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SOME CHARACTERIZATIONS OF FILED PRODUCT OF QUASI-ANTIORDERS

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Abstract It is known that the filed product of two quasi-antiorders need not to be a quasi-antiorder. After some preparations, we give some sufficient conditions in order that the filed product of two quasi-antiorder relations on the same set is a quasi-antiorder again.

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1. Introduction

Issues of commuting relations on sets draw attention for more years. Many authors are investigated commuting properties of equivalences, orders and quasi-orders ([4]-[12], [14], [15], [20]-[23]).

Setting of this article is the Constructive Mathematics, mathematics based on the Intuitionistic Logic, in the sense of books [1]-[3] and [13]. One of important relations in Constructive Mathematics is quasi-antiorder relation. For relation R in set $(X, =, \neq)$ with apartness we say that it is a quasi-antiorder relation on X if satisfies the following conditions:

$$R \subseteq \neq \text{ (consistency) and } R \subseteq R * R \text{ (cotransitivity)},$$

where the operation "$*$", the filled operation between relations R and S on set X, is defined by

$$S * R = \{(x, y) \in X \times X : (\forall t \in X)((x, t) \in R \lor (t, y) \in S)\}.$$

This author investigated characteristics of this relation in several of his papers, for example in [16]-[19].

In this article we investigate one of commuting problems of these relations. If R and S are quasi-antiorders, then their filed products need not to be quasi-antiorders again, in general case. After some preparations, we give some sufficient conditions in order that the filed product of two quasi-antiorder relations on the same set is a quasi-antiorder again.

2. A few basic facts on relations

As usual, a subset R of a product set $X^2 = X \times X$ is called a relation on X. In particular, the relation $\triangle = \{(x,x) : x \in X\}$ is called the identity relation on X, and $\nabla = \{(x,y) \in X^2 : x \neq y\}$ is the diversity relation on X. If R is a relation on X, and moreover $x \in X$, then the sets $xR = \{y \in X : (x,y) \in R\}$ and $Rx = \{z \in X : (z,x) \in R\}$ are called left and right classes of R generated by the element x. The relation $R = \{(y,x) \in X^2 : (x,y) \in R\}$ is the inverse of R and denoted by R^{-1}. Moreover, if R and S are relations on X, then the filled product of R and S are defined by the usual way as the relation

$$S * R = \{(x,y) \in X2 : (\forall t \in X)((x,t) \in R \vee (t,y) \in S)\}.$$

Since the filled product is associative, in particular, for all natural number $n \geq 2$, we put ${}^nR = R * {}^{(n-1)}R = {}^{(n-1)}R * R$ and ${}^1R = R$ and ${}^0R = \nabla$. A relation R on X is called:
(1) *consistent* if $R \subseteq \nabla$,
(2) *cotransitive* if $R \subseteq R * R$ and
(3) *linear* if $\nabla \subseteq R \cup R^{-1}$.
Moreover, a consistent and cotransitive relation is called a *quasi-antiorder* relation, and a linear quasi-antiorder relation is called an *anti-order relation* on set X. A consistent, symmetric and cotransitive relation is called a *coequivality* relation on X. For any relation R on X, we define $c(R) = \bigcap\{{}^nR : n \in \mathbf{N} \cup \{0\}\}$. Thus, $c(R)$ is the biggest quasi-antiorder relation on X contained in R (see, for example [16] or [19]).

For undefined notions and notations we refer on articles [16]-[19].

3. Characterizations of filed products

Theorem 1. *If R and S are relations on X, then the following assertions are equivalent:*
(1) $S * R \subseteq R * S$;
(2) $xR \cup Sy = X$ implies $xS \cup Ry = X$ for all $x, y \in X$.

Proof: To check this, note that for any $x, y \in X$ we have

$$\begin{aligned}(x, y) \in S * R &\Longleftrightarrow (\forall t \in X)((x, t) \in R \vee (t, y) \in S)\\ &\Longleftrightarrow (\forall t \in X)(t \in xR \cup Sy)\\ &\Longleftrightarrow xR \cup Sy = X,\end{aligned}$$

and similarly $(x, y) \in R * S \Longleftrightarrow xS \cup Ry = X$. □

Now, as some immediate consequences of Theorem 1, we can also state:

Colorallary 1. *If R is a relation on X, then the following assertions are equivalent:*
(1) $R^{-1} * R \subseteq R * R^{-1}$;
(2) $xR \cup yR = X$ implies $Rx \cup Ry = X$ for all $x, y \in X$.

Concerning cotransitive relations we can prove:

Theorem 2. *If R and S are cotransitive relations on X such that $S * R \subseteq R * S$, then $R * S$ is also a cotransitive relation on X.*

Proof: We evidently have

$$R * S \subseteq (R * R) * (S * S) = R * (R * S) * S \subseteq R * (S * R) * S = (R * S) * (R * S).$$

□

The following example shows that commuting property for cotransitive relations need not be satisfies.

Example: If $X = \{1, 2, 3\}$, and moreover

$$R = \{(1,1), (2,1), (2,2), (3,1), (3,2), (3,3)\} \text{ and}$$
$$S = \{(1,1), (2,1), (2,2), (2,3), (3,1), (3,3)\},$$

then it can be easily seen that R and S are cotransitive relations on X. We have that

$$S * R = \{(1,1), (1,3), (2,1), (2,3), (3,1), (3,2), (3,3)\},$$
$$R * S = \{(1,1), (1,2), (2,1), (2,2), (2,3), (3,1), (3,2)\},$$

and $S * R$ and $R * S$ are also cotransitive relations on X, but $\neg(S * R \subseteq R * S)$ and $\neg(R * S \subseteq S * R)$. ◇

4. Characterizations of filed product of quasi-antiorders

Despite example above, as a partial case, we can still prove:

Theorem 3. *If R and S are quasi-antiorders on X, then the following assertions are equivalent:*
(1) $S * R \subseteq R * S$;
(2) $R * S$ *is a quasi-antiorder;*
(3) $R * S = c(R \cap S)$.

Proof: Since $R * S \subseteq \nabla * \nabla = \nabla$, by Theorem 2 it is clear that the implication (1) $\Longrightarrow$ (2) is true. Moreover, by the corresponding properties of the operation c, (see, for example, [17]) it is clear that $c(R \cap S) \subseteq c(R) = R$ and $c(R \cap S) \subseteq c(S) = S$, and hence $c(R \cap S) = c(R \cap S) * c(R \cap S) \subseteq R * S$.

On the other hand, by the consistency of the relations R and S, it is clear that $R * S \subseteq \nabla * S = S$ and $R * S \subseteq R * \nabla = R$, and thus $R * S \subseteq R \cap S$. Since $c(R \cap S)$ is the biggest quasi-antiorder relation under $R \cap S$, we have to $R * S \subseteq c(R \cap S)$. Therefore, the implication (2) $\Longrightarrow$ (3) is also true.

Finally, from the inclusion $c(R \cap S) \subseteq R * S$ established above, it is clear that $S * R = c(S \cap R) = c(R \cap S) \subseteq R * S$. Therefore, the implication (3) $\Longrightarrow$ (1) is also true. $\square$

The following example shows that the equality cannot be stated in Theorem 3.

Example If $X = \{1, 2, 3\}$, and moreover

$$R = \{((1,3),(2,1),(2,3),(3,1),(3,2)\}$$
$$S = \{(1,2),(1,3),(2,1),(2,3),(3,2)\},$$

then it can be easily seen that R and S are quasi-antiorders on X such that $S * R = \{(1,3),(2,1),(2,3),(3,2)\}$ is a quasi-antiorder on X and $R*S = \{(1,3),(2,1),(2,3)\}$ is not a quasi-antiorder X, but $R * S \subset S * R$. $\diamond$

Now, as an immediate consequence of Theorem 3, we can also state:

Colorallary 2. *If R is a quasi-antiorder on X, then the following assertions are equivalent :*
(1) $R^{-1} * R \subseteq R * R^{-1}$;
(2) $R * R^{-1}$ *is a quasi-antiorder;*
(3) $R * R^{-1} = c(R \cap R^{-1})$

In addition to Theorem 3, it is also worth proving the following:

Theorem 4. *If R is a consistent relation and S is a quasi-antiorder on X, then the following assertions are equivalent:*
(1) $S \subseteq R$;
(2) $S = R * S$;
(3) $S = S * R$.

Proof. Suppose that the assertion (1) holds. Then it is clear that $S \subseteq S * S \subseteq R * S \subseteq \nabla * S = S$ and $S = S * S \subseteq S * R \subseteq S * \nabla = S$. Therefore, (2) and (3) also hold. Opposite, assume that condition (2) or (3) holds. Thus, we have $S = R * S \subseteq R * \nabla = R$, or $S = S * R \subseteq \nabla * R = R$. Therefore, the implications (2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (1) are also true. □

Now, as an immediate consequence of the above theorem, we can also state:

Colorallary 3. *If R is a consistent relation and S is a cotransitive relation on X such that $S \subseteq R$, then $R * S = S * R$.*

Proof: Note that now $S \subseteq R \subseteq \nabla$ also holds. Therefore, by Theorem 4, we have $R * S = S = S * R$. □

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