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COMMON FIXED POINTS FOR THREE MAPPINGS USING G-FUNCTIONS AND THE PROPERTY (E.A)

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Abstract. Here, using the property (E.A) (see [1]) and the class of G-functions, we provide an extension to a result recently obtained by P. N. Dutta and Binayak S. Choudhury in [3].

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1. Introduction

The Banach contraction mapping theorem is a fundamental result in mathematical analysis. Since the apparition of this result, mathematicians have produced many fixed and common fixed point theorems as generalizations and extensions of the banach contraction principle to more general settings (see for example [10]).

To study fixed and common fixed point theorems for self-mappings in metric spaces several conditions are considered. These conditions are in general of the following types:

(1) Contractive conditions.
(2) Continuity or a weak form of continuity of some (or all these) mappings.
(3) Containement conditions between some of these mappings.
(4) Topological conditions, like completeness (or compacteness) of the metric space or the ranges of some (or all of these) mappings.
(5) Commutativity or a weak form of Commutativity between some (or all these) mappings.

Amongst the references of this paper, we find some papers, where these conditions were used to establish fixed point or common fixed point results.

In connection with the first type, we recall that some papers were devoted to make comparison between different contractive conditions (see [24] and [11]).

Nowadays, almost all the results which are newly published in the field of Fixed point theory are obtained for mappings assumed to satisfy inequalities through given implicit relations (see [20], [21] and other papers the references).

In connection with the fifth type, we recall that, in 1982, S. Sessa (see [26]) have first introduced the concept of weak commutativity. Since then, several concepts expressing weak types of commutativity conditions for mappings were invented as tools helping to find their common fixed points. In this direction, we can quote, compatibility (see [6]), compatibility of type (A) (see [9]), compatibility of type (B) (see [17]), compatibility of type (C) (see [19]) and compatibility of type (P) (see [18]). We recall that all these concepts imply the weak compatibility (see [7], [15] and [16]).

Definition 1.1 ([7]) *Two self-mappings S and T of a metric space (X, d) are said to be weakly compatible if $Tu = Su$, for $u \in X$, then $STu = TSu$.*

Aamri and Moutawakil [1] introduced a generalization of the concept of noncompatible mappings.

Definition 1.2 ([1]) *Let S and T be two self mappings of a metric space (X, d). We say that T and S satisfy property (E.A) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.*

Remark 1.1. It is clear that two self mappings of a metric space (X, d) will be noncompatible if there exists at least a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$, but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is either nonzero or non exist. Therefore, two noncompatible self mappings of a metric space (X, d) satisfy property (E, A).

The concept of occasionally weak compatibility is newly introduced in (see [2]) by Al-Thagafi and N. Shahzad.

In [3], the following class of functions was introduced.

Definition 1.3 *A function $g : [0, \infty)^6 \to [0, \infty)$ is called a G-function, if it satisfies the following three conditions:*

(i) g is continuous.

(ii) g is nondecreasing in each variable.

(iii) If $h(r) = g(r, r, r, r, r)$, then the function $r \to r - h(r)$ is strictly increasing and positive in $(0, \infty)$.

Examples of G-type functions are given in [3].

By using these functions, the following result was proved in [3].

Theorem 1.1 ([3]) *Let (X, d) be a metric space and $A, B : X \to X$ be two self-mappings satisfying the inequality*

$$d(Ax, By) \leq g(d(x, y), d(Ax, x), d(By, y), d(Ax, y), d(By, x)) \tag{1.1}$$

where g belongs to the class of G-type functions.

Let $\{x_n\}$ be any sequence in X which satisfies

$$\lim_{n\to\infty} d(x_n, Ax_n) = 0 \tag{1.2}$$

If $\{x_n\}$ converges to a point x then any other sequence $\{y_n\}$ having the property that $\lim_{n\to\infty} d(y_n, By_n) = 0$ will also converge to x and x is a common fixed point of A and B.

We observe that the conditions assumed in the previous theorem imply that the pair of self-mapping $\{I, A\}$ satisfies the property (E.A).

The condition (1.1) is realised through a function of five variables and involves only two functions. The aim of this note is to extend this result to the case of thereee self-mappings, by using the propert (E.A).

2. Main result

The main result of this paper is the following.

Theorem 2.1. *Let (X, d) be a metric space and $A, B, J : X \to X$ be three self-mappings satisfying the inequality*

$$d(Ax, By) \leq g(d(Jx, Jy), d(Ax, Jx), d(By, Jy), d(Ax, Jy), d(By, Jx)) \tag{2.1}$$

for all $x, y \in X$, where g is a $G-$function.

We suppose that

(A 1) J is continuous and commutes with A or B.

(A 2) The pair $\{J, A\}$ satisfies the property (E.A).

(A 3) There exists a sequence $\{y_n\}$ such that $\lim_{n\to\infty} d(Jy_n, By_n) = 0$.

Then

(i) the mappings A, B and J have a unique common fixed point x in X.

(ii) If $\{w_n\}$ is any other sequence satisfying $\lim_{n\to\infty} d(Jw_n, Bw_n) = 0$, then the sequence $\{Jw_n\}$ converges to the unique common fixed point x.

(iii) The mappings A and B are continuous at the unique common fixed point x.

Proof. Since J and A satisfy the property (E.A), then there exists a point x in X and a sequence $\{x_n\}$ of points in X such that

$$\lim_{n\to\infty} Jx_n = \lim_{n\to\infty} Ax_n = x. \tag{2.2}$$

By assumption (A 3), there is a sequence $\{y_n\}$ of points in X such that

$$\lim_{n\to\infty} d(Jy_n, By_n) = 0. \tag{2.3}$$

(1) We want to show that the sequence $\{Jy_n\}$ converges to the point x. Since $\{Jx_n\}$ converges to a point x, it is equivalent to see that

$$\lim_{n\to\infty} d(Jx_n, Jy_n) = 0. \tag{2.4}$$

To get a contradiction, we suppose that (2.4) is false. In this case (by considering subsequences), we may suppose that there exists a positive number $\epsilon > 0$ such that

$$d(Jx_n, Jy_n) \geq \epsilon, \quad \forall n \in \mathcal{N}, \tag{2.5}$$

where $\mathcal{N}$ is the set of all non-negative integers.

Let δ be any number such that

$$0 < \delta < \frac{\epsilon - h(\epsilon)}{3}. \tag{2.6}$$

Then we can find a positive integer N_δ such that

$$\forall n \in \mathcal{N},\ n \geq N_\delta \Longrightarrow \max\{d(Jx_n, Ax_n), d(Jy_n, By_n)\} \leq \delta. \tag{2.7}$$

To simplify notations, for each integer $n \in \mathcal{N}$, we define

$$\alpha_n := d(Jx_n, Jy_n), \quad \beta_n := d(Ax_n, Jx_n) \quad \text{and} \quad \gamma_n := d(By_n, Jy_n). \tag{2.8}$$

With these notations, by using (2.1), for every n, we have

$$d(Ax_n, By_n) \leq g(\alpha_n, \beta_n, \gamma_n, d(Ax_n, Jy_n), d(By_n, Jx_n)). \tag{2.9}$$

For every non-negative integer $n \geq N_\delta$, we have the following inequalities:

$$\alpha_n - 2\delta \leq \alpha_n - \beta_n - \gamma_n \leq d(Ax_n, By_n),$$

$$d(Ax_n, Jy_n) \leq \alpha_n + \beta_n \leq \alpha_n + \delta,$$

and

$$d(By_n, Jx_n) \leq \alpha_n + \gamma_n \leq \alpha_n + \delta.$$

From the previous inequalities and the property (ii) of g, we obtain the following inequalities:

$$\alpha_n - 2\delta \leq g(\alpha_n, \delta, \delta, \alpha_n + \delta, \alpha_n + \delta) \leq h(\alpha_n + \delta), \tag{2.10}$$

for every non-negative integer $n \geq N_\delta$.

From (2.10) and (2.6) we obtain

$$\alpha_n + \delta - h(\alpha_n + \delta) \leq 3\delta < \epsilon - h(\epsilon). \tag{2.11}$$

By condition (iii) on the function g, we deduce that

$$\epsilon \leq \alpha_n + \delta < \epsilon,$$

which is a contradiction. Hence we have

$$x = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Jx_n = \lim_{n\to\infty} Jy_n = \lim_{n\to\infty} By_n. \tag{2.12}$$

Since J is continuous, then we have

$$Jx = \lim_{n\to\infty} JAx_n = \lim_{n\to\infty} J^2x_n = \lim_{n\to\infty} J^2y_n = \lim_{n\to\infty} JBy_n. \tag{2.13}$$

From the lines of proof above, we conclude that for any other sequence $\{w_n\}$ satisfying $\lim_{n\to\infty} d(Jw_n, Bw_n) = 0$, the sequence $\{Jw_n\}$ converges to the point x.

From (2.1), for every non-negative integer n, we have

$$d(Ax, By_n) \leq g(d(Jx, Jy_n), d(Ax, Jx), d(By_n, Jy_n), d(Ax, Jy_n), d(By_n, Jx)). \tag{2.14}$$

Making $n \to \infty$ and noting that g is continuous, we obtain from (2.14) that

$$d(Ax, x) \leq g(d(Jx, x), d(Ax, Jx), 0, d(Ax, x), d(x, Jx)). \tag{2.15}$$

(2) Suppose that J and A are commuting, then we have

$$Jx = \lim_{n\to\infty} JAx_n = \lim_{n\to\infty} AJx_n. \tag{2.16}$$

Again, using (2.1), we have

$$d(AJx_n, By_n) \leq g\left(d(J^2x_n, Jy_n), d(AJx_n, J^2x_n), d(By_n, Jy_n), d(AJx_n, Jy_n), d(By_n, J^2x_n)\right). \tag{2.17}$$

Making $n \to \infty$ and noting that g is continuous, we obtain from (2.17) that

$$d(Jx, x) \leq g(d(Jx, x), 0, 0, d(Jx, x), d(x, Jx)) \leq h(d(Jx, x)). \tag{2.18}$$

From the assumption (iii) on g and (2.18) we get $Jx = x$. Reporting this in equation (2.15), we obtain that

$$d(Ax, x) \leq g(0, d(Ax, x), 0, d(Ax, x), 0) \leq h(d(Ax, x)). \tag{2.19}$$

From the assumption (iii) on g and (2.19) we get $Ax = x$. So we have shown that

$$Ax = Jx = x.$$

Again, using (2.1), for every non-negative integer n, we have

$$d(Ax_n, Bx) \leq g\left(d(J^2x_n, Jx), d(Ax_n, Jx_n), d(Bx, Jx), d(Ax_n, Jx), d(Bx, Jx_n)\right). \tag{2.20}$$

Making $n \to \infty$ and noting that g is continuous, we obtain from (2.20) that

$$d(x, Bx) \leq g(0, 0, d(Bx, x), 0, d(Bx, x)) \leq h(d(Bx, x)). \tag{2.21}$$

From the assumption (iii) on g and (2.21) we get $Bx = x$. So we have shown that

$$Ax = Jx = x = Bx.$$

(3) The same results are obtained if we suppose that J and B commute. So, we omit the details.

(4) It remains to prove that A and B are continuous at the unique common fixed point x.

(a) To show that A is continuous at the unique common fixed point x, let $\{u_n\}$ be a sequence converging to x. Then by using the inequality (3.26), we have

$$d(Au_n, x) = d(Au_n, Bx) \leq g(d(Ju_n, x), d(Au_n, Ju_n), 0, d(Au_n, Jx), d(x, Ju_n)), \tag{2.22}$$

for all non-negative integer n. By using the propeties of the G-function g, we deduce from (2.22) that

$$\begin{aligned} d(Au_n, x) &= d(Au_n, Bx) \\ &\leq g(d(Ju_n, x), d(Au_n, x) + d(x, Ju_n), 0, d(Au_n, Jx), d(x, Ju_n)), \\ &\leq h(d(Au_n, x) + d(x, Ju_n)). \end{aligned} \tag{2.23}$$

From (2.23) we get

$$d(Au_n, x) + d(x, Ju_n) - h(d(Au_n, x) + d(x, Ju_n)) \leq d(x, Ju_n). \tag{2.24}$$

Let $\epsilon > 0$. Since J is continuous, then there exists an integer N_ϵ such that for all non-negative integer n, we have

$$n \geq N_\epsilon \Longrightarrow d(x, Ju_n) \leq \epsilon - h(\epsilon). \tag{2.25}$$

By the condition (iii) on the function g and by virtue of (2.24) and (2.25), we get

$$d(Au_n, x) + d(x, Ju_n) \leq \epsilon, \quad \forall n \geq N_\epsilon.$$

Thus A is continuous at the point x.

(b) To show that B is continuous at the unique common fixed point x, let $\{v_n\}$ be a sequence converging to x. Then by using the inequality (2.1), we have

$$d(x, Bv_n) = d(Ax, Bv_n) \leq g(d(x, Jv_n), 0, d(Bv_n, Jv_n), d(x, Jv_n), d(Bv_n, x)), \quad (2.26)$$

for all non-negative integer n. By using the properties of the G-function g, we deduce from (2.26) that

$$d(x, Bv_n) \leq g(d(x, Jv_n), 0, d(Bv_n, x) + d(x, Jv_n), d(x, Jv_n), d(Bv_n, x)),$$
$$\leq h(d(Bv_n, x) + d(x, Jv_n)). \quad (2.27)$$

From (2.27) we get

$$d(Bv_n, x) + d(x, Jv_n) - h(d(Bv_n, x) + d(x, Jv_n)) \leq d(x, Jv_n). \quad (2.28)$$

Let $\epsilon > 0$. Since J is continuous, then there exists an integer N_ϵ such that for all non-negative integer n, we have

$$n \geq N_\epsilon \Longrightarrow d(x, Jv_n) \leq \epsilon - h(\epsilon). \quad (2.29)$$

By the condition (iii) on the function g and by virtue of (2.28) and (2.29), we get

$$d(Bv_n, x) + d(x, Jv_n) \leq \epsilon, \quad \forall n \geq N_\epsilon.$$

Thus B is continuous at the point x. This completes the proof.

The next result is an easy consequence from Theorem 2.1.

Corollary 2.1. *Let (X, d) be a metric space and $A, B : X \to X$ be two self-mappings satisfying the inequality*

$$d(Ax, By) \leq g(d(x, y), d(Ax, x), d(By, y), d(Ax, y), d(By, x)),$$

for all $x, y \in X$, where g belongs to the class of G-type functions.

We suppose that:

(A 1) There exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} d(x_n, Ax_n) = 0,$$

and $\{x_n\}$ *converges to a point* x *in* X.
(A 2) There exists a sequence $\{y_n\}$ *in* X *such that* $\lim_{n\to\infty} d(y_n, By_n) = 0$.
Then,
(i) x *is the unique common fixed point of the mappings* A *and* B.
(ii) The sequence $\{y_n\}$ *converges to* x *and any other sequence* $\{z_n\}$ *having the property that* $\lim_{n\to\infty} d(z_n, Bz_n) = 0$ *will also converge to the unique common fixed point* x *of* A *and* B.
(iii) The mappings A *and* B *are continuous at their unique common fixed point.*

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