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DIFFERENTIAL OPERATORS FOR P-VALENT FUNCTIONS

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Abstract. Let $f(z) = D(F(z))$, where D is differential operator defined separtely in every result. Let $S(p)$ denote the class of functions $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n\, z^n$ which are analytic and p-valent in the unit disc $U = \{z : |z| < 1\}$. The purpose of this is paper to find out the disc in which the operator D transforms some classes of p-valent functions into the same. For example, if F is p-valent starlike of order $\lambda(0 \leq \lambda < p)$, then the disc in which f is also in the same class is found. We also discuss several special cases which can be derived from our main results.

Keywords: p-Valent, analytic, starlike, close-to-convex, Bazilevic functions.

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1.Introduction

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \tag{1}$$

which are analytic in the unit disc $U = \{z : |z| < 1\}$. Further let $S(p)$ be the subclass of $A(p)$ consisting of functions which are p-valent in U. A function $f(z) \in S(p)$ is said to be in the class $S_p^*(\lambda)$ of p-valently starlike functions of order $\lambda(0 \leq \lambda < p)$ if it satisfies

$$Re\left\{\frac{z\, f'(z)}{f(z)}\right\} > \lambda \ \ and \ \ \int_0^{2\pi} Re\left\{\frac{z\, f'(z)}{f(z)}\right\} d\theta = 2\pi p\ . \tag{2}$$

The class $S_p^*(\lambda)$ was studied recently by Owa [8] and Aouf et al. [2].

Let $K_p(\gamma,\lambda)$ denote the class of functions $F(z)\in A(p)$ which satisfy

$$Re\left\{\frac{z\,F^{'}(z)}{G(z)}\right\}>\gamma\quad (z\in U), \tag{3}$$

where $G(z)\in S_p^*(\lambda)$ and $0\le\gamma,\ \lambda<p$. The class $K_p(\gamma,\lambda)$ of p-valenty close-to-convex functions of order γ and type λ was studied by Aouf [1]. We note that $K_1(\gamma,\lambda)=K(\gamma,\lambda)$, the class of convex functions of order γ and type λ was studied by Libera [4] and $K_1(1,1)=K$, is the well-known class of close-to-convex functions, introduced by Kaplan [3].

A function $F(z)\in S(p)$ is said to be in the class $B_p(\beta,\lambda)$ if and only if there exists a function $G(z)\in S_p^*(\lambda), 0\le\lambda<p, \beta>0$, such that

$$Re\left\{\frac{zF^{'}(z)}{F^{1-\beta}(z)G^{\beta}(z)}\right\}>0\quad (z\in U). \tag{4}$$

The class $B_p(\beta,\lambda)$ is the subclass of p-valently Bazilevic functions in U.
We note that $B_p(\beta,0)=B_p(\beta)$, is the class of p-valently Bazilevic functions of type β and $B_1(\beta,0)=B(\beta)$, is the class of Bazilevic functions of type β (see [10]).

2.Main Results

We shall consider some differential operators and find out the disc in which the classes of p-valent functions defined by (2), (3) and (4), respectively, are preserved under these operators. We prove the following:
Theorem 1. *Let $F(z)\in S_p^*(\lambda)(0\le\lambda<p), \beta>0$ and $0<\alpha\le 1$. Let the function $f(z)$ be defined by the differential operator*

$$D_{\alpha,\beta,p}(F)=f^{\beta}(z)=\frac{1}{\alpha\beta p-\alpha+1}\left[(1-\alpha)F^{\beta}(z)+\alpha\,z(F^{\beta}(z))^{'}\right]. \tag{5}$$

Then $f(z)\in S_p^*(\lambda)$ for $|z|<r_0$, where

$$r_0=\frac{\alpha\beta p-\alpha+1}{[\alpha(\beta p-\beta\lambda+1)+\sqrt{\alpha^2(\beta p-\beta\lambda+1)^2+(\alpha\beta p-\alpha+1)(1-\alpha\beta p+2\alpha\beta\lambda-\alpha)}}. \tag{6}$$

The result is sharp.

Proof. We can write

$$f^{\beta}(z)=\frac{1}{\alpha\beta p-\alpha+1}\left[(1-\alpha)F^{\beta}(z)+\alpha\,z(F^{\beta}(z))^{'}\right], \tag{7}$$

as

$$f^{\beta}(z)=\frac{1}{\alpha\beta p-\alpha+1}\left[\alpha\,z^{2-\frac{1}{\alpha}}(z^{\frac{1}{\alpha}-1}\ F^{\beta}(z))^{'}\right],$$

and from this it follows that

$$F^{\beta}(z) = (\beta p + \frac{1}{\alpha} - 1)\, z^{1-\frac{1}{\alpha}} \int\limits_0^z z^{\frac{1}{\alpha}-2} f^{\beta}(z) d\,z \ .$$

(8)

Thus, we have

$$\beta \frac{zF'(z)}{F(z)} = (1 - \frac{1}{\alpha}) + \frac{z^{\frac{1}{\alpha}-1} f^{\beta}(z)}{\int_0^z z^{\frac{1}{\alpha}-2} f^{\beta}(z) d\,z} \ . \qquad (9)$$

Since $F(z) \in S_p^*(\lambda)$, we can write (9) as

$$\beta(p-\lambda)h(z) \int\limits_0^z z^{\frac{1}{\alpha}-2} f^{\beta}(z) d\,z + \beta\lambda \int\limits_0^z z^{\frac{1}{\alpha}-2} f^{\beta}(z) d\,z$$

$$= \ (1 - \frac{1}{\alpha}) \int\limits_0^z z^{\frac{1}{\alpha}-2} f^{\beta}(z) d\,z + z^{\frac{1}{\alpha}-1} f^{\beta}(z),$$

where $Re\, h(z) > 0$. Differentiating again with respect to z, we obtain

$$\frac{z\, f'(z)}{f(z)} = (p-\lambda)h(z) + \lambda + \frac{(p-\lambda)h'(z) \int_0^z z^{\frac{1}{\alpha}-2} f^{\beta}(z) d\,z}{z^{\frac{1}{\alpha}-2} f^{\beta}(z)} \ . \qquad (10)$$

Now, using a well-known result [4],

$$|h'(z)| \leq \frac{2Re(z)}{1-r^2} \ \ (|z| = r),$$

(11)

we have from (10

$$Re \left\{ \frac{z\, f'(z)}{f(z)} - \lambda \right\} \geq Re\, h(z) \left\{ (p-\lambda) - \right.$$

$$\frac{2(p-\lambda)}{1-r^2}\left|\frac{\int\limits_0^z z^{\frac{1}{\alpha}-2} f^{\beta}(z)d\,z}{z^{\frac{1}{\alpha}-2} f^{\beta}(z)}\right|\Bigg\}. \tag{12}$$

Also, from (7) and (8), we have

$$\begin{aligned}
\frac{z^{\frac{1}{\alpha}-1} f^{\beta}(z)}{\int\limits_0^z z^{\frac{1}{\alpha}-2} f^{\beta}(z)d\,z} &= \frac{\alpha\, z(z^{\frac{1}{\alpha}-1}F^{\beta}(z))'}{\alpha\,(z^{\frac{1}{\alpha}-1}F^{\beta}(z))} \\
&= (\frac{1}{\alpha}-1)+\beta\frac{z\,F'(z)}{F(z)} \\
&= (\frac{1}{\alpha}-1)+\beta\,[(p-\lambda)h(z)+\lambda],
\end{aligned}$$

since $F(z) \in S_p^*(\lambda)$. Thus we have

$$\begin{aligned}
\left|\frac{z^{\frac{1}{\alpha}-1} f^{\beta}(z)}{\int\limits_0^z z^{\frac{1}{\alpha}-2} f^{\beta}(z)d\,z}\right| &\geq Re\left\{(\frac{1}{\alpha}-1)+\beta[(p-\lambda)h(z)+\lambda]\right\} \\
&\geq (\frac{1}{\alpha}-1)+\beta\lambda+\beta(p-\lambda)\frac{1-r}{1+r} \\
&= \frac{(\frac{1}{\alpha}-1+\beta\lambda)(1+r)+(\beta p-\beta\lambda)(1-r)}{1+r}\,.
\end{aligned} \tag{13}$$

Hence, from (12) and (13), we have

$$\begin{aligned}
Re\left\{\frac{z\,f'(z)}{f(z)}-\lambda\right\} &\geq Re\,h(z)\,\{(p-\lambda)- \\
&\left[\frac{2(p-\lambda)}{(1+r)(1-r)}\right]\;\frac{r(1+r)}{(\frac{1}{\alpha}-1+\beta\lambda)(1+r)+(\beta p-\beta\lambda)(1-r)}\Bigg\} \\
&= (p-\lambda)\,Re\,h(z).
\end{aligned}$$

$$.\left\{\frac{(\beta p+\frac{1}{\alpha}-1)-2(\beta p-\beta\lambda+1)r-(\frac{1}{\alpha}-\beta p+2\beta\lambda-1)r^2}{(1-r)(\beta p+\frac{1}{\alpha}-1)+(\frac{1}{\alpha}-\beta p+2\beta\lambda-1)r}\right\}. \tag{14}$$

The right-hand side of (50) is positive for $r < r_0$, $0 < \alpha \leq 1, \beta > 0$ and $0 \leq \lambda < p$, where r_0 is given by the relation (6).

The function $f_0(z)$ defined by:

$$f_0^{\beta}(z) = \frac{1}{\alpha\beta p - \alpha + 1}\left[(1-\alpha)\ F_0^{\beta}(z) + \alpha\, z(F_0^{\beta}(z))^{'}\right], \tag{15}$$

where

$$F_0(z) = \frac{z^p}{(1-z)^{2(p-\lambda)}} \in S_p^*(\lambda) \tag{16}$$

shows that the result is sharp.

Putting $\beta = 1$ in Theorem 1, we obtain

Corollary 1. *Let $F(z) \in S_p^*(\lambda)(0 \le \lambda < p)$, and $0 < \alpha \le 1$. Let the function $f(z)$ be defined by the differential operator*

$$D_{\alpha,1,p}(F) = f(z) = \frac{1}{\alpha p - \alpha + 1}\left[(1-\alpha)F(z) + \alpha z F^{'}(z)\right]. \tag{17}$$

Then $f(z) \in S_p^*(\lambda)$ for $|z| < r_0^*$, where

$$r_0^* = \frac{\alpha p - \alpha + 1}{\alpha(p-\lambda+1) + \sqrt{\alpha^2(p-\lambda+1)^2 + (\alpha p - \alpha + 1)(1 - \alpha p + 2\alpha\lambda - \alpha)}}\ . \tag{18}$$

The result is sharp.

Putting $\beta = 1$ and $\lambda = 0$ in Theorem 1, we have

Corollary 2. *Let $F(z) \in S_p^*$ and $0 < \alpha \le 1$. Let the function $f(z)$ be defined by the differential operator* (53). *Then $f(z) \in S_p^*$ for $|z| < r_0^{**}$, where*

$$r_0^{**} = \frac{\alpha\, p - \alpha + 1}{\alpha(p+1) + \sqrt{\alpha^2(p+1)^2 + (\alpha p - \alpha + 1)(1 - \alpha\ p - \alpha)}}\ . \tag{19}$$

The result is sharp.

Remark 1. *(1) Putting $p = 1$ in Theorem 1, we obtain the result obtained by Noor* [6]*;*

(2) Putting $p = 1$ in Corollary 2, we obtain the result obtained By Noor et al. [7];

(3) Putting $p = 1$ and $\alpha = \frac{1}{2}$ in Corollary 2, we obtain the result obtained by Livingston [5]*;*

(4) Theorem 1 generalizes a result due to Padmanabhan [9] *when we take $p = \beta = 1$ and $\alpha = \frac{1}{2}$.*

Theorem 2. *Let $F(z) \in K_p(\gamma, \lambda)(0 \le \gamma, \lambda < p)$ and let $\alpha > 0$. Then the function $f(z)$, defined as*

$$D_{\alpha,1,p}(F) = f(z) = (1-\alpha)F(z) + \alpha\, zF^{'}(z) \tag{20}$$

belongs to the same class for $|z| < R_0$, $R_0 = \min(r_1, r_2)$, where r_1 and r_2 are the respective least positive roots of the equations

$$(p-1+\frac{1}{\alpha}) - 2(p+1-\lambda)r - (\frac{1}{\alpha} + 2\lambda - (p+1))r^2 = 0\ , \tag{21}$$

and

$$(p-1+\frac{1}{\alpha}) - 2(p+1-\gamma)r - (\frac{1}{\alpha} + 2\gamma - (p+1))r^2 = 0\ . \tag{22}$$

The result is sharp.

Proof. Since $F(z) \in K_p(\gamma,\lambda)$, then there exists a function $G(z) \in S_p^*(\lambda)$ such that $Re\left\{\frac{zF'(z)}{G(z)}\right\} > \gamma$, $z \in U$. Now let

$$D_{\alpha,1,p}(G) = g(z) = (1-\alpha)G(z) + \alpha z G'(z). \tag{23}$$

Then, from Theorem 1, it follows that $g(z) \in S_p^*(\lambda)$ for $|z| < r_1$, where r_1 is the least positive root of (57).

Using the same technique of Theorem 1, we can easily show that $Re\left\{\frac{zf'(z)}{g(z)}\right\} > \gamma$ for $|z| < R_0$, where $R_0 = \min(r_1, r_2)$ and r_2 is given by (58). The sharpness of the result can be seen as follows:

Let

$$F_1(z) = \frac{z^p}{(1-z)^{2(p-\gamma)}} \quad and \quad G_1(z) = \frac{z^p}{(1-z)^{2(p-\lambda)}}\ .$$

Then

$$Re\frac{zF_1'(z)}{G_1(z)} > \gamma\ ,\ G_1 \in S_p^*(\lambda) \Rightarrow F_1 \in K_p(\gamma,\lambda).$$

Let $f_1(z) = (1-\alpha)F_1(z) + \alpha z F_1'(z)$ and $g_1(z) = (1-\alpha)G_1(z) + \alpha z G_1'(z)$. Then $Re\frac{zf_1'(z)}{g_1(z)} > \gamma$ for $|z| < R_0$, where R_0 is as given in Theorem 2 and can not be improved.

Remark 2. *(1) Putting $p = 1$ in Theorem 2, we obtain the result obtained by Noor* [6]*;*

(2) Putting $p = 1$ and $\gamma = \lambda = 0$ in Theorem 2, we obtain the result obtained By Noor et al. [7]*;*

(3) Putting $p = 1, \gamma = \lambda = 0$ and $\alpha = \frac{1}{2}$ in Theorem 2, we obtain the result obtained by Livingston [5]*;*

(4) Putting $p = 1$ and $\alpha = \frac{1}{2}$ in Theorem 2, we obtain the result obtained by Padmanabhan[9].

Theorem 3. *Let $0 < \alpha \leq \beta \leq 1$ and $F(z) \in S_p^*(\lambda)(0 \leq \lambda < p)$. Then $f(z)$ defined as*

$$D^*_{\alpha,\beta,p}(F) = f^\alpha(z) = z^\alpha(z^{1-\beta}F^\beta(z))'. \tag{24}$$

belongs to $S^*_p(\lambda_1)$ for $|z| < R_1$, where R_1 is given by

$$R_1 = \frac{1}{(\beta p - \beta\lambda + 1) + \sqrt{\beta^2(p-\lambda)^2 + \beta(1-p)[\beta(1+p-2\lambda)-2]}}\,, \tag{25}$$

and

$$\lambda_1 = 1 - \frac{\beta}{\alpha}(1-\lambda).$$

The result is sharp.

Proof. From (2.20), we can write

$$F^\beta(z) = z^{\beta-1}\int_0^z (\frac{f(z)}{z})^\alpha\, dz\ ,$$

and so

$$\beta\frac{zF'(z)}{F(z)} = (\beta-1) + \frac{z\,(\frac{f(z)}{z})^\alpha}{\int_0^z (\frac{f(z)}{z})^\alpha\, dz}\ .$$

(26)

Since $F(z) \in S^*_p(\lambda)$, $\frac{z\,F'(z)}{F(z)} = (p-\lambda)h(z) + \lambda, 0 \le \lambda < p$, $Re\,h(z) > 0$, $z \in U$. Thus, from (2.22), we obtain

$$\beta(p-\lambda)h(z)\int_0^z (\frac{f(z)}{z})^\alpha\, dz + [\beta\lambda + (1-\beta)]\int_0^z (\frac{f(z)}{z})^\alpha\, dz = z(\frac{f(z)}{z})^\alpha.$$

Differentiating again with respect to z, we obtain

$$\beta(p-\lambda)h(z)(\frac{f(z)}{z})^\alpha + \beta(p-\lambda)h'(z)\int_0^z (\frac{f(z)}{z})^\alpha\, dz + [\beta\lambda + (1-\beta)](\frac{f(z)}{z})^\alpha$$

$$= \ \alpha\ z^{1-\alpha}f^{\alpha-1}(z)\,f'(z) + (1-\alpha)(\frac{f(z)}{z})^\alpha$$

or

$$\frac{z\,f'(z)}{f(z)} - [1 - \frac{\beta}{\alpha}(1-\lambda)]$$

$$= \frac{\beta}{\alpha}(p-\lambda)\left[h(z) + \frac{h'(z)\int_0^z (\frac{f(z)}{z})^\alpha\, dz}{(\frac{f(z)}{z})^\alpha}\right]. \tag{27}$$

Now, using the well-known result (11), we have from (2.23),

$$Re\ \frac{z\,f'(z)}{f(z)} - [1 - \frac{\beta}{\alpha}(1-\lambda)] \geq$$

$$\frac{\beta}{\alpha}(p-\lambda) Re\ h(z)\left[1 - \frac{2}{1-r^2}\left|\frac{\int_0^z (\frac{f(z)}{z})^\alpha\, dz}{(\frac{f(z)}{z})^\alpha}\right|\right].$$

Now

$$\begin{aligned} \frac{z\ (\frac{f(z)}{z})^\alpha}{\int_0^z (\frac{f(z)}{z})^\alpha\, dz} &= \frac{z\ (z^{1-\beta}F^\beta(z))'}{z^{1-\beta}\,F^\beta(z)} \\ &= \beta\frac{zF'(z)}{F(z)} + (1-\beta) \end{aligned}$$

$$= \beta[(p-\lambda)h(z) + \lambda] + (1-\beta),\ \ Re\, h(z) > 0,$$

and from this, it follows that

$$\begin{aligned} \left|\frac{z(\frac{f(z)}{z})^\alpha}{\int_0^z (\frac{f(z)}{z})^\alpha\, dz}\right| &\geq \beta(p-\lambda)Re\, h(z) + \beta\lambda + (1-\beta) \\ &\geq \beta(p-\lambda)\frac{1-r}{1+r} + \beta\lambda + (1-\beta) \\ &= \frac{\beta(p-\lambda)(1-r) + (\beta\lambda + (1-\beta))(1+r)}{(1+r)}. \end{aligned}$$

Hence

$$Re\,\frac{z\,f'(z)}{f(z)} - [1 - \frac{\beta}{\alpha}(1-\lambda)] \geq \frac{\beta}{\alpha}(p-\lambda)\,Re\,h(z).$$

$$\cdot\left\{1-\frac{2}{1-r^2}\;\frac{r(1+r)}{\beta(p-\lambda)(1-r)+(\beta\lambda+(1-\beta))(1+r)}\right\}$$

$$=\frac{\beta}{\alpha}(p-\lambda)Reh(z)\left\{\frac{(1-r)[\beta(p-\lambda)(1-r)+(\beta\lambda+(1-\beta))(1+r)]-2r}{(1-r)[\beta(p-\lambda)(1-r)+(\beta\lambda+(1-\beta))(1+r)]}\right\}. \quad (28)$$

The right-hand side of the inequality (2.24) is positive for $|z| < R_1$, where R_1 is the least positive root of the equation

$$(1-\beta p-\beta+2\beta\lambda)r^2+2(\beta p-\beta\lambda+1)r-(\beta p-\beta+1)=0, \quad (29)$$

and thus we obtain the required result. The function

$$F_1(z)=\frac{z^p}{(1-z)^{2(p-\lambda)}}\in S_p^*(\lambda) \quad (30)$$

shows that the result is best possible.

Putting $\lambda=\frac{1}{2}$ in Theorem 3, we obtain

Corollary 3. *Let* $0<\alpha\leq\beta\leq 1$ *and* $F(z)\in S_p^*(\frac{1}{2})$. *Then* $f(z)$ *defined by* (2.20) *belongs to* $S_p^*(1-\frac{\beta}{2\alpha})$ *for* $|z|<R_2$, *where* R_2 *is given by*

$$R_2=\frac{2}{(2\beta p-\beta+2)+\sqrt{\beta^2(2p-1)^2+8+\beta(1-p)(p-2)}}\,. \quad (31)$$

The result is sharp.

Remark 3. *(1) Putting* $p=1$ *in Theorem 3, we obtain the result obtained by Noor* [6]*;*

(2) Putting $\alpha=\beta$ *and* $F(z)\in S_p^*(\lambda)$. *Then* $f(z)$ *defined by* (2.20) *also belongs to* $S_p^*(\lambda)$ *for* $|z|<R_1$*;*

(3) Putting $p=1$ *in Corollary 3, we obtain the result obtained by Noor* [6].

Using the same technique of Theorem 3 we can prove the following theorem.

Theorem 4. *Let* $F(z)\in B_p(\beta,\lambda), 0<\beta\leq 1$ *and* $0\leq\lambda<p$. *Let* $f(z)$ *be defined as*

$$f^{\beta}(z)=z^{\beta}(z^{1-\beta}\;F^{\beta}(z))'\,.$$

Then $f(z)\in B_p(\beta,\lambda)$ for $|z|<R_3$, where R_3 is given by

$$R_3=\frac{1}{(\beta p-\beta+1)+\sqrt{\beta^2(p-\lambda)^2+2+\beta(1-p)[\beta(1+p-2\lambda)-2]}}\,. \quad (32)$$

The result is sharp.

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