y^3 + a_4 Y^4 + b_1 Y^{-1} + b_2 Y^{-2} + b_3 Y^{-3} + b_4 Y^{-4}. \tag{10}$$

Substituting (10) into (9) and using the Riccati equation (4), and because all coefficients of Y^i have to vanish, we obtain a system of algebraic equations in the unknowns $a_0, a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4,\ \alpha, \beta, \gamma, \mu$ and c of the following form:

$$\begin{aligned}
&\frac{1}{2}ab_4{}^2 - 840k\mu^4 b_4\alpha^4 = 0,\\
&ab_3b_4 - 360k\mu^4 b_3\alpha^4 - 2640k\mu^4 b_4\beta\alpha^3 = 0,\\
&\frac{1}{2}ab_3{}^2 + ab_2b_4 + 20b\mu^2 b_4\alpha^2 - 120k\mu^4 b_2\alpha^4 - 1080k\mu^4 b_3\beta\alpha^3 - 3020k\mu^4 b_4\beta^2\alpha^2\\
&-2080k\mu^4 b_4\gamma\alpha^3 = 0,\\
&ab_1b_4 + ab_2b_3 + 12b\mu^2 b_3\alpha^2 + 36b\mu^2 b_4\beta\alpha - 336k\mu^4 b_2\beta\alpha^3 - 24k\mu^4 b_1\alpha^4\\
&-1164k\mu^4 b_3\beta^2\alpha^2 - 816k\mu^4 b_3\gamma\alpha^3 - 1476k\mu^4 b_4\beta^3\alpha - 4608k\mu^4 b_4\beta\gamma\alpha^2 = 0,\\
&-cb_4 + \frac{1}{2}ab_2{}^2 + ab_1b_3 + ab_4a_0 + 6b\mu^2 b_2\alpha^2 + 16b\mu^2 b_4\beta^2 + 21b\mu^2 b_3\beta\alpha + 32b\mu^2 b_4\gamma\alpha\\
&-60k\mu^4 b_1\beta\alpha^3 - 330k\mu^4 b_2\beta^2\alpha^2 - 240k\mu^4 b_2\gamma\alpha^3 - 1680k\mu^4 b_3\beta\gamma\alpha^2 - 525k\mu^4 b_3\beta^3\alpha\\
&-256k\mu^4 b_4\beta^4 - 3232k\mu^4 b_4\beta^2\gamma\alpha - 1696k\mu^4 b_4\gamma^2\alpha^2 = 0,\\
&-cb_3 + aa_1b_4 + ab_1b_2 + ab_3a_0 + 9b\mu^2 b_3\beta^2 + 2b\mu^2 b_1\alpha^2 + 10b\mu^2 b_2\beta\alpha + 18b\mu^2 b_3\gamma\alpha\\
&+28b\mu^2 b_4\beta\gamma - 50k\mu^4 b_1\beta^2\alpha^2 - 40k\mu^4 b_1\gamma\alpha^3 - 130k\mu^4 b_2\beta^3\alpha - 440k\mu^4 b_2\beta\gamma\alpha^2\\
&-81k\mu^4 b_3\beta^4 - 576k\mu^4 b_3\gamma^2\alpha^2 - 1062k\mu^4 b_3\beta^2\gamma\alpha - 2240k\mu^4 b_4\beta\gamma^2\alpha - 700k\mu^4 b_4\beta^3\gamma\\
&= 0,\\
&-cb_2 + \frac{1}{2}ab_1{}^2 + aa_1b_3 + aa_2b_4 + ab_2a_0 + 4b\mu^2 b_2\beta^2 + 12b\mu^2 b_4\gamma^2 + 3b\mu^2 b_1\beta\alpha\\
&+8b\mu^2 b_2\gamma\alpha + 15b\mu^2 b_3\beta\gamma - 15k\mu^4 b_1\beta^3\alpha - 136k\mu^4 b_2\gamma^2\alpha^2 - 60k\mu^4 b_1\beta\gamma\alpha^2\\
&-232k\mu^4 b_2\beta^2\gamma\alpha - 16k\mu^4 b_2\beta^4 - 195k\mu^4 b_3\beta^3\gamma - 660k\mu^4 b_3\beta\gamma^2\alpha - 660k\mu^4 b_4\beta^2\gamma^2\\
&-480k\mu^4 b_4\gamma^3\alpha = 0,
\end{aligned} \tag{11}$$

$$
\begin{aligned}
&-cb_1+aa_3b_4+aa_1b_2+aa_2b_3+ab_1a_0+b\mu^2b_1\beta^2+6b\mu^2b_3\gamma^2+2b\mu^2b_1\gamma\alpha \\
&+6b\mu^2b_2\beta\gamma-16k\mu^4b_1\gamma^2\alpha^2-30k\mu^4b_2\beta^3\gamma-22k\mu^4b_1\beta^2\gamma\alpha-k\mu^4b_1\beta^4 \\
&-120k\mu^4b_2\beta\gamma^2\alpha-150k\mu^4b_3\beta^2\gamma^2-120k\mu^4b_3\gamma^3\alpha-240k\mu^4b_4\beta\gamma^3=0, \\
&-ca_0+2b\mu^2a_2\alpha^2+2b\mu^2b_2\gamma^2-24k\mu^4a_4\alpha^4-24k\mu^4b_4\gamma^4+aa_1b_1+\frac{1}{2}aa_0{}^2 \\
&+aa_2b_2+aa_3b_3+aa_4b_4+b\mu^2a_1\beta\alpha+b\mu^2b_1\beta\gamma-k\mu^4b_1\beta^3\gamma-14k\mu^4b_2\beta^2\gamma^2 \\
&-16k\mu^4b_2\gamma^3\alpha-14k\mu^4a_2\beta^2\alpha^2-8k\mu^4b_1\beta\gamma^2\alpha-36k\mu^4b_3\beta\gamma^3-8k\mu^4a_1\beta\gamma\alpha^2 \\
&-k\mu^4a_1\beta^3\alpha-16k\mu^4a_2\gamma\alpha^3-36k\mu^4a_3\alpha^3\beta=0, \\
&-ca_1-k\mu^4a_1\beta^4+aa_1a_0+aa_2b_1+aa_3b_2+aa_4b_3+b\mu^2a_1\beta^2+6b\mu^2a_3\alpha^2 \\
&+2b\mu^2a_1\gamma\alpha+6b\mu^2a_2\beta\alpha-240k\mu^4a_4\beta\alpha^3-150k\mu^4a_3\beta^2\alpha^2-120k\mu^4a_3\gamma\alpha^3 \\
&-30k\mu^4a_2\beta^3\alpha-120k\mu^4a_2\gamma\alpha^2\beta-16k\mu^4a_1\gamma^2\alpha^2-22k\mu^4a_1\beta^2\gamma\alpha=0, \\
&-ca_2+\frac{1}{2}aa_1{}^2+aa_2a_0+aa_3b_1+aa_4b_2+4b\mu^2a_2\beta^2+12b\mu^2a_4\alpha^2+3b\mu^2a_1\beta\gamma \\
&+8b\mu^2a_2\gamma\alpha+15b\mu^2a_3\beta\alpha-660k\mu^4a_4\beta^2\alpha^2-480k\mu^4a_4\gamma\alpha^3-195k\mu^4a_3\beta^3\alpha \\
&-660k\mu^4a_3\gamma\alpha^2\beta-136k\mu^4a_2\gamma^2\alpha^2-16k\mu^4a_2\beta^4-232k\mu^4a_2\beta^2\gamma\alpha-15k\mu^4a_1\beta^3\gamma \\
&-60k\mu^4a_1\beta\gamma^2\alpha=0, \\
&-ca_3+aa_4b_1+aa_1a_2+aa_3a_0+2b\mu^2a_1\gamma^2+9b\mu^2a_3\beta^2+10b\mu^2a_2\beta\gamma+18b\mu^2a_3\gamma\alpha \\
&+28b\mu^2a_4\beta\alpha-700k\mu^4a_4\beta^3\alpha-2240k\mu^4a_4\beta\gamma\alpha^2-81k\mu^4a_3\beta^4-576k\mu^4a_3\gamma^2\alpha^2 \\
&-1062k\mu^4a_3\beta^2\gamma\alpha-130k\mu^4a_2\beta^3\gamma-440k\mu^4a_2\beta\gamma^2\alpha-50k\mu^4a_1\beta^2\gamma^2-40k\mu^4a_1\gamma^3\alpha=0, \\
&-ca_4+\frac{1}{2}aa_2{}^2+aa_1a_3+aa_4a_0+6b\mu^2a_2\gamma^2+16b\mu^2a_4\beta^2+21b\mu^2a_3\beta\gamma+32b\mu^2a_4\gamma\alpha \\
&-1696k\mu^4a_4\gamma^2\alpha^2-256k\mu^4a_4\beta^4-3232k\mu^4a_4\beta^2\gamma\alpha-525k\mu^4a_3\beta^3\gamma-1680k\mu^4a_3\beta\gamma^2\alpha \\
&-330k\mu^4a_2\beta^2\gamma^2-240k\mu^4a_2\gamma^3\alpha-60k\mu^4a_1\beta\gamma^3=0, \\
&aa_2a_3+aa_1a_4+12b\mu^2a_3\gamma^2+36b\mu^2a_4\beta\gamma-1476k\mu^4a_4\beta^3\gamma-4608k\mu^4a_4\beta\gamma^2\alpha \\
&-1164k\mu^4a_3\beta^2\gamma^2-816k\mu^4a_3\gamma^3\alpha-336k\mu^4a_2\beta\gamma^3-24k\mu^4a_1\gamma^4=0, \\
&\frac{1}{2}aa_3{}^2+aa_2a_4+20b\mu^2a_4\gamma^2-3020k\mu^4a_4\beta^2\gamma^2-2080k\mu^4a_4\gamma^3\alpha-1080k\mu^4a_3\beta\gamma^3 \\
&-120k\mu^4a_2\gamma^4=0, \\
&aa_3a_4-2640k\mu^4a_4\beta\gamma^3-360k\mu^4a_3\gamma^4=0, \\
&\frac{1}{2}aa_4{}^2-840k\mu^4a_4\gamma^4=0.
\end{aligned} \tag{12}
$$

Case (1): By seting $\alpha=\beta=1$ and $\gamma=0$ in (11) and solving the resulting system, we obtain the following two sets of solutions:

- $a_0 = a_1 = a_2 = a_3 = a_4 = 0, \quad b_1 = 0, \quad b_2 = \dfrac{1680b^2}{169ka}, \quad b_3 = \dfrac{3360b^2}{169ka},$
$b_4 = \dfrac{1680b^2}{169ka}, \quad c = \dfrac{36b^2}{169k}, \quad \mu = \pm\sqrt{\dfrac{b}{13k}}, \quad \dfrac{b}{k} > 0,$
- $a_0 = \dfrac{-72b^2}{169ka}, \quad a_1 = a_2 = a_3 = a_4 = 0, \quad b_1 = 0, \quad b_2 = \dfrac{1680b^2}{169ka}, \quad b_3 = \dfrac{3360b^2}{169ka},$
$b_4 = \dfrac{1680b^2}{169ka}, \quad c = \dfrac{-36b^2}{169k}, \quad \mu = \pm\sqrt{\dfrac{b}{13k}}, \quad \dfrac{b}{k} > 0.$

Substituting these values and $Y = e^{\xi} - 1$ in (10), after some simplifications, we obtain

$$u_1(x,t) = \frac{1680k\mu^4}{a} \frac{e^{2\mu(x-\frac{36b^2}{169k}t)}}{(e^{\mu(x-\frac{36b^2}{169k}t)} - 1)^4}, \tag{13}$$

and

$$u_2(x,t) = \frac{24k\mu^4}{a} \frac{-3e^{4\mu(x+\frac{36b^2}{169k}t)} + 12e^{3\mu(x+\frac{36b^2}{169k}t)} + 52e^{2\mu(x+\frac{36b^2}{169k}t)} + 12e^{\mu(x+\frac{36b^2}{169k}t)} - 3}{(e^{\mu(x+\frac{36b^2}{169k}t)} - 1)^4}, \tag{14}$$

where $\mu = \pm\sqrt{\frac{b}{13k}}$, $\frac{b}{k} > 0$.

Case (2): By assuming $\alpha = 1/2$, $\gamma = -1/2$ and $\beta = 0$ in (11) and solving the obtained system and Substituting it's solution and $Y = \coth(\xi) \pm csch(\xi)$ or $Y = \tanh(\xi) \pm isech(\xi)$, in (10), we we have

$$u_3(x,t) = \frac{105b^2}{169ka}[1 - 2(\coth(\xi) \pm csch(\xi))^2 + (\coth(\xi) \pm csch(\xi))^4], \tag{15}$$

$$u_4(x,t) = \frac{105b^2}{169ka}[1 - 2(\tanh(\xi) \pm isech(\xi))^2 + (\tanh(\xi) \pm isech(\xi))^4], \tag{16}$$

where $\xi = \sqrt{\frac{b}{13k}}(x - \frac{36b^2t}{169k})$, $\frac{b}{k} > 0$.

$$u_5(x,t) = \frac{b^2}{169ka}[33 - 210(\coth(\xi) \pm csch(\xi))^2 + 105(\coth(\xi) \pm csch(\xi))^4], \tag{17}$$

$$u_6(x,t) = \frac{b^2}{169ka}[33 - 210(\tanh(\xi) \pm isech(\xi))^2 + 105(\tanh(\xi) \pm isech(\xi))^4], \tag{18}$$

where $\xi = \sqrt{\frac{b}{13k}}(x + \frac{36b^2t}{169k}),\ \frac{b}{k} > 0.$

$$u_7(x,t) = \frac{105b^2}{169ka}[1 - 2(\coth(\xi) \pm csch(\xi))^{-2} + (\coth(\xi) \pm csch(\xi))^{-4}], \quad (19)$$

$$u_8(x,t) = \frac{105b^2}{169ka}[1 - 2(\tanh(\xi) \pm isech(\xi))^{-2} + (\tanh(\xi) \pm isech(\xi))^{-4}], \quad (20)$$

where $\xi = \sqrt{\frac{b}{13k}}(x - \frac{36b^2t}{169k}),\ \frac{b}{k} > 0.$

$$u_9(x,t) = \frac{b^2}{169ka}[33 - 210(\coth(\xi) \pm csch(\xi))^{-2} + 105(\coth(\xi) \pm csch(\xi))^{-4}], \quad (21)$$

$$u_{10}(x,t) = \frac{b^2}{169ka}[33 - 210(\tanh(\xi) \pm isech(\xi))^{-2} + 105(\tanh(\xi) \pm isech(\xi))^{-4}], \quad (22)$$

where $\xi = \sqrt{\frac{b}{13k}}(x + \frac{36b^2t}{169k}),\ \frac{b}{k} > 0.$
Case (3): By solving (11) for $\alpha = \gamma = \pm 1/2$ and $\beta = 0$, and Substituting obtained values and $Y = \sec(\xi) \pm \tan(\xi)$ or $Y = \csc(\xi) \mp \cot(\xi)$, in (10), we get

$$u_{11}(x,t) = \frac{105b^2}{169ka}[1 + 2(\sec(\xi) \pm \tan(\xi))^2 + (\sec(\xi) \pm \tan(\xi))^4], \quad (23)$$

$$u_{12}(x,t) = \frac{105b^2}{169ka}[1 + 2(\csc(\xi) \mp \cot(\xi))^2 + (\csc(\xi) \mp \cot(\xi))^4], \quad (24)$$

where $\xi = \sqrt{\frac{-b}{13k}}(x - \frac{36b^2t}{169k}),\ \frac{b}{k} < 0.$

$$u_{13}(x,t) = \frac{b^2}{169ka}[33 + 210(\sec(\xi) \pm \tan(\xi))^2 + 105(\sec(\xi) \pm \tan(\xi))^4], \quad (25)$$

$$u_{14}(x,t) = \frac{b^2}{169ka}[33 + 210(\csc(\xi) \mp \cot(\xi))^2 + 105(\csc(\xi) \mp \cot(\xi))^4], \quad (26)$$

where $\xi = \sqrt{\frac{-b}{13k}}(x + \frac{36b^2t}{169k}),\ \frac{b}{k} < 0.$

$$u_{15}(x,t) = \frac{105b^2}{169ka}[1 + 2(\sec(\xi) \pm \tan(\xi))^{-2} + (\sec(\xi) \pm \tan(\xi))^{-4}], \quad (27)$$

$$u_{16}(x,t) = \frac{105b^2}{169ka}[1 + 2(\csc(\xi) \mp \cot(\xi))^{-2} + (\csc(\xi) \mp \cot(\xi))^{-4}], \quad (28)$$

where $\xi = \sqrt{\frac{-b}{13k}}(x - \frac{36b^2t}{169k}),\ \frac{b}{k} < 0.$

$$u_{17}(x,t) = \frac{b^2}{169ka}[33 + 210(\sec(\xi) \pm \tan(\xi))^{-2} + 105(\sec(\xi) \pm \tan(\xi))^{-4}], \quad (29)$$

$$u_{18}(x,t) = \frac{b^2}{169ka}[33 + 210(\csc(\xi) \mp \cot(\xi))^{-2} + 105(\csc(\xi) \mp \cot(\xi))^{-4}], \quad (30)$$

where $\xi = \sqrt{\frac{-b}{13k}}(x + \frac{36b^2t}{169k}),\ \frac{b}{k} < 0$.
Case (4): Let $\alpha = 1$, $\gamma = -1$ and $\beta = 0$. By solving (11) and Substituting it's solutions and $Y = \tanh(\xi)$ or $Y = \coth(\xi)$, in (10), we obtain

$$u_{19}(x,t) = \frac{105b^2}{169ka}[1 - 2\tanh^2(\xi) + \tanh^4(\xi)], \tag{31}$$

where $\xi = \frac{1}{2}\sqrt{\frac{b}{13k}}(x - \frac{36b^2t}{169k}),\ \frac{b}{k} > 0$.

$$u_{20}(x,t) = \frac{3b^2}{169ka}[11 - 70\tanh^2(\xi) + 35\tanh^4(\xi)], \tag{32}$$

where $\xi = \frac{1}{2}\sqrt{\frac{b}{13k}}(x + \frac{36b^2t}{169k}),\ \frac{b}{k} > 0$.

$$u_{21}(x,t) = \frac{105b^2}{169ka}[1 - 2\coth^2(\xi) + \coth^4(\xi)], \tag{33}$$

where $\xi = \frac{1}{2}\sqrt{\frac{b}{13k}}(x - \frac{36b^2t}{169k}),\ \frac{b}{k} > 0$.

$$u_{22}(x,t) = \frac{3b^2}{169ka}[11 - 70\coth^2(\xi) + 35\coth^4(\xi], \tag{34}$$

where $\xi = \frac{1}{2}\sqrt{\frac{b}{13k}}(x + \frac{36b^2t}{169k}),\ \frac{b}{k} > 0$.
The solutions (32) and (34) are same Eq. (33) and Eq. (34) in [24] respectively.
Case(5): By considering $\alpha = \gamma = \pm 1$, and $\beta = 0$ in (11) and solving the resulting system, we obtain unknown variables. By Substituting these values and $Y = \tan(\xi)$ or $Y = \cot(\xi)$, in (10) we derive

$$u_{23}(x,t) = \frac{105b^2}{169ka}(1 + 2\tan^2(\xi) + \tan^4(\xi)), \tag{35}$$

where $\xi = \frac{1}{2}\sqrt{\frac{-b}{13k}}(x - \frac{36b^2t}{169k}),\ \frac{b}{k} < 0$.

$$u_{24}(x,t) = \frac{3b^2}{169ka}(11 + 70\tan^2(\xi) + 35\tan^4(\xi)), \tag{36}$$

where $\xi = \frac{1}{2}\sqrt{\frac{-b}{13k}}(x + \frac{36b^2t}{169k}),\ \frac{b}{k} < 0$.

$$u_{25}(x,t) = \frac{105b^2}{169ka}(1 + 2\cot^2(\xi) + \cot^4(\xi)), \tag{37}$$

where $\xi = \frac{1}{2}\sqrt{\frac{-b}{13k}}(x - \frac{36b^2t}{169k})$, $\frac{b}{k} < 0$.

$$u_{26}(x,t) = \frac{3b^2}{169ka}(11 + 70\cot^2(\xi) + 35\cot^4(\xi)). \tag{38}$$

where $\xi = \frac{1}{2}\sqrt{\frac{-b}{13k}}(x + \frac{36b^2t}{169k})$, $\frac{b}{k} < 0$.

The solutions (36) and (38) are same Eq. (35) and Eq. (36) in [24] respectively.

Case (6): By letting $\alpha = 1$, $\gamma = -4$, and $\beta = 0$ in (11) and solving the resulting system and Substituting it's solutions and $Y = \frac{\tanh(\xi)}{1+\tanh^2(\xi)}$, in (10), after some simplifications, we get

$$u_{27}(x,t) = \frac{105b^2}{169ka}\frac{(1 - 4\tanh^2(\xi) + 6\tanh^4(\xi) - 4\tanh^6(\xi) + \tanh^8(\xi))}{(1 + \tanh^2(\xi))^4}, \tag{39}$$

where $\xi = \frac{1}{4}\sqrt{\frac{b}{13k}}(x - \frac{36b^2t}{169k})$, $\frac{b}{k} > 0$.

$$u_{28}(x,t) = \frac{3b^2}{169ka}\frac{(11 - 236\tanh^2(\xi) + 66\tanh^4(\xi) - 236\tanh^6(\xi) + 11\tanh^8(\xi))}{(1 + \tanh^2(\xi))^4}, \tag{40}$$

where $\xi = \frac{1}{4}\sqrt{\frac{b}{13k}}(x + \frac{36b^2t}{169k})$, $\frac{b}{k} > 0$.

$$u_{29}(x,t) = \frac{105b^2}{2704ka}(6 - 4\coth^2(\xi) - 4\tanh^2(\xi) + \coth^4(\xi) + \tanh^4(\xi)), \tag{41}$$

where $\xi = \frac{1}{4}\sqrt{\frac{b}{13k}}(x - \frac{36b^2t}{169k})$, $\frac{b}{k} > 0$.

$$u_{30}(x,t) = \frac{3b^2}{2704ka}(-174 - 140\coth^2(\xi) - 140\tanh^2(\xi) + 35\coth^4(\xi) + 35\tanh^4(\xi)). \tag{42}$$

where $\xi = \frac{1}{4}\sqrt{\frac{b}{13k}}(x + \frac{36b^2t}{169k})$, $\frac{b}{k} > 0$.

The solutions (41) and (42) are same Eq. (44) and Eq. (47) in [24] respectively.

Case (7): By solving (11) for $\alpha = 1$, $\gamma = 4$, and $\beta = 0$ we obtain unknown variables. By Substituting these values and $Y = \frac{\tan(\xi)}{1-\tan^2(\xi)}$, in (10), after some simplifications, we get

$$u_{31}(x,t) = \frac{105b^2}{169ka}\frac{(1 + 4\tan^2(\xi) + 6\tan^4(\xi) + 4\tan^6(\xi) + \tan^8(\xi))}{(-1 + \tan^2(\xi))^4}, \tag{43}$$

where $\xi = \frac{1}{4}\sqrt{\frac{-b}{13k}}(x - \frac{36b^2t}{169k})$, $\frac{b}{k} < 0$.

$$u_{32}(x,t) = \frac{3b^2}{169ka}\frac{(11 + 236\tan^2(\xi) + 66\tan^4(\xi) + 236\tan^6(\xi) + 11\tan^8(\xi))}{(-1 + \tan^2(\xi))^4}, \tag{44}$$

where $\xi = \frac{1}{4}\sqrt{\frac{-b}{13k}}(x + \frac{36b^2t}{169k})$, $\frac{b}{k} < 0$.

$$u_{33}(x,t) = \frac{105b^2}{2704ka}(6 + 4\cot^2(\xi) + 4\tan^2(\xi) + \cot^4(\xi) + \tan^4(\xi)), \tag{45}$$

where $\xi = \frac{1}{4}\sqrt{\frac{-b}{13k}}(x - \frac{36b^2t}{169k})$, $\frac{b}{k} < 0$.

$$u_{34}(x,t) = \frac{3b^2}{2704ka}(-174 + 140\cot^2(\xi) + 140\tan^2(\xi) + 35\cot^4(\xi) + 35\tan^4(\xi)). \tag{46}$$

where $\xi = \frac{1}{4}\sqrt{\frac{-b}{13k}}(x + \frac{36b^2t}{169k})$, $\frac{b}{k} < 0$.
The solutions (45) and (46) are same Eq. (45) and Eq. (48) in [24] respectively.
Case (8): By letting $\alpha = -1$, $\gamma = -4$ and $\beta = 0$ in (11) and solving the resulting system and Substituting it's solutions and $Y = \frac{\cot(\xi)}{1-cot^2(\xi)}$ in (10) we have

$$u_{35}(x,t) = \frac{105b^2}{169ka}\frac{(1 + 4\cot^2(\xi) + 6\cot^4(\xi) + 4\cot^6(\xi) + \cot^8(\xi))}{(-1 + \cot^2(\xi))^4}, \tag{47}$$

where $\xi = \frac{1}{4}\sqrt{\frac{-b}{13k}}(x - \frac{36b^2t}{169k})$, $\frac{b}{k} < 0$.

$$u_{36}(x,t) = \frac{3b^2}{169ka}\frac{(11 + 236\cot^2(\xi) + 66\cot^4(\xi) + 236\cot^6(\xi) + 11\cot^8(\xi))}{(-1 + \cot^2(\xi))^4}, \tag{48}$$

where $\xi = \frac{1}{4}\sqrt{\frac{-b}{13k}}(x + \frac{36b^2t}{169k})$, $\frac{b}{k} < 0$.

$$u_{37}(x,t) = \frac{105b^2}{2704ka}(6 + 4\tan^2(\xi) + 4\cot^2(\xi) + \tan^4(\xi) + \cot^4(\xi)), \tag{49}$$

where $\xi = \frac{1}{4}\sqrt{\frac{-b}{13k}}(x - \frac{36b^2t}{169k})$, $\frac{b}{k} < 0$.

$$u_{38}(x,t) = \frac{3b^2}{2704ka}(-174 + 140\tan^2(\xi) + 140\cot^2(\xi) + 35\tan^4(\xi) + 35\cot^4(\xi)). \tag{50}$$

where $\xi = \frac{1}{4}\sqrt{\frac{-b}{13k}}(x + \frac{36b^2t}{169k})$, $\frac{b}{k} < 0$.
The solutions (49) and (50) are same Eq. (45) and Eq. (48) in [24] respectively.
Case (9): By solving (11) for $\alpha = 1$, $\beta = \mp 2$ and $\gamma = 2$ Substituting these values and $Y = \frac{\tan(\xi)}{1 \pm \tan(\xi)}$, in (10) and after some simplifications, we get

$$u_{39}(x,t) = \frac{420b^2}{169ka}\frac{(1 + 2\tan^2(\xi) + \tan^4(\xi))}{(\pm 1 + \tan(\xi))^4}, \tag{51}$$

where $\xi = \frac{1}{2}\sqrt{\frac{-b}{13k}}(x - \frac{36b^2t}{169k})$, $\frac{b}{k} < 0$.

$$u_{40}(x,t) = \frac{12b^2}{169ka}\frac{(29 \mp 24\tan(\xi) + 34\tan^2(\xi) \mp 24\tan^3(\xi) + 29\tan^4(\xi))}{(\pm 1 + \tan(\xi))^4}, \qquad (52)$$

where $\xi = \frac{1}{2}\sqrt{\frac{-b}{13k}}(x + \frac{36b^2t}{169k})$, $\frac{b}{k} < 0$.

$$u_{41}(x,t) = \frac{105b^2}{169ka}(1 + 2\cot^2(\xi) + \cot^4(\xi)), \qquad (53)$$

where $\xi = \frac{1}{2}\sqrt{\frac{-b}{13k}}(x - \frac{36b^2t}{169k})$, $\frac{b}{k} < 0$.

$$u_{42}(x,t) = \frac{3b^2}{169ka}(11 + 70\cot^2(\xi) + 35\cot^4(\xi)). \qquad (54)$$

where $\xi = \frac{1}{2}\sqrt{\frac{-b}{13k}}(x + \frac{36b^2t}{169k})$, $\frac{b}{k} < 0$.
The solution (54) is same Eq. (36) in [24].
Case (10): By assuming $\alpha = -1$, $\beta = \pm 2$ and $\gamma = -2$ in (11) and solving the obtained system, then Substituting it's solutions and $Y = \frac{\cot(\xi)}{1 \pm \cot(\xi)}$ in (10), after some simplifications, we obtain

$$u_{43}(x,t) = \frac{420b^2}{169ka}\frac{(1 + 2\cot^2(\xi) + \cot^4(\xi))}{(\pm 1 + \cot(\xi))^4}, \qquad (55)$$

where $\xi = \frac{1}{2}\sqrt{\frac{-b}{13k}}(x - \frac{36b^2t}{169k})$, $\frac{b}{k} < 0$.

$$u_{44}(x,t) = \frac{12b^2}{169ka}\frac{(29 \mp 24\cot(\xi) + 34\cot^2(\xi) \mp 24\cot^3(\xi) + 29\cot^4(\xi))}{(\pm 1 + \cot(\xi))^4}, \qquad (56)$$

where $\xi = \frac{1}{2}\sqrt{\frac{-b}{13k}}(x + \frac{36b^2t}{169k})$, $\frac{b}{k} < 0$.

$$u_{45}(x,t) = \frac{105b^2}{169ka}(1 + 2\tan^2(\xi) + \tan^4(\xi)), \qquad (57)$$

where $\xi = \frac{1}{2}\sqrt{\frac{-b}{13k}}(x - \frac{36b^2t}{169k})$, $\frac{b}{k} < 0$.

$$u_{46}(x,t) = \frac{3b^2}{169ka}(11 + 70\tan^2(\xi) + 35\tan^4(\xi)). \qquad (58)$$

where $\xi = \frac{1}{2}\sqrt{\frac{-b}{13k}}(x + \frac{36b^2t}{169k})$, $\frac{b}{k} < 0$.
The solution (58) is same Eq. (35) in [24].
Not only our solutions cover all results obtained by wazwaz in [24], but also other new solutions appear.

4.The modified Kawahara equation

Let us consider the modified Kawahara equation

$$u_t + au^2u_x + bu_{xxx} - ku_{xxxxx} = 0, \tag{59}$$

where a, b and k are nonzero real constants. In order to obtain traveling wave solutions for Eq. (59), we use

$$u(x,t) = u(\xi), \quad \xi = \mu(x - ct), \tag{60}$$

Substituting (60) into (59), we obtain

$$-cu' + au^2u' + b\mu^2u''' - k\mu^4u^{(5)} = 0, \tag{61}$$

by once time integrating we find

$$-cu + \frac{a}{3}u^3 + b\mu^2u'' - k\mu^4u^{(4)} = 0. \tag{62}$$

Balancing the order of the nonlinear term u^3 with the highest order linear term $u^{(4)}$ in (62), we obtain N=2.
Thus, the solution of (3) has the form

$$u(\xi) = a_0 + a_1Y + a_2Y^2 + b_1Y^{-1} + b_2Y^{-2}. \tag{63}$$

Substituting (63) into (62) and using the Riccati equation (4), and because all coefficients of Y^i have to vanish, we obtain a system of algebraic equations in the unknowns $a_0, a_1, a_2, b_1, b_2, \alpha, \beta, \gamma, \mu$ and c of the following form:

$$\frac{1}{3}ab_2{}^3 - 120k\mu^4b_2\alpha^4 = 0,$$
$$ab_1b_2{}^2 - 24k\mu^4b_1\alpha^4 - 336k\mu^4b_2\beta\alpha^3 = 0,$$
$$ab_1{}^2b_2 + ab_2{}^2a_0 + 6b\mu^2b_2\alpha^2 - 60k\mu^4b_1\beta\alpha^3 - 330k\mu^4b_2\beta^2\alpha^2 - 240k\mu^4b_2\gamma\alpha^3 = 0,$$
$$\frac{1}{3}ab_1{}^3 + aa_1b_2{}^2 + 2ab_1b_2a_0 + 10b\mu^2b_2\beta\alpha + 2b\mu^2b_1\alpha^2 - 50k\mu^4b_1\beta^2\alpha^2 - 40k\mu^4b_1\gamma\alpha^3$$
$$-130k\mu^4b_2\beta^3\alpha - 440k\mu^4b_2\beta\gamma\alpha^2 = 0,$$
$$aa_2b_2{}^2 + ab_1{}^2a_0 + ab_2a_0{}^2 - cb_2 + 2aa_1b_1b_2 + 3b\mu^2b_1\beta\alpha + 8b\mu^2b_2\gamma\alpha + 4b\mu^2b_2\beta^2$$
$$-16k\mu^4b_2\beta^4 - 15k\mu^4b_1\beta^3\alpha - 60k\mu^4b_1\beta\gamma\alpha^2 - 136k\mu^4b_2\gamma^2\alpha^2 - 232k\mu^4b_2\beta^2\gamma\alpha = 0,$$
$$aa_1b_1{}^2 + ab_1a_0{}^2 - cb_1 + 2aa_1b_2a_0 + 2aa_2b_1b_2 + 2b\mu^2b_1\gamma\alpha + 6b\mu^2b_2\beta\gamma + b\mu^2b_1\beta^2$$
$$-16k\mu^4b_1\gamma^2\alpha^2 - 22k\mu^4b_1\beta^2\gamma\alpha - k\mu^4b_1\beta^4 - 30k\mu^4b_2\beta^3\gamma - 120k\mu^4b_2\beta\gamma^2\alpha = 0,$$

$$-ca_0+\frac{1}{3}aa_0{}^3+2aa_1b_1a_0+2aa_2b_2a_0+aa_2b_1{}^2+aa_1{}^2b_2+2b\mu^2a_2\alpha^2+2b\mu^2b_2\gamma^2$$
$$+b\mu^2b_1\beta\gamma+b\mu^2a_1\beta\alpha-k\mu^4a_1\beta^3\alpha-14k\mu^4a_2\beta^2\alpha^2-16k\mu^4a_2\gamma\alpha^3-8k\mu^4a_1\beta\gamma\alpha^2$$
$$-k\mu^4b_1\beta^3\gamma-8k\mu^4b_1\beta\gamma^2\alpha-14k\mu^4b_2\beta^2\gamma^2-16k\mu^4b_2\gamma^3\alpha=0, \qquad (64)$$
$$aa_1{}^2b_1+aa_1a_0{}^2-ca_1+2aa_1a_2b_2+2aa_2b_1a_0+2b\mu^2a_1\gamma\alpha+6b\mu^2a_2\beta\alpha+b\mu^2a_1\beta^2$$
$$-22k\mu^4a_1\beta^2\gamma\alpha-k\mu^4a_1\beta^4-16k\mu^4a_1\gamma^2\alpha^2-30k\mu^4a_2\beta^3\alpha-120k\mu^4a_2\gamma\alpha^2\beta=0,$$
$$aa_1{}^2a_0+aa_2{}^2b_2+aa_2a_0{}^2-ca_2+2aa_1a_2b_1+3b\mu^2a_1\beta\gamma+8b\mu^2a_2\gamma\alpha+4b\mu^2a_2\beta^2$$
$$-15k\mu^4a_1\beta^3\gamma-16k\mu^4a_2\beta^4-60k\mu^4a_1\beta\gamma^2\alpha-136k\mu^4a_2\gamma^2\alpha^2-232k\mu^4a_2\beta^2\gamma\alpha=0,$$
$$aa_2{}^2b_1+\frac{1}{3}aa_1{}^3+2aa_1a_2a_0+10b\mu^2a_2\beta\gamma+2b\mu^2a_1\gamma^2-50k\mu^4a_1\beta^2\gamma^2-40k\mu^4a_1\gamma^3\alpha$$
$$-130k\mu^4a_2\beta^3\gamma-440k\mu^4a_2\beta\gamma^2\alpha=0,$$
$$aa_1{}^2a_2+aa_2{}^2a_0+6b\mu^2a_2\gamma^2-60k\mu^4a_1\beta\gamma^3-330k\mu^4a_2\beta^2\gamma^2-240k\mu^4a_2\gamma^3\alpha=0,$$
$$aa_1a_2{}^2-24k\mu^4a_1\gamma^4-336k\mu^4a_2\beta\gamma^3=0,$$
$$\frac{1}{3}aa_2{}^3-120k\mu^4a_2\gamma^4=0.$$

Case (1): By assuming $\alpha=\beta=1$ and $\gamma=0$ in (64), and solving the resulting system, we obtain

- $a_0=a_1=a_2=0,\ \ b_1=b_2=\mp\dfrac{12b}{\sqrt{10ka}},\ \ c=\dfrac{4b^2}{25k},\ \ \mu=\pm\sqrt{\dfrac{b}{5k}},\ \ \dfrac{b}{k}>0.$

Substituting these values and $Y=e^{\xi}-1$ in (63), after some simplifications, we obtain

$$u_1(x,t)=\mp\frac{12b}{\sqrt{10ka}}\frac{e^{\pm\sqrt{\frac{b}{5k}}(x-\frac{4b^2}{25k}t)}}{(e^{\pm\sqrt{\frac{b}{5k}}(x-\frac{4b^2}{25k}t)}-1)^2}. \qquad (65)$$

Case (2): By considering $\alpha=\frac{1}{2}$, $\gamma=-\frac{1}{2}$, and $\beta=0$ in (64) and solving the obtained system and Substituting it's solution and $Y=\coth\xi\pm csch\xi$ or $Y=\tanh\xi\pm isech\xi$ in (63), after some simplifications, we have

$$u_2(x,t)=\mp\frac{6b}{\sqrt{10ka}}\frac{(1\pm\cosh(\mu(x-\frac{4b^2}{25k}t)))}{\sinh^2(\mu(x-\frac{4b^2}{25k}t))}, \qquad (66)$$

$$u_3(x,t)=\pm\frac{6b}{\sqrt{10ka}}\frac{(1\mp i\sinh(\mu(x-\frac{4b^2}{25k}t)))}{\cosh^2(\mu(x-\frac{4b^2}{25k}t))}, \qquad (67)$$

$$u_4(x,t) = \pm\frac{6b}{\sqrt{10ka}}\frac{1}{(1 \pm \cosh(\mu(x - \frac{4b^2}{25k}t)))}, \quad (68)$$

$$u_5(x,t) = \pm\frac{6b}{\sqrt{10ka}}\frac{(i\sinh(\mu(x - \frac{4b^2}{25k}t)) \mp 1)}{(2i\sinh(\mu(x - \frac{4b^2}{25k}t)) \mp 2 \pm \cosh^2(\mu(x - \frac{4b^2}{25k}t)))}, \quad (69)$$

where $\mu = \sqrt{\frac{b}{5k}}$, $\frac{b}{k} > 0$.

Case (3): by solving (64) for $\alpha = \gamma = \pm\frac{1}{2}$, and $\beta = 0$ we obtain unknown variables. By substituting these values and $Y = \sec\xi \pm tan\xi$ or $Y = \csc\xi \mp \cot\xi$ in (63), after some simplifications, we derive

$$u_6(x,t) = \pm\frac{6b}{\sqrt{10ka}}\frac{1}{(1 \mp \sin(\mu(x - \frac{4b^2}{25k}t)))}, \quad (70)$$

$$u_7(x,t) = \pm\frac{6b}{\sqrt{10ka}}\frac{1}{(1 \pm \cos(\mu(x - \frac{4b^2}{25k}t)))}, \quad (71)$$

where $\mu = \sqrt{\frac{-b}{5k}}$, $\frac{b}{k} < 0$.

The solution (69) is same Eq. (43) in [25].

Case (4): By letting $\alpha = 1$, $\gamma = -1$, and $\beta = 0$ in (64) and solving the resulting system and Substituting it's solution and $Y = \tanh\xi$ or $Y = \coth(\xi)$ in (63), after some straightforward computations, we obtain

$$u_8(x,t) = \pm\frac{3b}{\sqrt{10ka}}sech^2(\mu(x - \frac{4b^2}{25k}t)), \quad (72)$$

$$u_9(x,t) = \mp\frac{3b}{\sqrt{10ka}}csch^2(\mu(x - \frac{4b^2}{25k}t)), \quad (73)$$

where $\mu = \frac{1}{2}\sqrt{\frac{b}{5k}}$, $\frac{b}{k} > 0$.

The solution (71) and (72) are same Eq. (23) and Eq. (30) in [25] respectively.

Case (5): By assuming $\alpha = \gamma = \pm 1$, and $\beta = 0$ in (64) and solving the resulting system and Substituting obtained values and $Y = \tan(\xi)$ and $Y = \cot(\xi)$ in (63), after some simplifications, we get

$$u_{10}(x,t) = \pm\frac{3b}{\sqrt{10ka}}sec^2(\mu(x - \frac{4b^2}{25k}t)), \quad (74)$$

$$u_{11}(x,t) = \pm\frac{3b}{\sqrt{10ka}}csc^2(\mu(x - \frac{4b^2}{25k}t)), \quad (75)$$

where $\mu = \frac{1}{2}\sqrt{\frac{-b}{5k}}$, $\frac{b}{k} < 0$.

The solution (73) and (74) are same Eq. (21) and Eq. (32) in [25] respectively.

Case (6): By solving (64) for $\alpha = 1$, $\gamma = -4$, and $\beta = 0$ in (64) we derive unknown variables. By Substituting obtained results and $Y = \frac{\tanh(\xi)}{1+\tanh^2(\xi)}$ in (63), after some straightforward computations, we obtain

$$u_{12}(x,t) = \pm\frac{3b}{\sqrt{10ka}}\frac{1-2\tanh^2(\mu(x-\frac{4b^2}{25k}t))+\tanh^4(\mu(x-\frac{4b^2}{25k}t))}{(1+\tanh^2(\mu(x-\frac{4b^2}{25k}t)))^2}, \quad (76)$$

$$u_{13}(x,t) = \mp\frac{3b}{4\sqrt{10ka}}(-2+\coth^2(\mu(x-\frac{4b^2}{25k}t))+\tanh^2(\mu(x-\frac{4b^2}{25k}t))), \quad (77)$$

where $\mu = \frac{1}{4}\sqrt{\frac{b}{5k}}$, $\frac{b}{k} > 0$.
The solution (76) is same Eq. (38) in [25].
Case (7): By assuming $\alpha = 1$, $\gamma = 4$, and $\beta = 0$ in (64) and solving the resulting system and Substituting it's solutions and $Y = \frac{\tan(\xi)}{1-\tan^2(\xi)}$ in (63), after some simplifications, we obtain

$$u_{14}(x,t) = \pm\frac{3b}{\sqrt{10ka}}\frac{1+2\tan^2(\mu(x-\frac{4b^2}{25k}t))+\tan^4(\mu(x-\frac{4b^2}{25k}t))}{(\tan^2(\mu(x-\frac{4b^2}{25k}t))-1)^2}, \quad (78)$$

$$u_{15}(x,t) = \pm\frac{3b}{4\sqrt{10ka}}(2+\cot^2(\mu(x-\frac{4b^2}{25k}t))+\tan^2(\mu(x-\frac{4b^2}{25k}t))), \quad (79)$$

where $\mu = \frac{1}{4}\sqrt{\frac{-b}{5k}}$, $\frac{b}{k} < 0$.
The solution (78) is same Eq. (39) in [25].
Case (8): By considering $\alpha = -1$, $\gamma = -4$, and $\beta = 0$ in (64) and solving the obtained system and Substituting it's solutions and $Y = \frac{\cot(\xi)}{1-\cot^2(\xi)}$ in (63) after some straightforward computations, we get

$$u_{16}(x,t) = \pm\frac{3b}{\sqrt{10ka}}\frac{1+2\cot^2(\mu(x-\frac{4b^2}{25k}t))+\cot^4(\mu(x-\frac{4b^2}{25k}t))}{(\cot^2(\mu(x-\frac{4b^2}{25k}t))-1)^2}, \quad (80)$$

$$u_{17}(x,t) = \pm\frac{3b}{4\sqrt{10ka}}(2+\tan^2(\mu(x-\frac{4b^2}{25k}t))+\cot^2(\mu(x-\frac{4b^2}{25k}t))), \quad (81)$$

where $\mu = \frac{1}{4}\sqrt{\frac{-b}{5k}}$, $\frac{b}{k} < 0$.
The solution (80) is same Eq. (39) in [25].
Case (9): By solving (64) for $\alpha = 1$, $\gamma = 2$, and $\beta = -2$ we obtain unknown variables. By Substituting these values and $Y = \frac{\tan(\xi)}{1+\tan(\xi)}$ in (63), after some simplifications,

we obtain

$$u_{18}(x,t) = \pm\frac{6b}{\sqrt{10ka}}\frac{(1+\tan^2(\mu(x-\frac{4b^2}{25k}t)))}{(1+\tan(\mu(x-\frac{4b^2}{25k}t)))^2}, \tag{82}$$

$$u_{19}(x,t) = \pm\frac{3b}{\sqrt{10ka}}\csc^2(\mu(x-\frac{4b^2}{25k}t)), \tag{83}$$

where $\mu = \frac{1}{2}\sqrt{\frac{-b}{5k}}$, $\frac{b}{k} < 0$.
The solution (82) is same Eq. (22) and Eq. (32) in [25].
Case (10): By letting $\alpha = 1$, $\gamma = 2$, and $\beta = 2$ in (64) and solving the resulting system and Substituting it's solutions and $Y = \frac{\tan(\xi)}{1-\tan(\xi)}$ in (63), after some simplifications, we obtain

$$u_{20}(x,t) = \pm\frac{6b}{\sqrt{10ka}}\frac{(1+\tan^2(\mu(x-\frac{4b^2}{25k}t)))}{(\tan(\mu(x-\frac{4b^2}{25k}t))-1)^2}, \tag{84}$$

$$u_{21}(x,t) = \pm\frac{3b}{\sqrt{10ka}}\csc^2(\mu(x-\frac{4b^2}{25k}t)), \tag{85}$$

where $\mu = \frac{1}{2}\sqrt{\frac{-b}{5k}}$, $\frac{b}{k} < 0$.
The solution (84) is same Eq. (22) and Eq. (32) in [25].
Case (11): By considering $\alpha = -1$, $\gamma = -2$, and $\beta = 2$ in (64) and solving the obtained system and Substituting it's solutions and $Y = \frac{\cot(\xi)}{1+\cot(\xi)}$ in (63), after some straightforward computations, we obtain

$$u_{22}(x,t) = \pm\frac{6b}{\sqrt{10ka}}\frac{(1+\cot^2(\mu(x-\frac{4b^2}{25k}t)))}{(1+\cot(\mu(x-\frac{4b^2}{25k}t)))^2}, \tag{86}$$

$$u_{23}(x,t) = \pm\frac{3b}{\sqrt{10ka}}\sec^2(\mu(x-\frac{4b^2}{25k}t)), \tag{87}$$

where $\mu = \frac{1}{2}\sqrt{\frac{-b}{5k}}$, $\frac{b}{k} < 0$.
The solution (86) is same Eq. (21) and Eq. (31) in [25].
Case (12): By solving (64) for $\alpha = -1$, $\gamma = -2$, and $\beta = -2$ we obtain unknown variable. By Substituting resulting solutions and $Y = \frac{\cot(\xi)}{1-\cot(\xi)}$ in (63), after some simplifications, we get

$$u_{24}(x,t) = \pm\frac{6b}{\sqrt{10ka}}\frac{(1+\cot^2(\mu(x-\frac{4b^2}{25k}t)))}{(\cot(\mu(x-\frac{4b^2}{25k}t))-1)^2}, \tag{88}$$

$$u_{25}(x,t) = \pm\frac{3b}{\sqrt{10ka}}\sec^2(\mu(x-\frac{4b^2}{25k}t)), \tag{89}$$

where $\mu = \frac{1}{2}\sqrt{\frac{-b}{5k}}$. $\frac{b}{k} < 0$.
The solution (88) is same Eq. (21) and Eq. (31) in [25].
Not only our solutions for the modified Kawahara equation (59) cover all results obtained by Wazwaz in [25], but also other solutions appear.

5.Conclusions

In this article, the modified tanh-coth method has been successfully implemented to find new traveling wave solutions for two nonlinear PDEs, namely, the Kawahara and the modified Kawahara equations. The results show that this method is a powerful Mathematical tool for obtaining exact solutions for the Kawahara and modified Kawahara equations. It is also a promising method to solve other nonlinear partial differential equations.

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Azizeh Jabbari and Hossein Kheiri
Faculty of mathematical sciences
University of Tabriz
Tabriz
Iran
email:*jabbari.azizeh@gmail.com, h-kheiri@tabrizu.ac, kheirihossein@yahoo.com*