Acta Universitatis Apulensis No. 23/2010
ISSN: 1582-5329 pp. 79-89

SOME PROPERTIES FOR CERTAIN INTEGRAL OPERATORS

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Abstract. Recently Breaz and Breaz [4] and Breaz et.al[5] introduced two general integral operators F_n and $F_{\alpha_1,\dots,\alpha_n}$. Considering the classes $\mathcal{N}(\gamma)$, $\mathcal{MT}(\mu,\beta)$ and $KD(\mu,\beta)$ we derived some properties for F_n and $F_{\alpha_1,\dots,\alpha_n}$. Two new subclasses $KDF_n(\mu,\beta,\alpha_1,...,\alpha_n)$ and $KDF_{\alpha_1,\dots,\alpha_n}(\mu,\beta,\alpha_1,...,\alpha_n)$ are defined. Necessary and sufficient conditions for a family of functions f_j to be in the $KDF_n(\mu,\beta,\alpha_1,...,\alpha_n)$ and $KDF_{\alpha_1,\dots,\alpha_n}(\mu,\beta,\alpha_1,...,\alpha_n)$ are determined.

AMS Subject Classification: 30C45

Keywords and phrases: Analytic and univalent functions, convex functions, integral operators.

1. Introduction.

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unite disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. We also denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathcal{U}$. Furthermore, we denote by $\mathcal{T}$ the subclass of $\mathcal{S}$ consisting of functions whose nonzero coefficients, from the second one, are negative and has the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0.$$

A function $f \in \mathcal{A}$ is the convex function of order α, $0 \leq \alpha < 1$, if f satisfies the following inequality

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha,\ z \in \mathcal{U}$$

and we denote this class by $\mathcal{K}(\alpha)$.

Similarly, if $f \in \mathcal{A}$ satisfies the following inequality:

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha,\ z \in \mathcal{U}$$

for some $\alpha,\ 0 \leq \alpha < 1$, then f is said to be starlike of order α and we denote this class by $\mathcal{S}^*(\alpha)$.

Let $\mathcal{N}(\gamma)$ be the subclass of $\mathcal{A}$ consisting of the functions f which satisfy the inequality

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) < \gamma,\ z \in \mathcal{U},\ \gamma > 1.$$

This class was studied by Owa and Srivastava [8].

Let $\mathcal{MT}(\mu, \beta)$ be the subclass of $\mathcal{A}$ consisting of the functions f which satisfy the analytic characterization

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \beta \left|\mu\frac{zf'(z)}{f(z)} + 1\right|$$

for some $0 < \beta \leq 1$, and $0 \leq \mu < 1$,

Definition 1.([9]) *A function f is said to be in the class $KD(\mu, \beta)$, if satisfies the following inequality:*

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) \geq \mu \left|\frac{zf''(z)}{f'(z)}\right| + \beta$$

for some $\mu \geq 0$ and $0 \leq \beta < 1$.

For $f_j(z) \in A$ and $\alpha_j > 0$ for all $j \in \{1, 2, 3, ..., n\}$, D. Breaz and N. Breaz [4] introduced the following integral operator

$$F_n(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{f_j(t)}{t}\right)^{\alpha_j} dt, \tag{1}$$

Recently Breaz et.al [5] introduced the following integral operator

$$F_{\alpha_1,\ldots,\alpha_n}(z) = \int_0^z \prod_{j=1}^{n} \left[f_j'(t)\right]^{\alpha_j} dt, \tag{2}$$

where $f_j \in \mathcal{A}$ and $\alpha_j > 0$, for all $j \in \{1, 2, 3, ..., n\}$.

For univalence, starlike and convexity of these integral operators see ([4]-[7]), see also ([1]-[3]) for several properties.

Now by using the equations (1) and (2) and the Definition 1, we introduce the following two new subclasses of $KD(\mu,\beta)$.

Definition 2. *A family of functions f_j, $j \in \{1,...,n\}$ is said to be in the class $KDF_n(\mu,\beta,\alpha_1,...,\alpha_n)$, if satisfies the inequality:*

$$\mathrm{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right) \geq \mu\left|\frac{zF_n''(z)}{F_n'(z)}\right|+\beta, \tag{3}$$

for some $\mu \geq 0$ and $0 \leq \beta < 1$, where F_n is defined in (1).

Definition 3. *A family of functions f_j, $j \in \{1,...,n\}$ is said to be in the class $KDF_{\alpha_1,...,\alpha_n}(\mu,\beta,\alpha_1,...,\alpha_n)$ if satisfies the inequality:*

$$\mathrm{Re}\left(\frac{zF_{\alpha_1,...,\alpha_n}''(z)}{F_{\alpha_1,...,\alpha_n}'(z)}+1\right) \geq \mu\left|\frac{zF_{\alpha_1,...,\alpha_n}''(z)}{F_{\alpha_1,...,\alpha_n}'(z)}\right|+\beta, \tag{4}$$

for some $\mu \geq 0$ and $0 \leq \beta < 1$, where $F_{\alpha_1,...,\alpha_n}$ is defined as in (2).

2. Main results

Our first result is the following:

Theorem 1. *Let $\alpha_j \in \mathbb{R}$, $\alpha_j > 0$ for $j \in \{1,...,n\}$ and $f_j \in \mathcal{A}$ and suppose that $\left|\frac{f_j'(z)}{f_j(z)}\right| < M_j$. If $f_j \in MT(\mu_j,\beta_j)$ then $F_n \in N(\sigma)$, where $\sigma = \sum_{j=1}^{n} \alpha_j\beta_j(\mu_j M_j+1)+1$.*

Proof. From (1), we observe that $F_n \in A$. On the other hand, it is easy to see that

$$F_n'(z) = \prod_{j=1}^{n}\left(\frac{f_j(z)}{z}\right)^{\alpha_j}. \tag{5}$$

Differentiating (5) logarithmically and multiply by z , we obtain

$$\frac{zF_n''(z)}{F_n'(z)} = \sum_{j=1}^{n}\alpha_j\left[\frac{zf_j'(z)}{f_j(z)}-1\right].$$

Thus we have
$$\frac{zF_n''(z)}{F_n'(z)}+1=\sum_{j=1}^{n}\alpha_j\left[\frac{zf_j'(z)}{f_j(z)}-1\right]+1.$$

We calculate the real part from both terms of the above expression and obtain
$$\mathrm{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)=\sum_{j=1}^{n}\alpha_j\mathrm{Re}\left[\frac{zf_j'(z)}{f_j(z)}-1\right]+1. \tag{6}$$

Since $\Re w\leq |\,w|$, then
$$\mathrm{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)\leq\sum_{j=1}^{n}\alpha_j\left|\frac{zf_j'(z)}{f_j(z)}-1\right|+1.$$

Since $f_j\in\mathcal{M}T(\mu_j,\beta_j)$ for $j\in\{1,...,n\}$, we have
$$\mathrm{Re}\left(\frac{zF_n''(z)}{F_n'(z)}+1\right)\leq\sum_{j=1}^{n}\alpha_j\beta_j\left|\mu_j\frac{zf_j'(z)}{f_j(z)}+1\right|+1\leq\sum_{j=1}^{n}\alpha_j\beta_j\mu_j\left|\frac{f_j'(z)}{f_j(z)}\right|+\sum_{j=1}^{n}\alpha_j\beta_j+1$$
$$<\sum_{j=1}^{n}\alpha_j\beta_j\mu_jM_j+\sum_{j=1}^{n}\alpha_j\beta_j+1=\sum_{j=1}^{n}\alpha_j\beta_j\left(\mu_jM_j+1\right)+1.$$

Hence $F_n\in N(\sigma)$, $\sigma=\sum\limits_{j=1}^{n}\alpha_j\beta_j\left(\mu_jM_j+1\right)+1.$

Letting $n=1,\ \alpha_1=\alpha,\ \alpha_2=...=\alpha_n=0,\ M_1=M$ and $f_1=f$, in the Theorem 1, we have

Corollary 1. *Let $\alpha\in R$,$\alpha>0,\ f\in A$ and suppose that* $\left|\frac{f'(z)}{f(z)}\right|<M$, M fixed. *If $f\in MT(\mu,\beta)$ then* $F_1(z)=\int_0^z\left(\frac{f(t)}{t}\right)^{\alpha}dt\in\mathrm{N}(\sigma)$, $\sigma=\alpha\beta(\mu M+1)+1.$

Letting $\alpha=1$ *in Corollary 1, we have*

Corollary 2. *Let $f\in A$ and suppose that* $\left|\frac{f'(z)}{f(z)}\right|<M$, M fixed. *If $f\in MT(\mu,\beta)$ then* $F_1(z)=\int_0^z\left(\frac{f(t)}{t}\right)dt\in\mathrm{N}(\sigma)$, $\sigma=\beta(\mu M+1)+1.$

Theorem 2. *Let $\alpha_j>0$ for $j\in\{1,...,n\}$, let $\beta_j>0$ be real number with the* *property* $0 \leq \beta_j < 1$ *and let* $f_j \in KD(\mu_j, \beta_j)$ *for* $j \in \{1, ..., n\}$, $\mu_j \geq 0$. *If* $0 < \sum_{j=1}^{n} \alpha_j(1-\beta_j) \leq 1$ *then the functions* $F_{\alpha_1,...,\alpha_n}$ *given by (2) is convex of order* $\rho = 1 - \sum_{j=1}^{n} \alpha_j(1-\beta_j)$.

Proof. From *(2)*, we observe that $F_{\alpha_1,...,\alpha_n} \in A$ and

$$\frac{zF''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)} + 1 = \sum_{j=1}^{n} \alpha_j \left(z\frac{f''_j(z)}{f'_j(z)} + 1 \right) - \sum_{j=1}^{n} +1.$$

We calculate the real part from both terms of the above expression and obtain

$$\text{Re}\left(\frac{zF''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)} + 1 \right) = \sum_{j=1}^{n} \alpha_j \text{Re}\left(z\frac{f''_j(z)}{f'_j(z)} + 1 \right) - \sum_{j=1}^{n} \alpha_j + 1.$$

Since $f_j \in KD(\mu_j, \beta_j)$ for $j = \{1, ..., n\}$, we have

$$\text{Re}\left(\frac{zF''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)} + 1 \right) > \sum_{j=1}^{n} \alpha_j \left(\mu_j \left| z\frac{f''_j(z)}{f'_j(z)} \right| + \beta_j \right) - \sum_{j=1}^{n} \alpha_j + 1.$$

This relation is equivalent to

$$\text{Re}\left(\frac{zF''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)} + 1 \right) > \sum_{j=1}^{n} \alpha_j \mu_j \left| z\frac{f''_j(z)}{f'_j(z)} \right| + \sum_{j=1}^{n} \alpha_j \left(\beta_j - 1 \right) + 1.$$

Since $\alpha_j \mu_j \left| z\frac{f''_j(z)}{f'_j(z)} \right| > 0$ we obtain

$$\text{Re}\left(\frac{zF''_{\alpha_1,...\alpha_n}(z)}{F'_{\alpha_1,...\alpha_n}(z)} + 1 \right) \geq 1 - \sum_{j=1}^{n} \alpha_j(1-\beta_j),$$

which implies that $F_{\alpha_1,...,\alpha_n}$ is convex of order $\rho = 1 - \sum_{j=1}^{n} \alpha_j(1-\beta_j)$.

Letting $n = 1$, $\alpha_1 = \alpha$, $\alpha_2 = ... = \alpha_n = 0$ and $f_1 = f$, in the Theorem 2, we have

Corollary 3. *Let* α *be a real number,* $\alpha > 0$. *Suppose that the function* $f_j \in$

$KD(\mu,\beta)$ and $o \leq \alpha(1-\beta) < 1$. In these conditions the function $F_\alpha(z) = \int\limits_0^z (f'(t))^\alpha \, dt$ is convex of order $1-(1-\beta)\alpha$.

Letting $\alpha = 1$ in Corollary 3, we have

Corollary 4. *Let $f \in KD(\mu,\beta)$ and consider the integral operator $F_1(z) = \int\limits_0^z f'(t)dt$. In this condition F_1 is convex of order β.*

A necessary and sufficient condition for a family of analytic functions $f_j \in KDF_n(\mu,\beta,\alpha_1,...,\alpha_n)$

In this section, we give a necessary and sufficient condition for a family of functions $f_j \in KDF_n(\mu,\beta,\alpha_1,...,\alpha_n)$. Before embarking on the proof of our result, let us calculate the expression $\frac{zF''(z)}{F_n'(z)}$, required for proving our result.

Recall that, from (1), we have

$$F_j'(z) = \prod_{j=1}^{n} \left(\frac{f_j(z)}{z}\right)^{\alpha_j}, \; z \in \mathfrak{U}.$$

After some calculation, we obtain that

$$\frac{F_n''(z)}{F_n'(z)} = \sum_{j=1}^{n} \alpha_j \left(\frac{f_j'(z)}{f_j(z)} - \frac{1}{z}\right),$$

that is equivalent to

$$\frac{zF_n''(z)}{F_n'(z)} = \sum_{j=1}^{n} \alpha_j \left(\frac{zf_j'(z)}{f_j(z)} - 1\right).$$

Let $f_j(z) = z - \sum\limits_{n=2}^{\infty} a_{n,j} z^n$. Then $f_j'(z) = 1 - \sum\limits_{n=2}^{\infty} n a_{n,j} z^{n-1}$ and we get

$$\frac{zF_n''(z)}{F_n'(z)} = \sum_{j=1}^{n} \alpha_j \left[\frac{z - \sum\limits_{n=2}^{\infty} n a_{n,j} z^n}{z - \sum\limits_{n=2}^{\infty} a_{n,j} z^n} - 1\right] =$$

$$= \sum_{j=1}^{n} \alpha_j \left[\frac{1 - \sum\limits_{n=2}^{\infty} n a_{n,j} z^{n-1} - 1 + \sum\limits_{n=2}^{\infty} a_{n,j} z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_{n,j} z^{n-1}}\right] = -\sum_{j=1}^{n} \alpha_j \left[\frac{\sum\limits_{n=2}^{\infty} (n-1) a_{n,j} z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_{n,j} z^{n-1}}\right]. \tag{7}$$

Theorem 3. *Let the function $f_j \in \mathcal{T}$ for $j \in \{1,...,n\}$. Then the functions $f_j \in KDF_n(\mu,\beta,\alpha_1,...,\alpha_n)$ for $j \in \{1,...,n\}$ if and only if*

$$\sum_{j=1}^{n}\left[\frac{\sum\limits_{n=2}^{\infty}\alpha_j\,(n-1)\,(\mu+1)\,a_{n,j}}{1-\sum\limits_{n=2}^{\infty}a_{n,j}}\right]\leq 1-\beta. \tag{8}$$

Proof. First consider

$$\mu\left|\frac{zF_n''(z)}{F_n'(z)}\right|-\mathrm{Re}\left(1+\frac{zF_n''(z)}{F_n'(z)}\right)\leq(\mu+1)\left|\frac{zF_n''(z)}{F_n'(z)}\right|.$$

From (7) we obtain

$$(\mu+1)\left|\frac{zF_n''(z)}{F_n'(z)}\right|=(\mu+1)\left|\sum_{n=2}^{\infty}\alpha_j\left[\frac{\sum\limits_{n=2}^{\infty}(n-1)\,a_{n,j}z^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_{n,j}z^{n-1}}\right]\right|\leq$$

$$\leq(\mu+1)\sum_{j=1}^{n}\left[\frac{\sum\limits_{n=2}^{\infty}\alpha_j\,(n-1)\,|a_{n,j}|\,|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}|a_{n,j}|\,|z|^{n-1}}\right]\leq(\mu+1)\sum_{j=1}^{n}\left[\frac{\sum\limits_{n=2}^{\infty}\alpha_j\,(n-1)\,a_{n,j}}{1-\sum\limits_{n=2}^{\infty}a_{n,j}}\right].$$

If (8) holds then the above expression is bounded by $1-\beta$, and consequently

$$\mu\left|\frac{zF_n''(z)}{F_n'(z)}\right|-\mathrm{Re}\left(1+\frac{zF_n''(z)}{F_n'(z)}\right)<-\beta,$$

which equivalent to

$$\mathrm{Re}\left(1+\frac{zF_n''(z)}{F_n'(z)}\right)\geq\mu\left|\frac{zF_n''(z)}{F_n'(z)}\right|+\beta.$$

Hence $f_j \in KDF_n(\mu,\beta,\alpha_1,...,\alpha_n)$ *for* $j \in \{1,...,n\}$.

Conversely. Let $f_j \in KDF_n(\mu,\beta,\alpha_1,...,\alpha_n)$ *for* $j \in \{1,...,n\}$ *and prove that (8) holds. If* $f_j \in KDF_n(\mu,\beta,\alpha_1,...,\alpha_n)$ *for* $j \in \{1,...,n\}$ and z is real, we get from (3) and (7)

$$1-\sum_{j=1}^{n}\alpha_j\left[\frac{\sum\limits_{n=2}^{\infty}(n-1)\,a_{n,j}z^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_{n,j}z^{n-1}}\right]\geq\mu\left|\sum_{j=1}^{\infty}\alpha_j\left[\frac{\sum\limits_{n=2}^{\infty}(n-1)\,a_{n,j}z^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_{n,j}z^{n-1}}\right]\right|+\beta\geq$$

$$\mu \sum_{j=1}^{n} \alpha_j \left[\frac{\sum\limits_{n=2}^{\infty} (n-1)\, a_{n,j} z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_{n,j} z^{n-1}} \right] + \beta.$$

That is equivalent to

$$\sum_{j=1}^{n} \left[\frac{\sum\limits_{n=2}^{\infty} \alpha_j \mu\, (n-1)\, a_{n,j} z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_{n,j} z^{n-1}} \right] + \sum_{j=1}^{n} \left[\frac{\sum\limits_{n=2}^{\infty} \alpha_j\, (n-1)\, a_{n,j} z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_{n,j} z^{n-1}} \right] \leq 1 - \beta.$$

The above inequality reduce to

$$\sum_{j=1}^{n} \left[\frac{\sum\limits_{n=2}^{\infty} \alpha_j (\mu+1)\, (n-1)\, a_{n,j} z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_{n,j} z^{n-1}} \right] \leq 1 - \beta.$$

Let $z \to 1^-$ along the real axis, then we get

$$\sum_{j=1}^{n} \left[\frac{\sum\limits_{n=2}^{\infty} \alpha_j (\mu+1)\, (n-1)\, a_{n,j}}{1 - \sum\limits_{n=2}^{\infty} a_{n,j}} \right] \leq 1 - \beta.$$

Which give the required result.

A necessary and sufficient condition for a family of analytic functions $f_j \in KDF_{\alpha_1,\ldots,\alpha_n}(\mu, \beta, \alpha_1, ..., \alpha_n)$

In this section, we give a necessary and sufficient condition for a family of analytic functions $f_j \in KDF_{\alpha_1,\ldots,\alpha_n}(\mu, \beta, \alpha_1, ..., \alpha_n)$. Let us first calculate the expression, $\frac{zF''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)}$, required for proving our result. From (2) we obtain

$$F'_{\alpha_1,\ldots,\alpha_n}(z) = \prod_{j=1}^{n} \left[f'_j(z) \right]^{\alpha_j}.$$

After some calculus, we obtain

$$\frac{zF''_{\alpha_1,\ldots,\alpha_n}(z)}{F'_{\alpha_1,\ldots,\alpha_n}(z)} = \sum_{j=1}^{n} \alpha_j \frac{zf'_j(z)}{f'_j(z)}.$$

Let $f_j(z) = z - \sum\limits_{n=2}^{\infty} a_{n,j} z^n$, $f_j'(z) = 1 - \sum\limits_{n=2}^{\infty} n a_{n,j} z^{n-1}$ and $f_j''(z) = -\sum\limits_{n=2}^{\infty} n(n-1) a_{n,j} z^{n-2}$ we get

$$\frac{zF''_{\alpha_1,\dots,\alpha_n}(z)}{F'_{\alpha_1,\dots,\alpha_n}(z)} = -\sum_{j=1}^{n} \alpha_j \left[\frac{\sum\limits_{n=2}^{\infty} n(n-1) a_{n,j} z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} n a_{n,j} z^{n-1}} \right]. \tag{9}$$

Theorem 4. *Let the functions $f_j \in \mathcal{T}$ for $j \in \{1, \dots, n\}$. Then the functions $f_j \in KDF_{\alpha_1,\dots,\alpha_n}(\mu, \beta, \alpha_1, \dots, \alpha_n)$ for $j \in \{1, \dots, n\}$ if and only if*

$$\sum_{j=1}^{n} \left[\frac{\sum\limits_{n=2}^{\infty} \alpha_j n(n-1)(\mu+1) a_{n,j}}{1 - \sum\limits_{n=2}^{\infty} n a_{n,j}} \right] \le 1 - \beta. \tag{10}$$

Proof. First consider

$$\mu \left| \frac{zF''_{\alpha_1,\dots,\alpha_n}(z)}{F'_{\alpha_1,\dots,\alpha_n}(z)} \right| - \operatorname{Re}\left(1 + \frac{zF''_{\alpha_1,\dots,\alpha_n}(z)}{F'_{\alpha_1,\dots,\alpha_n}(z)}\right) \le (\mu+1) \left| \frac{zF''_{\alpha_1,\dots,\alpha_n}(z)}{F'_{\alpha_1,\dots,\alpha_n}(z)} \right|.$$

From (9) we obtain

$$(\mu+1) \left| \frac{zF''_{\alpha_1,\dots,\alpha_n}(z)}{F'_{\alpha_1,\dots,\alpha_n}(z)} \right| = (\mu+1) \left| \sum_{n=2}^{\infty} \alpha_j \left[\frac{\sum\limits_{n=2}^{\infty} n\,(n-1)\, a_{n,j} z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} n a_{n,j} z^{n-1}} \right] \right| \le$$

$$\le (\mu+1) \sum_{j=1}^{n} \left[\frac{\sum\limits_{n=2}^{\infty} \alpha_j n\,(n-1)\, |a_{n,j}|\, |z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty} n\, |a_{n,j}|\, |z|^{n-1}} \right] \le (\mu+1) \sum_{j=1}^{n} \left[\frac{\sum\limits_{n=2}^{\infty} \alpha_j n\,(n-1)\, a_{n,j}}{1 - \sum\limits_{n=2}^{\infty} n a_{n,j}} \right].$$

If (10) holds then the above expression is bounded by $1-\beta$ and consequently

$$\mu \left| \frac{zF''_{\alpha_1,\dots,\alpha_n}(z)}{F'_{\alpha_1,\dots,\alpha_n}(z)} \right| - \operatorname{Re}\left(1 + \frac{zF''_{\alpha_1,\dots,\alpha_n}(z)}{F'_{\alpha_1,\dots,\alpha_n}(z)}\right) < -\beta,$$

which equivalent to

$$\operatorname{Re}\left(1 + \frac{zF''_{\alpha_1,\dots,\alpha_n}(z)}{F'_{\alpha_1,\dots,\alpha_n}(z)}\right) \ge \mu \left| \frac{zF''_{\alpha_1,\dots,\alpha_n}(z)}{F'_{\alpha_1,\dots,\alpha_n}(z)} \right| + \beta.$$

Hence $f_j \in KDF_{\alpha_1,...,\alpha_n}(\mu, \beta, \alpha_1, ..., \alpha_n)$ for $j \in \{1, ..., n\}$.

Conversely,Let $f_j \in KDF_{\alpha_1,...,\alpha_n}(\mu, \beta, \alpha_1, ..., \alpha_n)$ and prove that (10) holds. If $f_j \in KDF_{\alpha_1,...,\alpha_n}(\mu, \beta, \alpha_1, ..., \alpha_n)$ and z is real we get from (4) and (9)

$$1 - \sum_{j=1}^{n} \alpha_j \left[\frac{\sum\limits_{n=2}^{\infty} n\,(n-1)\, a_{n,j} z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} n a_{n,j} z^{n-1}} \right] \geq \mu \left| \sum_{j=1}^{n} \alpha_j \left[\frac{\sum\limits_{n=2}^{\infty} n\,(n-1)\, a_{n,j} z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} n a_{n,j} z^{n-1}} \right] \right| + \beta \geq$$

$$\geq \mu \sum_{j=1}^{\infty} \alpha_j \left[\frac{\sum\limits_{n=2}^{\infty} n\,(n-1)\, a_{n,j} z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} n a_{n,j} z^{n-1}} \right] + \beta,$$

which is equivalent to

$$\sum_{j=1}^{n} \left[\frac{\sum\limits_{n=2}^{\infty} \alpha_j \mu n\,(n-1)\, a_{n,j} z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} n a_{n,j} z^{n-1}} \right] + \sum_{j=1}^{n} \left[\frac{\sum\limits_{n=2}^{\infty} \alpha_j n\,(n-1)\, a_{n,j} z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} n a_{n,j} z^{n-1}} \right] \leq 1 - \beta.$$

The above inequality reduce to

$$\sum_{j=1}^{n} \left[\frac{\sum\limits_{n=2}^{\infty} \alpha_j n(\mu+1)\,(n-1)\, a_{n,j} z^{n-1}}{1 - \sum\limits_{n=2}^{\infty} n a_{n,j} z^{n-1}} \right] \leq 1 - \beta.$$

Let $z \to 1^-$ along the real axis, then we get

$$\sum_{j=1}^{n} \left[\frac{\sum\limits_{n=2}^{\infty} \alpha_j n(\mu+1)\,(n-1)\, a_{n,j}}{1 - \sum\limits_{n=2}^{\infty} n a_{n,j}} \right] \leq 1 - \beta,$$

which give the required result.

Acknowledgement: The work here is supported by UKM-ST-06-FRGS0107-2009.

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