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AN ELEMENTARY PROOF OF THE FERMAT COMPOSITE CONJECTURE AND THE CONNECTION WITH THE GOLDBACH CONJECTURE

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Abstract. The Goldbach conjecture (see [3] or [4] or [5] or [6] or [7]) states that every even integer $e \geq 4$ is of the for $e = p + r$, where (p, r) is a couple of prime(s). In this paper, we use elementary arithmetic congruences, elementary arithmetic calculus, elementary complex analysis; and we give an original proof of a simple Theorem which immediately implies that: there are infinitely many Fermat composite. Moreover, our simple Theorem, immediately implies that the Fermat composite conjecture (that we solved) was only an elementary special case of the famous Goldbach conjecture [[we recall (see [1] or [2] or [3]) that a Fermat composite is a non prime number of the form $F_n = 2^{2^n} + 1$, where n is an integer ≥ 1. It is known (see [1] or [2] or [3]) that F_5 and F_6 are composite, and it is conjectured that there are infinitely many Fermat composite numbers]].

Key words: goldbach; of type 37, n-conform.

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1. Prologue

In Section.*1*, we introduce some non-standard denotations and elementary properties. In Section.*2*, using only simple definitions, elementary arithmetic congruences, elementary arithmetic calculus, elementary complex analysis, reasoning by reduction to absurd and two elementary properties of a simple Proposition of Section.*1*, we prove a Theorem which implies that there are infinitely many Fermat composite numbers; moreover, our Theorem clearly implies that the Fermat composite conjecture that we solved, was only an elementary consequence of the Goldbach conjecture. That being so, this manuscript is original, and therefore, is not related to all strong investigations that have been done on the Fermat composite numbers conjecture and the Goldbach conjecture in the past.

2. Introduction, non-standard denotations, and simple properties

The prime numbers are well-known, and we say that e is goldbach, if e is an even integer ≥ 4 and is of the form $e = p + q$, where (p, q) is a couple of prime(s). The Goldbach conjecture (see Abstract and definitions) states that every even integer $e \geq 4$ is goldbach. Now, for every integer $n \geq 2$, we define $\mathcal{G}(n)$, g''_n, $\mathcal{X}(\mathcal{FCO})(n)$, $\mathcal{FCO}(n)$, o_n, and $o_{n,1}$ as follows: $\mathcal{G}(n) = \{g''; 1 < g'' \leq 2n,\, and\, g''\, is\, goldbach\}$; $g''_n = \max_{g'' \in \mathcal{G}(n)} g''$; $\mathcal{X}(\mathcal{FCO})(n) = \{F_5\} \bigcup \{x; 1 < x < 2n,\, and\, x\, is\, a\, Fermat\, composite\}$, $\mathcal{FCO}(n) = \{o; o \in \mathcal{X}(\mathcal{FCO})(n)\}$ [observing (by using Abstract and definitions) that $F_5 = 2^{2^5} + 1$ and F_5 is a Fermat composite, then it becomes immediate to deduce that for every integer $n \geq 2$, $F_5 \in \mathcal{FCO}(n)$], $o_n = \max_{o \in \mathcal{FCO}(n)} o$, and $o_{n,1} = 11 + 185 o_n^{o_n}$.

Using the previous definitions and denotations, let us propose.

Proposition 1.1 *Let n be an integer ≥ 2; consider $\mathcal{X}(\mathcal{FCO})(n)$, $\mathcal{FCO}(n)$, o_n and $o_{n,1}$. We have the following three properties.*

(1.1.0.) $F_5 - 1 < o_n < o_{n,1}$; $o_{n,1} = 11 + 185 o_n^{o_n}$; $o_{n,1} \equiv 11\ mod[185]$; $o_{n,1} > 10 + 185 F_5^{F_5}$; *and* $o_{n,1} \equiv 11\ mod[37]$.

(1.1.1.) If $o_n < n$, *then:* $n > 200$ *and* $o_n = o_{n-1}$ *and* $o_{n,1} = o_{n-1,1}$.

(1.1.2.) If there exists an integer $y \geq 0$ *such that* $o_n < n - y$, *then:* $n > y + 200$ *and* $o_n = o_{n-y-1}$ *and* $o_{n,1} = o_{n-y-1,1}$ *[it is clear that property (1.1.2) is a considerable generalization of property (1.1.1)].*

Proof. Property (1.1.0) is immediate [the first four assertions of property (1.1.0) are immediate (it suffices to use the definition of o_n and $o_{n,1}$); the last assertion of property (1.1.0) is also immediate (indeed, since $o_{n,1} \equiv 11\ mod[185]$, observing that $185 \equiv 0\ mod[37]$, then, using the previous two congruences, it becomes trivial to deduce that $o_{n,1} \equiv 11\ mod[37]$)].

(1.1.1.) If $o_n < n$, clearly $n > 200$ [use the definition of o_n] and so $o_n < n < 2n - 2$ [since $n > 200$ (by the previous) and $o_n < n$ (by the hypothesis)]; consequently

$$o_n < 2n - 2 \qquad (1.1.1.0).$$

Inequality *(1.1.1.0)* immediately implies that $\mathcal{X}(\mathcal{FCO})(n) = \mathcal{X}(\mathcal{FCO})(n - 1)$; consequently $\mathcal{FCO}(n) = \mathcal{FCO}(n - 1)$ and therefore

$$o_n = o_{n-1} \qquad (1.1.1.1).$$

Equality *(1.1.1.1)* immediately implies that $o_{n,1} = o_{n-1,1}$. Property (1.1.1) follows.

(1.1.2.) If there exists an integer $y \geq 0$ such that $o_n < n - y$, clearly

$$n > y + 200 \qquad (1.1.2.0),$$

by using the definition of o_n. Now using inequality *(1.1.2.0)* and the fact that $o_n < n - y$, then it becomes immediate to deduce that $o_n < n - y < 2n - 2y - 2$ [since $n > y + 200$ (by *(1.1.2.0)*) and since y is an integer ≥ 0 such that $o_n < n - y$ (by the hypothesis)]; consequently

$$o_n < 2n - 2y - 2 \qquad (1.1.2.1).$$

Inequality *(1.1.2.1)* immediately implies that $\mathcal{X}(\mathcal{FCO})(n) = \mathcal{X}(\mathcal{FCO})(n - y - 1)$; consequently $\mathcal{FCO}(n) = \mathcal{FCO}(n - y - 1)$ and therefore

$$o_n = o_{n-y-1} \qquad (1.1.2.2).$$

Equality *(1.1.2.2)* immediately implies that $o_{n,1} = o_{n-y-1,1}$. Property (1.1.2) follows and Proposition 1.1 immediately follows. □

Using the definition of goldbach and the definition of g''_{n+1} (via the definition of g''_n) and the definition of $o_{n,1}$, then we have the following proposition.

Proposition 1.2. *We have the following five simple properties.*
(1.2.0) *For every integer $n \geq 1$, $2n + 2$ is goldbach if and only if $g''_{n+1} = 2n + 2$.*
(1.2.1) *The Goldbach conjecture holds if and only if for every integer $n \geq 1$, we have $g''_{n+1} = 2n + 2$.*
(1.2.2) *If $\lim_{n\to+\infty} 30o_{n,1} = +\infty$, then there are infinitely many Fermat composite numbers.*
(1.2.3) *If for every integer n of the form $n = 37k$ (where k is an integer ≥ 3), we have $30o_{n,1} > n$, then are infinitely many Fermat composite numbers.*
(1.2.4) *If for every integer n of the form $n = 37k$ (where k is an integer ≥ 3), we have $30o_{n,1} \geq g''_{n+1}$, then the Fermat composite conjecture is a special case of the Goldbach conjecture.*
Proof. Property (1.2.0) is immediate (it suffices to use the definition of g''_{n+1}, via the definiton of g''_n); property (1.2.1) is obvious (it suffices to use the definition of g''_{n+1} (via the definiton of g''_n) and the meaning of the Goldbach conjecture), and property (1.2.2) is immediate (indeed, it suffices to use the definition of $o_{n,1}$). Property (1.2.3) is also immediate (indeed, if for every integer n of the form $n = 37k$ (where k is an integer ≥ 3), we have $30o_{n,1} > n$, clearly $\lim_{n\to+\infty} 30o_{n,1} = +\infty$, and therefore there are infinitely many Fermat composite numbers (by using the previous equality and property (1.2.2)); and Property (1.2.4) is easy (indeed, suppose that the Goldbach conjecture holds, then (by using property (1.2.1)) we have $g''_{n+1} = 2n+2$ (for every integer $n \geq 1$); so for every integer n of the form $n = 37k$ (where k is an integer ≥ 3), we have $30o_{n,1} \geq g''_{n+1} > 2n > n$, and clearly

$30o_{n,1} > n,\ \ for\ every\ integer\ \ n\ \ of\ \ the\, form\, n = 37k, where\, k\, is\, an\, integer\ \geq 3$ *(1.2.4.0)*.

Consequently, there are infinitely many Fermat composite numbers (by using *(1.2.4.0)* and property (1.2.3)). Proposition 1.2 follows.□

Properties (1.2.4) and (1.2.3) of Proposition 1.2 clearly say that: if for every integer n of the form $n = 37k$ (where k is an integer ≥ 3), we have $30o_{n,1} \geq g''_{n+1}$ and $30o_{n,1} > n$, then the Fermat composite conjecture is an obvious special case of the Goldbach conjecture and there are infinitely many Fermat composite numbers. This is what we are going to do in Section *2*.

3. The Fermat composite conjecture is an obvious special case of the Goldbach conjecture, and there are infinitely many Fermat composite numbers

Here, using only simple definitions, elementary arithmetic congruences, elementary arithmetic calculus, elementary complex analysis, reasoning by reduction to absurd and properties 1.2.3 and 1.2.4 of Proposition 1.2, we prove a Theorem which implies that there are infinitely many Fermat composite numbers; and the Fermat composite conjecture that we solved, was only an elementary consequence of the Goldbach conjecture. In this Section, the definition of o_n and $o_{n,1}$ [see Section *1*] are *crucial.*

Definitions 2.0 *(Fundamental.1). We recall that θ is a complex number, if $\theta = x + iy$, where x and y are real, and where i is the complex entity satisfying $i^2 = -1$. Now let n be an integer ≥ 2 and let $o_{n,1}$; then $\phi(o_{n,1})$ is defined by the following simple equation:*

$$\phi(o_{n,1}) = in^2(\ o^3_{n,1} - o_{n,1} + 1\ ) + 48877n.$$

That being so, let n'; we say that n' is of type 37, if $n' = 37k$, where k is an integer ≥ 3. It is immediate that the previous definitions get sense; and it is also immediate that for every integer $n \geq 2$, $\phi(o_{n,1})$ exists and is well defined. We will see that the using of $\phi(o_{n,1})$ will crucify and will make surrender the Fermat composite conjecture.

Recall 2.1. We recall that r is a relative integer if r is an integer ≥ 0 or if r is an integer ≤ 0

Example 1. -11, -13, -108, 0, 7 and 24 are all relative integers. $\frac{51}{8}$ is not a relative integer.

Now let (u, v, W), where u and v are real, and W is an integer > 0; we recall that $u \equiv v\ mod(W)$ if and only if there exists a relative integer k such that $u - v = kW$

Example 2. Let n be of type 37 and let $o_{n,1}$. Then $o_{n,1} + 26 \equiv 0 \ mod(37)$. Moreover, if $o_{n,1} < n + 11$, then $o_{n,1} + 26 \leq n$.
Proof. Indeed consider the quantity $o_{n,1} + 26$; observing (by using property (1.1.0) of Proposition 1.1) that $o_{n,1} \equiv 11 \ mod(37)$, then the previous congruence immediately implies that

$$o_{n,1} + 26 \equiv 11 + 26 \ mod(37) \qquad (2.1.0).$$

Clearly $11 + 26 = 37$ and congruence *(2.1.0)* clearly says that

$$o_{n,1} + 26 \equiv 37 \ mod(37) \qquad (2.1.1).$$

Congruence *(2.1.1)* immediately implies that

$$o_{n,1} + 26 \equiv 0 \ mod(37) \qquad (2.1.2).$$

Now, to prove Example.2, it suffices to prove that if $o_{n,1} < n+11$, then $o_{n,1}+26 \leq n$. For that, let $n + 11$; recalling that n is of type 37, clearly (by using the definition of type 37), $n \equiv 0 \ mod(37)$ and using the pevious congruence, we immediately deduce that

$$n + 11 \equiv 0 + 11 \ mod(37) \qquad (2.1.3).$$

Congruence *(2.1.3)* clearly says that

$$n + 11 \equiv 11 \ mod(37) \qquad (2.1.4).$$

Now look at $o_{n,1}$ and observe (by using property (1.1.0) of Proposition 1.1) that

$$o_{n,1} \equiv 11 \ mod(37) \qquad (2.1.5).$$

That being so, if $o_{n,1} < n + 11$, then, using congruences *(2.1.5)* and *(2.1.4)*, it becomes trivial to deduce that the previous inequality immedialely implies that

$$o_{n,1} \leq n + 11 - 37 \qquad (2.1.6),$$

since $n + 11 \equiv 11 \ mod(37)$ *and* $o_{n,1} \equiv 11 \ mod(37)$ *and* $o_{n,1} < n + 11$. Inequality *(2.1.6)* clearly says that $o_{n,1} \leq n - 26$ and consequently $o_{n,1} + 26 \leq n$. Example.2 follows]. That being so, let $\theta = x + iy$ be a complex number (see definitions 2.0); we recall that x is called the real part of θ and y is called the imaginary part of θ. The real part of a complex number θ is denoted by $R[\theta]$ and the imaginary part of a complex number θ is denoted by $I[\theta]$.

Example 3. Let n be an integer ≥ 2 and let $o_{n,1}$. Now put

$$Z(n) = 1369i(o_{n,1}^3 - o_{n,1} + 1) - 1808449.$$

Then $R[Z(n)] + I[Z(n)] \equiv 0 \ mod(185)$.
Proof. Indeed, let $Z(n) = 1369i(o_{n,1}^3 - o_{n,1} + 1) - 1808449$ and look at $R[Z(n)]$ and $I[Z(n)]$; clearly

$$R[Z(n)] = -1808449 \ and \ I[Z(n)] = 1369(o_{n,1}^3 - o_{n,1} + 1) \qquad (2.1.7).$$

Now consider the quantity $R[Z(n)] + I[Z(n)]$; then, using *(2.1.7)*, it becomes immediate to deduce that

$$R[Z(n)] + I[Z(n)] = -1808449 + 1369(o_{n,1}^3 - o_{n,1} + 1) \qquad (2.1.8).$$

That being so, noticing [by using property (1.1.0) of Proposition 1.1] that $o_{n,1} \equiv 11 \ mod(185)$, then the previous congruence immediately implies that

$$-1808449+1369(o_{n,1}^3-o_{n,1}+1) \equiv -1808449+ 1369(11\times11\times11 - 11 +1) \ mod(185) \qquad (2.1.9).$$

Clearly $-1808449 + 1369(11 \times 11 \times 11 - 11 + 1) = -1808449 + 1369(1331 - 11 + 1) = -1808449 + 1369(1321) = -1808449 + 1808449 = 0$; so $-1808449+ 1369(11\times11\times11 - 11 +1) = 0$ and congruence *(2.1.9)* clearly says that

$$-1808449 + 1369(o_{n,1}^3 - o_{n,1} + 1) \equiv 0 \ mod(185) \qquad (2.1.10).$$

Now using congruence *(2.1.10)* and equality *(2.1.8)*, then it becomes trivial to deduce that $R[Z(n)] + I[Z(n)] \equiv 0 \ mod(185)$. Example.3 follows. □]

Now we are quasily ready to state a simple Theorem which implies that there are infinitely many Fermat composite numbers and the Fermat composite conjecture is an obvious special case of the Goldbach conjecture. But before, let us introduce.

Definition 2.2. *(Fundamental.2). We recall (see definitions* 2.0*) that n is of type* 37*, if n is an integer of the form $n = 37k$, where k is integer ≥ 3. That being so, we say that $q(n)$ is n-conform, if $q(n)$ is of the form $q(n) = q_0(n) + nq_1(n) + in^2q_2(n)$, where n is of type* 37 *and where for every $j \in \{0, 1, 2\}$, $q_j(n)$ is a complex number satisfying $q_j(n) = q_j(n-37)$. It is immediate that the previous definition gets sense.*

Now using definition 2.2, let us remark.

Remark 2.3. *We have the following two assertions (*2.3.0*) and (*2.3.1*).*
*(*2.3.0*). Let n be of type* 37*. Then* 0 *is n-conform.*
*(*2.3.1*). Let n be of type* 37 *and let $o_{n,1}$. Now look at equation $\phi(o_{n,1})$ [see*

definitions 2.0 *for the equation of* $\phi(o_{n,1})$*]. If* $o_{n,1} = o_{n-37,1}$*, then* $\phi(o_{n,1})$ *is* n*-conform.*
Proof. (2.3.0). Let n be of type 37 and look at 0; clearly $0 = q_0(n)+nq_1(n)+in^2q_2(n)$, where for every $j \in \{0, 1, 2\}$, we have $q_j(n) = 0$ (clearly $q_j(n) = q_j(n-37) = 0$, for every $j \in \{0, 1, 2\}$). Now using the previous , then it becomes trivial to deduce that 0 is of the form $0 = q_0(n)+nq_1(n)+in^2q_2(n)$, where n is of type 37 and where for every $j \in \{0, 1, 2\}$, $q_j(n)$ is a complex number satisfying $q_j(n) = q_j(n-37)$; so 0 is n-conform. Assertion (2.3.0) follows.
(2.3.1). Indeed, consider $\phi(o_{n,1})$ and observe (by using definitions 2.0) that

$$\phi(o_{n,1}) = in^2(o_{n,1}^3 - o_{n,1} + 1) + 48877n \qquad (2.3.1.0).$$

Now look at equation *(2.3.1.0)* and put

$$\phi_2(n) = o_{n,1}^3 - o_{n,1} + 1;\ \phi_1(n) = 48877;\ \textit{and}\ \phi_0(n) = 0 \qquad (2.3.1.1);$$

then, using *(2.3.1.1)*, it becomes trivial to deduce that equation *(2.3.1.0)* is of the form

$$\phi(o_{n,1}) = in^2\,\phi_2(n) + n\,\phi_1(n) + \phi_0(n) \qquad (2.3.1.2).$$

Using the three equalities of *(2.3.1.1)*, we immediately deduce that

$$\phi_2(n-37) = o_{n-37,1}^3 - o_{n-37,1} + 1;\ \phi_1(n-37) = 48877;\ \textit{and}\ \phi_0(n-37) = 0 \qquad (2.3.1.3).$$

Now noticing (via the hypotheses) that $o_{n,1} = o_{n-37,1}$, then it becomes trivial to deduce that *(2.3.1.3)* is of the form

$$\phi_2(n-37) = o_{n,1}^3 - o_{n,1} + 1;\ \phi_1(n-37) = 48877;\ \textit{and}\ \phi_0(n-37) = 0 \quad (2.3.1.4)$$

Using *(2.3.1.1)* and *(2.3.1.4)*, then we immediately deduce that

$$\phi_2(n) = \phi_2(n-37) = o_{n,1}^3 - o_{n,1} + 1;\ \phi_1(n) = \phi_1(n-37) = 48877;\ \textit{and}\ \phi_0(n) = \phi_0(n-37) = 0 \qquad (2.3.1.5).$$

That being so look at equation *(2.3.1.2)*, then, using *(2.3.1.5)*, it becomes trivial to deduce that equation *(2.3.1.2)* is of the form

$$\phi(o_{n,1}) = in^2\,\phi_2(n) + n\,\phi_1(n) + \phi_0(n) \qquad (2.3.1.6),$$

$$\textit{where for}\ j = 0, 1, 2,\ \phi_j(n)\ \textit{is a complex number satisfying}\ \phi_j(n) = \phi_j(n-37) \qquad (2.3.1.7).$$

Recalling (by the hypotheses) that n is of type 37, then it becomes trivial that *(2.3.1.6)* and *(2.3.1.7)* clearly say that

$\phi(o_{n,1})$ is n-conform. Assertion 2.3.1 follows and Remark 2.3 immediately follows. □

The following Proposition characterizes those which are n-conform.

Proposition 2.4 *(Fundamental.3: The unicity). Let n be of type 37 and let $q(n)$ be n-conform. We have the following two properties.*
(2.4.0). $q(n) = q_0(n) + nq_1(n) + in^2q_2(n)$ [where for every $j \in \{0, 1, 2\}$, $q_j(n)$ is a complex number satisfying $q_j(n) = q_j(n-37)$].
(2.4.1). If $q(n) = x_0(n) + nx_1(n) + in^2x_2(n)$ [where for every $j \in \{0, 1, 2\}$, $x_j(n)$ is a complex number satisfying $x_j(n) = x_j(n-37)$], then for every $j \in \{0, 1, 2\}$, we have $x_j(n) = q_j(n)$ [where $q_0(n)$, $q_1(n)$ and $q_2(n)$ are known, via property (2.4.0)].
Proof. (2.4.0). Immediate and follows by using definition 2.2.
(2.4.1). Indeed, look at property (2.4.0), and observe that

$$q(n) = q_0(n) + nq_1(n) + in^2q_2(n) \qquad (2.4.1.0),$$

where for every $j \in \{0, 1, 2\}$, $q_j(n)$ is a complex number satisfying $q_j(n) = q_j(n-37)$. Now, if $q(n)$ is also of the form

$$q(n) = x_0(n) + nx_1(n) + in^2x_2(n) \qquad (2.4.1.1),$$

where for every $j \in \{0, 1, 2\}$, $x_j(n)$ is a complex number satisfying $x_j(n) = x_j(n-37)$, then, using equations *(2.4.1.0)* and *(2.4.1.1)*, it becomes immediate to deduce that $0 = q_0(n) - x_0(n) + n(\, q_1(n) - x_1(n)\,) + in^2(\, q_2(n) - x_2(n)\,)$; consequently

$$x_0(n) - q_0(n) = n(\ q_1(n) - x_1(n)\ ) + in^2(\ q_2(n) - x_2(n)\ ) \qquad (2.4.1.2).$$

That being so, it becomes clear that:
to show property (2.4.1) it suffices to show that for every $j \in \{0, 1, 2\}$, we have

$$x_j(n) = q_j(n) \qquad (0.).$$

For that, observing [by using property (2.4.0)] that $q_0(n) = q_0(n-37)$, and noticing [by using the hypotheses on property (2.4.1)] that $x_0(n) = x_0(n-37)$, then using the previous two equalities, it becomes trivial to deduce that

$$x_0(n) - q_0(n) = x_0(n-37) - q_0(n-37) \qquad (2.4.1.3).$$

Now consider $n-37$ and look at $x_0(n-37) - q_0(n-37)$; then, using equation *(2.4.1.2)*, it becomes trivial to deduce that

$$x_0(n-37) - q_0(n-37) = (n-37)(q_1(n-37) - x_1(n-37)) + i(n-37)^2(q_2(n-37) - x_2(n-37)) \qquad (1.).$$

Using equality *(2.4.1.3)* and using equation (1.), then it becomes trivial to deduce that

$$x_0(n) - q_0(n) = x_0(n-37) - q_0(n-37) =$$
$$= (n-37)(q_1(n-37) - x_1(n-37)) + i(n-37)^2(q_2(n-37) - x_2(n-37)) \quad (2.4.1.4).$$

Now observing [by using again property (2.4.0)] that for every $j \in \{1, 2\}$, we have $q_j(n) = q_j(n-37)$, and noticing [by using the hypotheses on property (2.4.1)] that for every $j \in \{1, 2\}$, we have $x_j(n) = x_j(n-37)$, then, using the previous, it becomes immediate to see that we can write that
for every $j \in \{1, 2\}$, we have

$$q_j(n) = q_j(n-37) \text{and} x_j(n) = x_j(n-37) \quad (2.4.1.5).$$

Look at *(2.4.1.4)*; then using *(2.4.1.5)*, it becomes trivial to deduce that *(2.4.1.4)* clearly implies that
$x_0(n) - q_0(n) = x_0(n-37) - q_0(n-37) = (n-37)(q_1(n) - x_1(n)) + i(n-37)^2(q_2(n) - x_2(n))$, and consequently

$$x_0(n) - q_0(n) = (n-37)(q_1(n) - x_1(n)) + i(n-37)^2(q_2(n) - x_2(n)) \quad (2.4.1.6).$$

Now using equation *(2.4.1.2)*, then it becomes trivial to deduce that equation *(2.4.1.6)* clearly says that

$$n(q_1(n) - x_1(n)) + in^2(q_2(n) - x_2(n)) = (n-37)(q_1(n) - x_1(n)) + i(n-37)^2(q_2(n) - x_2(n)) \quad (2.4.1.7).$$

It is immediate to see that equation *(2.4.1.7)* implies that

$$i(\ n^2 - (n-37)^2\ )(\ q_2(n) - x_2(n)\ ) = -37(\ q_1(n) - x_1(n)\ ) \quad (2.4.1.8).$$

Clearly $i(\ n^2 - (n-37)^2\ ) = i(74n - 1369)$ and equation *(2.4.1.8)* clearly says that

$$i(74n - 1369)(\ q_2(n) - x_2(n)\ ) = -37(\ q_1(n) - x_1(n)\ ) \quad (2.4.1.9).$$

Consider the quantity $-37(\ q_1(n-37) - x_1(n-37)\ )$, then using equation *(2.4.1.9)*, it becomes trivial to deduce that

$$-37(\ q_1(n-37) - x_1(n-37)\ ) = i(\ 74(n-37) - 1369\ )(\ q_2(n-37) - x_2(n-37)\ ) \quad (2.4.1.10).$$

That being so, observing [by using property (2.4.0)] that $q_1(n) = q_1(n-37)$, and noticing [by using the hypotheses on property (2.4.1)] that $x_1(n) = x_1(n-37)$, then the previous two equalities imply that

$$q_1(n) - x_1(n) = q_1(n-37) - x_1(n-37) \qquad (2.4.1.11).$$

Using equality *(2.4.1.11)*, then it becomes immediate to deduce that equation *(2.4.1.10)* clearly says that

$$-37(\ q_1(n) - x_1(n)\ ) = i(\ 74(n-37) - 1369\ )(\ q_2(n-37) - x_2(n-37)\ ) \qquad (2.4.1.12).$$

Now using equations *(2.4.1.9)* and *(2.4.1.12)*, then it becomes trivial to deduce that

$$i(74n - 1369)(\ q_2(n) - x_2(n)\ ) = i(\ 74(n-37) - 1369\ )(\ q_2(n-37) - x_2(n-37)\ ) \qquad (2.4.1.13).$$

That being so, observing [by using property (2.4.0)] that $q_2(n) = q_2(n-37)$, and noticing [by using the hypotheses on property (2.4.1)] that $x_2(n) = x_2(n-37)$, then the previous two equalities imply that

$$q_2(n) - x_2(n) = q_2(n-37) - x_2(n-37) \qquad (2.4.1.14).$$

Using equality *(2.4.1.14)*, then it becomes trivial to deduce that equation *(2.4.1.13)* clearly says that

$$i(74n - 1369)(\ q_2(n) - x_2(n)\ ) = i(\ 74(n-37) - 1369\ )(\ q_2(n) - x_2(n)\ ) \qquad (2.4.1.15).$$

It is immediate to check that equation *(2.4.1.15)* clearly says that

$$i(\ 74n - 1369 - 74(n-37) + 1369\ )(\ q_2(n) - x_2(n)\ ) = 0 \qquad (2.4.1.16).$$

Clearly $i(\ 74n - 1369 - 74(n-37) + 1369\ ) = 2738i$ and equation *(2.4.1.16)* clearly says that

$$2738i(\ q_2(n) - x_2(n)\ ) = 0 \qquad (2.4.1.17).$$

Using equation *(2.4.1.17)*, it becomes trivial to deduce that

$$q_2(n) = x_2(n) \qquad (2.4.1.18).$$

Having proved that $q_2(n) = x_2(n)$ (use equality *(2.4.1.18)*), we are going now to show that $q_1(n) = x_1(n)$. For that, look at equation *(2.4.1.2)*; then using equality *(2.4.1.18)*, then it becomes immediate to deduce that equation *(2.4.1.2)* clearly say that

$$x_0(n) - q_0(n) = n(\ q_1(n) - x_1(n)\ ) + 0 \qquad (2.4.1.19).$$

Now look at $x_0(n-37) - q_0(n-37)$, then, using equation *(2.4.1.19)*, it becomes trivial to deduce that

$$x_0(n-37) - q_0(n-37) = (n-37)(\ q_1(n-37) - x_1(n-37)\ ) + 0 \qquad (2.4.1.20).$$

Observing [by using property (2.4.0)] that $q_0(n) = q_0(n-37)$, and noticing [by using the hypotheses on property (2.4.1)] that $x_0(n) = x_0(n-37)$, then, the previous two equalities immediately imply that

$$x_0(n) - q_0(n) = x_0(n-37) - q_0(n-37) \qquad (2.4.1.21).$$

Using equality *(2.4.1.21)*, then it becomes trivial to deduce that equation *(2.4.1.20)* clearly says that

$$x_0(n) - q_0(n) = (n-37)(\ q_1(n-37) - x_1(n-37)\ ) + 0 \qquad (2.4.1.22).$$

Observing [by using property (2.4.0)] that $q_1(n) = q_1(n-37)$, and noticing [by using the hypotheses on property (2.4.1)] that $x_1(n) = x_1(n-37)$, then, the previous two equalities immediately imply that

$$q_1(n) - x_1(n) = q_1(n-37) - x_1(n-37) \qquad (2.4.1.23).$$

Using equality *(2.4.1.23)*, then it becomes trivial to deduce that equation *(2.4.1.22)* clearly says that

$$x_0(n) - q_0(n) = (n-37)(\ q_1(n) - x_1(n)\ ) + 0 \qquad (2.4.1.24).$$

Now using equations *(2.4.1.19)* and *(2.4.1.24)*, then it becomes trivial to deduce that

$$n(\ q_1(n) - x_1(n)\ ) + 0 = (n-37)(\ q_1(n) - x_1(n)\ ) + 0 \qquad (2.4.1.25).$$

It is immediate to check that equation *(2.4.1.25)* clearly says that

$$(n - n + 37)(\,q_1(n) - x_1(n)\,) = 0 \qquad (2.4.1.26).$$

Clearl $n-n+37 = 37$ and equation *(2.4.1.26)* clearly says that $37(\,q_1(n)-x_1(n)\,) = 0$; consequently $q_1(n) - x_1(n) = 0$ and therefore

$$q_1(n) = x_1(n) \qquad (2.4.1.27).$$

Now using equalities *(2.4.1.18)* and *(2.4.1.27)*, then it becomes trivial to deduce that

$$q_2(n) = x_2(n)\, and\, q_1(n) = x_1(n) \qquad (2.4.1.28).$$

That being so, noticing (by *(2.4.1.28)*) that $q_2(n) = x_2(n)$ and $q_1(n) = x_1(n)$, then using (0.) [(0.) is situated above equality *(2.4.1.3)*], it becomes trivial to see that to show property (2.4.1), it suffices to show that $q_0(n) = x_0(n)$. For that, look at equation *(2.4.1.2)*; then using the two equalities of *(2.4.1.28)*, it becomes trivial to deduce that $q_0(n) = x_0(n)$. Property (2.4.1) follows, and Proposition 2.4 immediately follows. □

Using Proposition 2.4, then the following remark becomes trivial.

Remark 2.5 *Let* n *be of type* 37 *and let* $q(n)$ *be* n*-conform. Then* $q(n) = q_0(n) + nq_1(n) + in^2q_2(n)$ *(where for every* $j \in \{0, 1, 2\}$, $q_j(n)$ *is a complex number satisfying* $q_j(n) = q_j(n-37)$*), and the previous writing is unique, once* n *is fixed.*
Proof. Immediate (indeed, it is a trivial consequence of the two properties of Proposition 2.4). □

Using Remark 2.5, let us define.
Definitions 2.6 *(Fundamental.4). Let* n *be of type* 37 *and let* $q(n)$ *be* n*-conform; clearly (by Remark* 2.5*), we know that*

$$q(n) = q_0(n) + nq_1(n) + in^2q_2(n),$$

where for every $j \in \{0, 1, 2\}$, $q_j(n)$ *is a complex number satisfying* $q_j(n) = q_j(n-37)$*. Then* $q_0(n)$ *is called the fixed part of* $q(n)$ *and is denoted by* $Fix[q(n)]$ *; and* $nq_1(n) + in^2q_2(n)$ *is called the complex part of* $q(n)$ *and is denoted by* $C[q(n)]$*.* Using Remark 2.5, then it becomes trivial to see that the previous definitions and denotations get sense, since $q(n)$ is unique, once n is fixed.

We will only use $Fix[q(n)]$, where n is of type 37, and where $q(n)$ is n-conform [using Remark 2.5, then it becomes trivial to see that $Fix[q(n)]$ is unique, once n is fixed]. That being so, using definitions 2.6, let us remark.

Remark 2.7. *Let* n *be of type* 37 *and let* $y(n)$ *be* n*-conform. Now let* $z(n)$ *be* n*-conform. We have the following two elementary properties.*
(2.7.0). $y(n) = y_0(n) + ny_1(n) + in^2y_2(n)$ *[where for every* $j \in \{0, 1, 2\}$, $y_j(n)$ *is a complex number satisfying* $y_j(n) = y_j(n-37)$*] and* $Fix[y(n)] = y_0(n)$ *;*$z(n) = z_0(n) + nz_1(n) + in^2z_2(n)$ *[where for every* $j \in \{0, 1, 2\}$, $z_j(n)$ *is a complex number satisfying* $z_j(n) = z_j(n-37)$*] and* $Fix[z(n)] = z_0(n)$*.*
(2.7.1). $y(n) + z(n)$ *is* n*-conform and* $Fix[y(n) + z(n)] = Fix[y(n)] + Fix[z(n)]$*.*
Proof. (2.7.0). Immediate, by observing that $y(n)$ is n-conform and $z(n)$ is n-conform, and by using Remark 2.5 and definitions 2.6.
(2.7.1). Immediate. Indeed let $y(n) + z(n)$, then, using property (2.7.0), it becomes trivial to deduce that

$$y(n)+z(n)=y_0(n)+z_0(n)+n(\ y_1(n)+z_1(n)\ )+in^2(\ y_2(n)+z_2(n)\ )\quad (2.7.1.0),$$

where for every $j=0,1,2$, $y_j(n)+z_j(n)$ is a complex number satisfying

$$y_j(n)+z_j(n)=y_j(n-37)+z_j(n-37)\qquad (2.7.1.1).$$

Recalling (via the hypotheses) that n is of type 37, then it becomes trivial to see that *(2.7.1.0)* and *(2.7.1.1)* clearly say that $y(n)+z(n)$ is n-conform and

$$Fix[y(n)+z(n)]=y_0(n)+z_0(n)\qquad (2.7.1.2).$$

Looking at $y(n)$ and $z(n)$, then using property (2.7.0), it becomes trivial to see that $Fix[y(n)]=y_0(n)$ and $Fix[z(n)]=z_0(n)$; now, using the previous two equalities, then *(2.7.1.2)* clearly says that $y(n)+z(n)$ is n-conform and $Fix[y(n)+z(n)]=Fix[y(n)]+Fix[z(n)]$. Property (2.7.1) follows and Proposition 2.7 immediately follows. □

Before stating Theorem which implies that there are infinitely many Fermat composite numbers, let us give two simple remarks which are only elementary using of the parameter $Fix[.]$ introduced in definitions 2.6.

Remark 2.8. *(The first elementary using of the parameter $Fix[.]$). Let n be of type 37 and let $o_{n,1}$; now put*

$$Q(n)=i(o_{n,1}+26)^2(o_{n,1}^3-o_{n,1}+1)+48877(o_{n,1}+26).$$

If $o_{n,1}=o_{n-37,1}$, then $Q(n)$ is n-conform and $Fix[Q(n)]=Q(n)$.

Proof. Indeed, observe (by using the definition of $Q(n)$ given above) that

$$Q(n)=i(o_{n,1}+26)^2(o_{n,1}^3-o_{n,1}+1)+48877(o_{n,1}+26)\qquad (2.8.0).$$

Now look at equation *(2.8.0)* and put

$$\phi_2(n)=0;\ \phi_1(n)=0;\ \textit{and}\ \phi_0(n)=i(o_{n,1}+26)^2(o_{n,1}^3-o_{n,1}+1)+48877(o_{n,1}+26)\qquad (2.8.1);$$

then, using *(2.8.1)*, it becomes trivial to deduce that equation *(2.8.0)* is of the form

$$Q(n)=in^2\,\phi_2(n)+n\,\phi_1(n)+\ \phi_0(n)\qquad (2.8.2).$$

Using the three equalities of *(2.8.1)*, we immediately deduce that

$$\phi_2(n-37)=0;\phi_1(n-37)=0;$$

and

$$\phi_0(n-37)=i(o_{n-37,1}+26)^2(o_{n-37,1}^3-o_{n-37,1}+1)+48877(o_{n-37,1}+26)\quad (2.8.3).$$

Now noticing (via the hypotheses) that $o_{n,1} = o_{n-37,1}$, then it becomes trivial to deduce that *(2.8.3)* clearly says that

$$\phi_2(n-37) = 0; \phi_1(n-37) = 0;$$

and

$$\phi_0(n-37) = i(o_{n,1}+26)^2(o_{n,1}^3 - o_{n,1} + 1) + 48877(o_{n,1}+26) \qquad (2.8.4)$$

Using *(2.8.1)* and *(2.8.4)*, then we immediately deduce that

$$\phi_2(n) = \phi_2(n-37);\ \phi_1(n) = \phi_1(n-37); \textit{and}\ \phi_0(n) = \phi_0(n-37) \qquad (2.8.5).$$

That being so look at equation *(2.8.2)*, then, using *(2.8.5)*, it becomes trivial to deduce that equation *(2.8.2)* is of the form

$$Q(n) = in^2\,\phi_2(n) + n\,\phi_1(n) + \phi_0(n) \qquad (2.8.6),$$

where for $j = 0, 1, 2$, $\phi_j(n)$ is a complex number satisfying

$$\phi_j(n) = \phi_j(n-37) \qquad (2.8.7).$$

Recalling (by the hypotheses) that n is of type 37, then it becomes trivial that *(2.8.6)* and *(2.8.7)* clearly say that

$$Q(n)\,is\,n-conform\,and\,Fix[Q(n)] = \phi_0(n) \qquad (2.8.8).$$

Now recalling (by using *(2.8.1)*) that $\phi_0(n) = i(o_{n,1}+26)^2(o_{n,1}^3 - o_{n,1} + 1) + 48877(o_{n,1}+26)$ and remarking [by using equation *(2.8.0)*] that $Q(n) = i(o_{n,1}+26)^2(o_{n,1}^3 - o_{n,1} + 1) + 48877(o_{n,1}+26)$, then using the previous two equations, it becomes trivial to deduce that *(2.8.8)* clearly says that $Q(n)$ is n-conform and $Fix[Q(n)] = Q(n)$. Remark 2.8 follows. □

Remark 2.9 *(The second elementary using of the parameter $Fix[.]$). Let n be of type* 37 *and let $o_{n,1}$. Now define $q(n)$ as follows.*

$$q(n) = i(9o_{n,1} - 25)^2(\,o_{n,1}^3 - o_{n,1} + 1) + 48877(9o_{n,1} - 25).$$

If $o_{n,1} = o_{n-37,1}$, then $q(n)$ is n-conform and $Fix[q(n)] = q(n)$.

Proof. Simple and rigorously analogous to the proof of Remark 2.8. □

The previous simple remarks made, now the following Theorem immediately implies that there are infinitely many Fermat composite numbers and the Fermat composite conjecture is only an obvious special case of the Goldbach conjecture.

Theorem 2.10. *Let n be of type 37 and let $o_{n,1}$. Then, at least one of the following two properties (2.10.0) and (2.10.1) is satisfied by n.*
(2.10.0.) $o_{n,1} \geq n+11$.
(2.10.1) $R[Fix[\phi(o_{n,1})]] + I[Fix[\phi(o_{n,1})]] \not\equiv 74 \; mod(185)$ and $n < 12 + 9o_{n,1}$ and $\phi(o_{n,1})$ is n-conform (see definitions 2.0 for the equation of $\phi(o_{n,1})$, and we recall (see Recall 2.1) that $R[Fix[\phi(o_{n,1})]]$ is the real part of $Fix[\phi(o_{n,1})]$ and $I[Fix[\phi(o_{n,1})]]$ is the imaginary part of $Fix[\phi(o_{n,1})]$).

That being so, to prove simply Theorem 2.10, we use the following fundamental proposition.

Proposition 2.11. *(Fundamental.5). Let n be of type 37 and let $o_{n,1}$; look at $\phi(o_{n,1})$ [see definitions 2.0 for the equation of $\phi(o_{n,1})$], and put*

$$Z'(n) = i(-74n + 1369)(o_{n,1}^3 - o_{n,1} + 1) - 1808449.$$

Now via $\phi(o_{n,1})$, consider $\phi(o_{n-37,1})$ [this consideration gets sense, since n is of type 37; in particular $n - 37 > 2$ and therefore $o_{n-37,1}$ clearly exists]. If $o_{n,1} = o_{n-37,1}$, then we have the following two properties.
(2.11.0). $\phi(o_{n-37,1}) = \phi(o_{n,1}) + Z'(n)$.
(2.11.1). $Z'(n)$ is n-conform and $Fix[Z'(n)] = 1369i(o_{n,1}^3 - o_{n,1} + 1) - 1808449$.
Proof. (2.11.0). Indeed, by the equation of $\phi(o_{n,1})$ (see definitions 2.0), we clearly have

$$\phi(o_{n,1}) = in^2(o_{n,1}^3 - o_{n,1} + 1) + 48877n \qquad (2.11.0.0).$$

Now consider $\phi(o_{n-37,1})$; then using equation *(2.11.0.0)* , we immediately deduce that

$$\phi(o_{n-37,1}) = i(n-37)^2(o_{n-37,1}^3 - o_{n-37,1} + 1) + 48877(n-37) \qquad (2.11.0.1).$$

Noticing [via the hypotheses] that $o_{n,1} = o_{n-37,1}$, then it becomes immediate to deduce that equation *(2.11.0.1)* is of the form

$$\phi(o_{n-37,1}) = i(n-37)^2(o_{n,1}^3 - o_{n,1} + 1) + 48877(n-37) \qquad (2.11.0.2).$$

Now look at $i(n-37)^2(o_{n,1}^3 - o_{n,1} + 1) + 48877(n-37)$; it is trivial to check (by elementary calculation) that

$$i(n-37)^2(o_{n,1}^3 - o_{n,1} + 1) + 48877(n-37) = in^2(o_{n,1}^3 - o_{n,1} + 1) + 48877n + Z'(n) \qquad (2.11.0.3),$$

where $Z'(n) = i(-74n + 1369)(o_{n,1}^3 - o_{n,1} + 1) - 1808449$. That being so, look at $\phi(o_{n-37,1})$; then, using *(2.11.0.3)* and *(2.11.0.2)*, it becomes trivial to deduce that

$$\phi(o_{n-37,1}) = in^2(o_{n,1}^3 - o_{n,1} + 1) + 48877n + Z'(n) \qquad (2.11.0.4).$$

Now using equations *(2.11.0.0)* and *(2.11.0.4)*, then it becomes immediate to deduce that $\phi(o_{n-37,1}) = \phi(o_{n,1}) + Z'(n)$. Property (2.11.0) follows.
(2.11.1). Indeed, recall [by using the definition of $Z'(n)$] that

$$Z'(n) = i(-74n + 1369)(o_{n,1}^3 - o_{n,1} + 1) - 1808449 \qquad (2.11.1.0).$$

It is trivial to see that equation *(2.11.1.0)* is of the form

$$Z'(n) = -74in(o_{n,1}^3 - o_{n,1} + 1) + 1369i(o_{n,1}^3 - o_{n,1} + 1) - 1808449 \qquad (2.11.1.1).$$

Now put

$$\phi_2(n) = 0;\ \phi_1(n) = -74i(o_{n,1}^3 - o_{n,1} + 1);\ and\ \phi_0(n) = 1369i(o_{n,1}^3 - o_{n,1} + 1) - 1808449 \qquad (2.11.1.2);$$

then, using *(2.11.1.2)*, it becomes trivial to deduce that equation *(2.11.1.1)* is of the form

$$Z'(n) = in^2\,\phi_2(n) + n\,\phi_1(n) + \phi_0(n) \qquad (2.11.1.3).$$

Consider $n - 37$; then, using the three equalities of *(2.11.1.2)*, we immediately deduce that

$$\phi_2(n-37) = 0; \phi_1(n-37) = -74i(o_{n-37,1}^3 - o_{n-37,1} + 1)$$

and

$$\phi_0(n-37) = 1369i(o_{n-37,1}^3 - o_{n-37,1} + 1) - 1808449 \qquad (2.11.1.4).$$

Now observe (via the hypotheses) that

$$o_{n,1} = o_{n-37,1} \qquad (2.11.1.5).$$

Look at the three equalities of *(2.11.1.4)*, then using equality *(2.11.1.5)*, it becomes trivial to deduce that *(2.11.1.4)* is of the form

$$\phi_2(n-37) = 0; \phi_1(n-37) = -74i(o_{n,1}^3 - o_{n,1} + 1)$$

and

$$\phi_0(n-37) = 1369i(o_{n,1}^3 - o_{n,1} + 1) - 1808449 \qquad (2.11.1.6).$$

Now using *(2.11.1.2)* and *(2.11.1.6)*, then we immediately deduce that

$$\phi_2(n) = \phi_2(n-37);\ \phi_1(n) = \phi_1(n-37);\ and\ \ \phi_0(n) = \phi_0(n-37) \qquad (2.11.1.7).$$

That being so look at equation *(2.11.1.3)*, then, using *(2.11.1.7)*, it becomes trivial to deduce that equation *(2.11.1.3)* is of the form

$$Z'(n) = in^2\,\phi_2(n) + n\,\phi_1(n) + \phi_0(n) \qquad (2.11.1.8),$$

where for $j = 0, 1, 2$, $\phi_j(n)$ is a complex number satisfying

$$\phi_j(n) = \phi_j(n - 37) \qquad (2.11.1.9).$$

Recalling (by using the hypotheses) that n is of type 37, then it becomes trivial that *(2.11.1.8)* and *(2.11.1.9)* clearly say that

$$Z'(n)\text{is n-conform and} Fix[Z'(n)] = \phi_0(n) \qquad (2.11.1.10).$$

Now recalling (by using *(2.11.1.2)*) that $\phi_0(n) = 1369i(o_{n,1}^3 - o_{n,1} + 1) - 1808449$, then it becomes trivial to deduce that *(2.11.1.10)* clearly says that $Z'(n)$ is n-conform and $Fix[Z'(n)] = 1369i(o_{n,1}^3 - o_{n,1} + 1) - 1808449$. Property (2.11.1) follows and Proposition 2.11 immediately follows. □

That being so, we will also use the following four elementary Propositions.

Proposition 2.12. *Let n be an integer ≥ 3 and let $o_{n,1}$. Define $Q(n)$ and $q(n)$ as follows.*

$$Q(n) = i(o_{n,1} + 26)^2(o_{n,1}^3 - o_{n,1} + 1) + 48877(o_{n,1} + 26);$$

and

$$q(n) = i(9o_{n,1} - 25)^2(\,o_{n,1}^3 - o_{n,1} + 1) + 48877(9o_{n,1} - 25).$$

Then we have the following two properties.
*(*2.12.0.*)* $R[Q(n)] + I[Q(n)] \not\equiv 74\ mod(185)$.
*(*2.12.1.*)* $R[q(n)] + I[q(n)] \equiv 74\ mod(185)$.

Proof. Property (2.12.0) is simple [indeed, look at $R[Q(n)]$ and $I[Q(n)]$ (we recall that $R[Q(n)]$ is the real part of $Q(n)$ and $I[Q(n)]$ is the imaginary part of $Q(n)$); recalling that $Q(n) = i(o_{n,1} + 26)^2(o_{n,1}^3 - o_{n,1} + 1) + 48877(o_{n,1} + 26)$ and using the previous equation, then it becomes trivial to deduce that

$$R[Q(n)] = 48877(o_{n,1}+26) \text{ and } I[Q(n)] = (o_{n,1}+26)^2(o_{n,1}^3 - o_{n,1}+1) \qquad (2.12.0.0).$$

Using *(2.12.0.0)*, then it becomes immediate to deduce that

$$R[Q(n)] + I[Q(n)] = 48877(o_{n,1} + 26) + (o_{n,1} + 26)^2(o_{n,1}^3 - o_{n,1} + 1) \qquad (2.12.0.1).$$

Now look at the quantity $48877(o_{n,1} + 26) + (o_{n,1} + 26)^2(o_{n,1}^3 - o_{n,1} + 1)$; noticing [by using property (1.1.0) of Proposition 1.1] that $o_{n,1} \equiv 11\ mod(185)$, then the previous congruence immediately implies that

$$48877(o_{n,1}+26)+(o_{n,1}+26)^2(o_{n,1}^3-o_{n,1}+1)$$
$$\equiv 48877(11+26)+(11+26)^2(11\times 11\times 11-11+1) mod(185) \qquad (2.12.0.2).$$

Clearly $48877(11+26)+(11+26)^2(11\times 11\times 11-11+1)=48877(11+26)+(11+26)^2(1321)=1808449+1808449=3616898$; so $48877(11+26)+(11+26)^2(11\times 11\times 11-11+1)=3616898$ and congruence *(2.12.0.2)* clearly says that

$$48877(o_{n,1}+26)+(o_{n,1}+26)^2(o_{n,1}^3-o_{n,1}+1) \equiv 3616898 \ mod(185) \quad (2.12.0.3).$$

Clearly

$$3616898 \equiv 148 \ mod(185) \qquad (2.12.0.4),$$

since $3616898-148=19550\times 185$. Now using congruences *(2.12.0.3)* and *(2.12.0.4)*, then it becomes trivial to deduce that

$$48877(o_{n,1}+26)+(o_{n,1}+26)^2(o_{n,1}^3-o_{n,1}+1) \equiv 148 \ mod(185) \qquad (2.12.0.5).$$

It is trivial to deduce that congruence *(2.12.0.5)* implies that

$$48877(o_{n,1}+26)+(o_{n,1}+26)^2(o_{n,1}^3-o_{n,1}+1) \not\equiv 74 \ mod(185) \qquad (2.12.0.6).$$

That being so, using *(2.12.0.6)* and *(2.12.0.1)*, then it becomes trivial to deduce that $R[Q(n)]+I[Q(n)] \not\equiv 74 \ mod(185)$. Property (2.12.0) follows.]. Property (2.12.1) is also simple [indeed look at $R[q(n)]$ and $I[q(n)]$ (we recall that $R[q(n)]$ is the real part of $q(n)$ and $I[q(n)]$ is the imaginary part of $q(n)$); recalling that $q(n)=i(9o_{n,1}-25)^2(o_{n,1}^3-o_{n,1}+1)+48877(9o_{n,1}-25)$ and using the previous equation, then it becomes trivial to deduce that

$$R[q(n)]=48877(9o_{n,1}-25) \ and \ I[q(n)]=(9o_{n,1}-25)^2(o_{n,1}^3-o_{n,1}+1) \quad (2.12.1.0).$$

Using *(2.12.1.0)*, then it becomes immediate to deduce that

$$R[q(n)]+I[q(n)]=48877(9o_{n,1}-25)+(9o_{n,1}-25)^2(o_{n,1}^3-o_{n,1}+1) \quad (2.12.1.1).$$

Now look at the quantity $48877(9o_{n,1}-25)+(9o_{n,1}-25)^2(o_{n,1}^3-o_{n,1}+1)$; noticing [by using property (1.1.0) of Proposition 1.1] that $o_{n,1} \equiv 11 \ mod(185)$, then the previous congruence immediately implies that

$$48877(9o_{n,1}-25)+(9o_{n,1}-25)^2(o_{n,1}^3-o_{n,1}+1)$$
$$\equiv 48877(99-25)+(99-25)^2(1331-11+1) \ mod(185) \qquad (2.12.1.2).$$

Clearly $48877(99 - 25) + (99 - 25)^2(1331, -11 + 1) = 48877(99 - 25) + (99 - 25)^2(1321) = 3616898 + 7233796 = 10850694$; so $48877(99 - 25) + (99 - 25)^2(1331 - 11 + 1) = 10850694$ and congruence *(2.12.1.2)* clearly says that

$$48877(9o_{n,1}-25)+(9o_{n,1}-25)^2(o_{n,1}^3-o_{n,1}+1) \equiv 10850694 \ mod(185) \quad (2.12.1.3).$$

Clearly

$$10850694 \equiv 74 \ mod(185) \qquad (2.12.1.4),$$

since $10850694 - 74 = 58652 \times 185$. Now using congruences *(2.12.1.3)* and *(2.12.1.4)*, then it becomes trivial to deduce that

$$48877(9o_{n,1} - 25) + (9o_{n,1} - 25)^2(o_{n,1}^3 - o_{n,1} + 1) \equiv 74 \ mod(185) \quad (2.12.1.5).$$

That being so, using *(2.12.1.5)* and *(2.12.1.1)*, then it becomes trivial to deduce that $R[q(n)] + I[q(n)] \equiv 74 \ mod(185)$. Property (2.12.1) follows and Proposition 2.12 immediatelly follows.]. □

Proposition 2.13. *Let n be of type 37 and let $o_{n,1}$; we have the following five properties.*

(2.13.0.)*If $o_{n,1} \geq n + 11$, then Theorem* 2.10 *is satisfied by n.*

(2.13.1.)*If $n \leq 185 \times F_5^{F_5}$, then Theorem* 2.10 *is satisfied by n.*

(2.13.2.) *If $o_n \geq n - 74$* [see section.1 for the meaning of o_n] *then Theorem* 2.10 *is satisfied by n.*

(2.13.3.)*If $n = o_{n,1} + 26$ and if $o_{n,1} = o_{n-37,1}$, then $\phi(o_{n,1})$ is n-conform and $n < 12 + 9o_{n,1}$ and $R[Fix[\phi(o_{n,1})]] + I[Fix[\phi(o_{n,1})]] \not\equiv 74 \ mod(185)$.*

(2.13.4.) *If $n = o_{n,1} + 26$ and $o_{n,1} = o_{n-37,1}$, then Theorem* 2.10 *is satisfied by n.*

Proof. Property (2.13.0) is immediate [indeed, let n be an integer of type 37; if $o_{n,1} \geq n + 11$, then property (2.10.0) of Theorem 2.10 is clearly satisfied by n; therefore Theorem 2.10 is satisfied by n]. Property (2.13.1) is also immediate [indeed observing (by using property (1.1.0) of Proposition 1.1) that

$$o_{n,1} > 10 + 185F_5^{F_5} \qquad (2.13.1.0),$$

if $n \leq 185F_5^{F_5}$, then, using *(2.13.1.0)*, we immediately deduce that

$$o_{n,1} \geq n + 11 \qquad (2.13.1.1).$$

Therefore Theorem 2.10 is satisfied by n , by using *(2.13.1.1)* and property (2.13.0). Property (2.13.1) follows]. Property (2.13.2) is easy [indeed, observe (by using property (1.1.0) of Proposition 1.1) that

$$F_5 - 1 < o_n < o_{n,1} \text{ and } o_{n,1} = 11 + 185o_n^{o_n} \qquad (2.13.2.0).$$

That being so, if $o_n \geq n - 74$ (recall that n is of type 37; so $n = 37k$ where k is an integer ≥ 3 and therefore $n \geq 111$), then, using *(2.13.2.0)*, it becomes immediate to deduce that $o_{n,1} > 10 + 185o_n^{o_n} \geq 10 + 185(n-74)^{n-74} > n + 11$; consequently

$$o_{n,1} > n + 11 \qquad (2.13.2.1).$$

Therefore Theorem 2.10 is satisfied by n , by using *(2.13.2.1)* and property (2.13.0). Property (2.13.2) follows]. Property (2.13.3) is trivial [indeed, let $\phi(o_{n,1})$; then (by using definitions 2.0), we clearly have

$$\phi(o_{n,1}) = in^2(o_{n,1}^3 - o_{n,1} + 1) + 48877n \qquad (2.13.3.0).$$

Since $n = o_{n,1} + 26$, then it becomes immediate to deduce that equation (2.13.3.0) is of the form

$$\phi(o_{n,1}) = i(o_{n,1} + 26)^2(o_{n,1}^3 - o_{n,1} + 1) + 48877(o_{n,1} + 26) \qquad (2.13.3.1).$$

Look at equation *(2.13.3.1)*; observing (by the hypotheses) that $n = o_{n,1} + 26$ and $o_{n,1} = o_{n-37,1}$, then using Remark 2.8, it becomes trivial to deduce that

$$\phi(o_{n,1})\ is\ n-conform\ and\ \ Fix[\phi(o_{n,1})] = \phi(o_{n,1}) \qquad (2.13.3.2),$$

since $Q(n) = \phi(o_{n,1})$, by using equation *(2.13.3.1)* and the equation of $Q(n)$ given in Remark 2.8. Now using equation *(2.13.3.1)* and property (2.12.0) of Proposition 2.12, then it becomes trivial to deduce that

$$R[\phi(o_{n,1})] + I[\phi(o_{n,1})] \not\equiv 74\ mod(185) \qquad (2.13.3.3),$$

since $Q(n) = \phi(o_{n,1})$, by using equation *(2.13.3.1)* and the equation of $Q(n)$ given in Proposition 2.12. Observing (by using *(2.13.3.2)*) that $Fix[\phi(o_{n,1})] = \phi(o_{n,1})$, then it becomes trivial to deduce that *(2.13.3.3)* clearly says that

$$R[Fix[\phi(o_{n,1})]] + I[Fix[\phi(o_{n,1})]] \not\equiv 74\ mod(185) \qquad (2.13.3.4).$$

Now noticing (by using again *(2.13.3.2)*) that $\phi(o_{n,1})$ is n-conform , then, using *(2.13.3.4)* and the previous, we immediately deduce that

$$\phi(o_{n,1})\text{is n-conform and} R[Fix[\phi(o_{n,1})]] + I[Fix[\phi(o_{n,1})]] \not\equiv 74\ mod(185) \qquad (2.13.3.5).$$

That being so, noticing (by using property (1.1.0) of Proposition 1.1) that $o_{n,1} > 10 + 185F_5^{F_5}$ and recalling (via the hypotheses) that $n = o_{n,1} + 26$, clearly $n < 12 + 9o_{n,1}$; now using the previous inequality and using *(2.13.3.5)*, then we immedialetly deduce that $\phi(o_{n,1})$ is n-conform and $R[Fix[\phi(o_{n,1})]] +$

$I[Fix[\phi(o_{n,1})]] \not\equiv 74\ mod(185)$ and $n < 12 + 9o_{n,1}$. Property (2.13.3) follows]. Property (2.13.4) is immediate [indeed, if $n = o_{n,1} + 26$ and $o_{n,1} = o_{n-37,1}$, then, using property (2.13.3), we easily deduce that $\phi(o_{n,1})$ is n-conform and $R[Fix[\phi(o_{n,1})]] + I[Fix[\phi(o_{n,1})]] \not\equiv 74\ mod(185)$ and $n < 12 + 9o_{n,1}$; the previous clearly implies that property (2.10.1) of Theorem 2.10 is satisfied by n; therefore Theorem 2.10 is satisfied by n. Property (2.13.4) follows and Proposition 2.13 immediately follows]. □

From Proposition 2.13, it comes:

Proposition 2.14. *We have the following two assertions (*2.14.0*) and (*2.14.1*).*
*(*2.14.0*).*
Suppose that Theorem 2.10 *is false. Then there exists an integer n of type* 37 *such that n is a minimum counter-example to Theorem* 2.10.
*(*2.14.1*).Suppose that Theorem* 2.10 *is false, and let n be an integer of type* 37 *such that n is a minimum counter-example to Theorem* 2.10 [such a n exists, by assertion (2.14.0)]; look at $o_{n,1}$. *Then, the following two properties (*2.14.1.0*) and (*2.14.1.1*) are simultaneously satisfied by $o_{n,1}$.*

*(*2.14.1.0*).* $o_{n,1} < n + 11$.

*(*2.14.1.1*).* $\phi(o_{n,1})$ *is not n-conform or* $R[Fix[\phi(o_{n,1})]] + I[Fix[\phi(o_{n,1})]] \equiv 74\ mod(185)$ *or* $n \geq 12 + 9o_{n,1}$.

Proof. Assertion (2.14.0) is immediate and assertion (2.14.1) immediately results from assertion (2.14.0).□

Using Proposition 2.14, then the following fundamental definition immediately comes:

Definition 2.15 *(Fundamental.6). Let n be an integer ≥ 3 and let $o_{n,1}$. We say that $o_{n,1}$ is a remarkable element , if the following three assertions (*2.15.0*),* (2.15.1*) and* (2.15.2*) are simultaneously satisfied.*

*(*2.15.0*).* *n is of type* 37 *and is a minimum counter-example to Theorem* 2.10.

*(*2.15.1*).* $o_{n,1} < n + 11$.

*(*2.15.2*).* $\phi(o_{n,1})$ *is not n-conform or* $R[Fix[\phi(o_{n,1})]] + I[Fix[\phi(o_{n,1})]] \equiv 74\ mod(185)$ *or* $n \geq 12 + 9o_{n,1}$.□

It is immediate to see that if Theorem 2.10 is false, then there exists n and there exists *a remarkable element* $o_{n,1}$, by using Proposition 2.14.

Proposition 2.16 (*Application of Proposition* 2.14 *and definition* 2.15). *Suppose that Theorem* 2.10 *is false; and let $o_{n,1}$ be a remarkable element [such a $o_{n,1}$ exists, by using Proposition* 2.14 *and definition* 2.15*]. Fix once and for all $o_{n,1}$, and look at o_n [see Section.1 for the meaning of o_n]. Then we have the following four properties.*

(2.16.0.) $o_{n,1} \leq n+10$ *and* $n \geq 11 + 185F_5^{F_5}$ *and* $o_{n-37,1}$ *exists and* $n-37$ *is of type* 37.
(2.16.1.) $o_n < n-74$.
(2.16.2.) $o_{n,1} = o_{n-37,1}$.
(2.16.3.) $o_{n,1} \leq n-63$.
Proof. (2.16.0). $o_{n,1} \leq n+10$. Otherwise [we reason by reduction to absurd] $o_{n,1} > n+10$; observing that $o_{n,1}$ and $n+10$ are integers, then the previous inequality implies that $o_{n,1} \geq n+11$, and we have a contradiction, since $o_{n,1}$ is a remarkable element [see definition 2.15 for the meaning of a remarkable element]. Having proved this fact, we have $n \geq 11 + 185F_5^{F_5}$. Otherwise [we reason by reduction to absurd] clearly

$$n < 11 + 185F_5^{F_5} \qquad (2.16.0.0).$$

Now observing that $n \equiv 0 \ mod(37)$ (since n is of type 37), and remarking that $11 + 185F_5^{F_5} \equiv 11 \ mod(37)$; then, using the previous two congruences, it becomes trivial to deduce that inequality *(2.16.0.0)* implies that

$$n \leq 11 + 185F_5^{F_5} - 11 \qquad (2.16.0.1).$$

Clearly $11 + 185F_5^{F_5} - 11 = 185F_5^{F_5}$ and inequality *(2.16.0.1)* clearly says that

$$n \leq 185F_5^{F_5} \qquad (2.16.0.2).$$

Now using the previous inequality and property (2.13.1) of Proposition 2.13, then we immediately deduce that Theorem 2.10 is satisfied by n, and we have a contradiction, since $o_{n,1}$ is a remarkable element, and in particular n is a minimum counter-example to Theorem 2.10. So $n \geq 11 + 185F_5^{F_5}$. That being so, to prove property (2.16.0), it suffices to prove that $o_{n-37,1}$ exists and $n-37$ is of type 37. For that, observing (by using the previous inequality) that $n \geq 11 + 185F_5^{F_5}$, then we immediately deduce that $o_{n-37,1}$ clearly exists and $n - 37 \geq -26 + 185F_5^{F_5} > F_5^{F_5} > 111$; consequently

$$n - 37 > 111 \qquad (2.16.0.3).$$

Recalling that n is of type 37, then, using inequality *(2.16.0.3)*, it becomes trivial to deduce that $n-37$ is of type 37. Property (2.16.0) follows.
(2.16.1). We have $o_n < n-74$. Otherwise [we reason by reduction to absurd], clearly

$$o_n \geq n - 74 \qquad (2.16.1.0).$$

Now observing [by using property (1.1.0) of Proposition 1.1] that $o_{n,1} = 11 + 185o_n^{o_n}$ and noticing [by using property (2.16.0)] that $n \geq 11 + 185F_5^{F_5}$, then using the previous and using inequality *(2.16.1.0)*, it becomes immediate to deduce that

$$o_{n,1} > 10 + 185 o_n^{o_n} > 9 + 185(n-74)^{n-74} > n + 11 \qquad (2.16.1.1)$$

[since $n \geq 11 + 185 F_5^{F_5}$]. *(2.16.1.1)* clearly implies that $o_{n,1} > n + 11$ and this contradicts property (2.16.0). So $o_n < n - 74$. Property (2.16.1) follows.
(2.16.2). Indeed, observing [by using property (2.16.1)] that

$$o_n < n - 74 \qquad (2.16.2.0),$$

then, using property (1.1.2) of Proposition 1.1 (where we replace y by 74), it becomes immediate to deduce that inequality *(2.16.2.0)* clearly implies that

$$o_{n,1} = o_{n-75,1} \qquad (2.16.2.1).$$

Using equality *(2.16.2.1)*, then it becomes trivial to deduce that

$$o_{n,1} = o_{n-1,1} = o_{n-2,1} = ... = o_{n-37,1} = ... = o_{n-74,1} = o_{n-75,1} \qquad (2.16.2.2).$$

(2.16.2.2) clearly implies that $o_{n,1} = o_{n-37,1}$. Property (2.16.2) follows.
(2.16.3). Otherwise [we reason by reduction to absurd]

$$o_{n,1} > n - 63 \qquad (2.16.3.0).$$

Now observing [by using property (1.1.0) of Proposition 1.1] that

$$o_{n,1} \equiv 11 \ mod(37) \qquad (2.16.3.1),$$

and remarking that

$$n - 63 \equiv 11 \ mod(37) \qquad (2.16.3.2)$$

[since n is of type 37, clearly $n \equiv 0 \ mod(37)$ and therefore $n-63 \equiv 11 \ mod(37)$], then, using congruences *(2.16.3.1)* and *(2.16.3.2)*, inequality *(2.16.3.0)* immediately implies that

$$o_{n,1} \geq n - 63 + 37 \qquad (2.16.3.3),$$

[since $o_{n,1} \equiv 11 \ mod(37)$ and $n-63 \equiv 11 \ mod(37)$ and $o_{n,1} > n-63$]. Inequality *(2.16.3.3)* clearly says that $o_{n,1} \geq n - 26$ and consequently

$$o_{n,1} + 26 \geq n \qquad (2.16.3.4).$$

That being so, we have this fact.
Fact: $o_{n,1} + 26 = n$.

Otherwise [we reason by reduction to absurd], using inequality *(2.16.3.4)*, we immediately deduce that

$$o_{n,1} + 26 > n \qquad (2.16.3.5).$$

Now observing [by using Example.2 of Recall 2.1] that $o_{n,1} + 26 \equiv 0 \ mod(37)$ and noticing that $n \equiv 0 \ mod(37)$ [since n is of type 37], then, using the previous two trivial congruences, inequality *(2.16.3.5)* immediately implies that

$$o_{n,1} + 26 \geq n + 37 \qquad (2.16.3.6)$$

[since $o_{n,1} + 26 \equiv 0 \ mod(37)$ and $n \equiv 0 \ mod(37)$ and $o_{n,1} + 26 > n$]. Inequality *(2.16.3.6)* clearly says that $o_{n,1} \geq n+11$ and this contradicts property (2.16.0). The Fact follows.
This fact made, observing (by the previous Fact) that $o_{n,1}+26 = n$ and noticing [by property (2.16.2)] that $o_{n,1} = o_{n-37,1}$, then, using property (2.13.3) of Proposition 2.13, we immediately deduce that$\phi(o_{n,1})$ is n-conform and

$$R[Fix[\phi(o_{n,1})]] + I[Fix[\phi(o_{n,1})]] \not\equiv 74 mod(185) and n < 12 + 9o_{n,1} \qquad (2.16.3.7).$$

Clearly *(2.16.3.7)* does not satisfied property (2.15.2) of definition 2.15, and we have a contradiction, since $o_{n,1}$ is a remarkable element [see definition 2.15 for the meaning of a remarkable element]. Property (2.16.3) follows, and Proposition 2.16 immediately follows.□

Now, we are ready to give an elementary proof that there are infinitely many Fermat composite numbers; but before, let us propose the following last two simple propositions.
Proposition 2.17. *(Fundamental.7: [The elementary using of the minimality of n]). Suppose that Theorem* 2.10 *is false; and let* $o_{n,1}$ *be a remarkable element [such a* $o_{n,1}$ *exists, by using Proposition* 2.14 *and definition* 2.15*]. Fix once and for all* $o_{n,1}$ *[*$o_{n,1}$ *is fixed once and for all; so* $o_{n,1}$ *does not move anymore]. Now consider* $\phi(o_{n,1})$ *[see definitions* 2.0 *for the equation of* $\phi(o_{n,1})$ *], and via* $\phi(o_{n,1})$*, look at* $\phi(o_{n-37,1})$*. Then* $R[Fix[\phi(o_{n-37,1})]] + I[Fix[\phi(o_{n-37,1})]] \not\equiv 74 \ mod(185)$ *and* $n < 49 + 9o_{n,1}$*.*
Proof. Indeed, look at $n - 37$; then we observe the following.
Observation.2.17.0. *n is a minimun counter-example to Theorem* 2.10.
Indeed, this Observation is immediate, by recalling that $o_{n,1}$ is a remarkable element, and by using the definition of a remarkable element [see definition 2.15].
Observation.2.17.1. $n - 37 < n$ *and* $n - 37$ *is of type* 37*, and* $n - 37 \geq -26 + 185F_5^{F_5}$.
Indeed, it is immediate that $n - 37 < n$; moreover, using property (2.16.0) of Proposition 2.16, then it becomes trivial to deduce that $n - 37$ is of type 37, and

$n-37 \geq -26+185F_5^{F_5}$. Observation.2.17.1 follows.
*Observation.*2.17.2. *Look at* $n-37$*; then at least one of properties (*2.10.0*) and (*2.10.1*) of Theorem* 2.10 *is satisfied by* $n-37$.

Indeed, remarking [by Observation.2.17.0] that n is a minimum counter-example to Theorem 2.10, and noticing that $n-37$ is of type 37 and $n-37 < n$ (where $n-37 \geq -26+185F_5^{F_5}$) [use Observation.2.17.1], then, by the minimality of n, $n-37$ is not a counter-example to Theorem 2.10; therefore at least one of properties (2.10.0) and (2.10.1) of Theorem 2.10 is satisfied by $n-37$. Observation.2.17.2 follows.

Observation.2.17.3. *Look at* $o_{n,1}$ *[recall that* $o_{n,1}$ *is a remarkable element]; then* $o_{n,1} \leq n-63$.

This observation is immediate, by noticing that $o_{n,1}$ is a remarkable element, and by using property (2.16.3) of Proposition 2.16.

*Observation.*2.17.4. *Look at* $n-37$*; then property (*2.10.1*) of Theorem* 2.10 *is satisfied by* $n-37$.

Otherwise [we reason by reduction to absurd], using Observation.2.17.2, then we immediately deduce that property (2.10.0) of Theorem 2.10 is satisfied by $n-37$, and in particular, we clearly have $o_{n-37,1} \geq (n-37)+11$ [since property (2.10.0) of Theorem 2.10 is satisfied by $n-37$]; the previous inequality clearly says that

$$o_{n-37,1} \geq n-26 \qquad (2.17.4.0).$$

Noticing [by recalling that $o_{n,1}$ is a remarkable element and by using property (2.16.2) of Proposition 2.16] that
$o_{n,1} = o_{n-37,1}$, then inequality *(2.17.4.0)* immediately becomes $o_{n,1} \geq n-26$ and this contradicts
Observation.2.17.3. Observation 2.17.4 follows.

Observation 2.17.5. *Look at* $n-37$ *and consider* $o_{n-37,1}$ [recall that $o_{n,1}$ is a remarkable element]. *Now let* $\phi(o_{n-37,1})$*; then* $R[Fix[\phi(o_{n-37,1})]]+I[Fix[\phi(o_{n-37,1})]] \not\equiv 74mod(185)$ *and* $n < 49+9o_{n,1}$.

Indeed, observing [by Observation.2.17.4] that property (2.10.1) of Theorem 2.10 is satisfied by $n-37$, then in particular, we clearly have $\phi(o_{n-37,1})$ is $n-37$ conform and

$$R[Fix[\phi(o_{n-37,1})]]+I[Fix[\phi(o_{n-37,1})]] \not\equiv 74mod(185)$$

and

$$n-37 < 12+9o_{n-37,1} \qquad (a.),$$

because property (2.10.1) of Theorem 2.10 is satisfied by $n-37$. (a.) clearly says that $\phi(o_{n-37,1})$ is $n-37$ conform and

$$R[Fix[\phi(o_{n-37,1})]]+I[Fix[\phi(o_{n-37,1})]] \not\equiv 74mod(185)$$

and

$$n < 49 + 9o_{n-37,1} \qquad (2.17.5.0).$$

(2.17.5.0) clearly implies that $R[Fix[\phi(o_{n-37,1})]]+I[Fix[\phi(o_{n-37,1})]] \not\equiv 74mod(185)$ and
$n < 49 + 9o_{n-37,1}$. Observation.2.17.5 follows.

These observations made, look at $o_{n-37,1}$ and consider $\phi(o_{n-37,1})$; observing [by Observation.2.17.5] that

$$R[Fix[\phi(o_{n-37,1})]] + I[Fix[\phi(o_{n-37,1})]] \not\equiv 74mod(185) and\, n < 49 + 9o_{n-37,1} \qquad (2.17.6),$$

and remarking [by recalling that $o_{n,1}$ is a remarkable element and by using property (2.16.2) of Proposition 2.16] that $o_{n-37,1} = o_{n,1}$, then it becomes immediate to deduce that *(2.17.6)* clearly says that $n < 49 + 9o_{n,1}$ and $R[Fix[\phi(o_{n-37,1})]] + I[Fix[\phi(o_{n-37,1})]] \not\equiv 74mod(185)$. Proposition 2.17 follows. □

Proposition 2.18. *(Fundamental.8: [The non obvious using of the minimality of n]). Suppose that Theorem* 2.10 *is false; and let* $o_{n,1}$ *be a remarkable element [such a* $o_{n,1}$ *exists , by using Proposition* 2.14 *and definition* 2.15*]. Fix once and for all* $o_{n,1}$ *[* $o_{n,1}$ *is fixed once and for all; so* $o_{n,1}$ *does not move anymore]. Then* $n < 12 + 9o_{n,1}$.

Proof. Otherwise [we reason by reduction to absurd], clearly

$$n \geq 12 + 9o_{n,1} \qquad (2.18.0),$$

and we observe the following.

Observation.2.18.1. $n = 12 + 9o_{n,1}$.

Indeed, remark [by recalling that $o_{n,1}$ is a remarkable element and by using Proposition 2.17] that

$$n < 49 + 9o_{n,1} \qquad (2.18.1.0).$$

Look at $49 + 9o_{n,1}$; observing [by using property (1.1.0) of Proposition 1.1] that $o_{n,1} \equiv 11\ mod(37)$, then the previous congruence immediately implies that

$$49 + 9o_{n,1} \equiv 49 + 9 \times 11\ mod(37) \qquad (2.18.1.1).$$

Clearly $49 + 9 \times 11 = 148$ and congruence *(2.18.1.1)* clearly says that

$$49 + 9o_{n,1} \equiv 148\ mod(37) \qquad (2.18.1.2).$$

Congruence *(2.18.1.2)* immediately implies that

$$49 + 9o_{n,1} \equiv 0 \ mod(37) \qquad (2.18.1.3),$$

since $148 = 4 \times 37$. Now consider n; recalling that n is of type of type 37, then [by using the definition of type 37 we immediately deduce that

$$n \equiv 0 \ mod(37) \qquad (2.18.1.4).$$

Now using congruences *(2.18.1.4)* and *(2.18.1.3)*, then it becomes trivial to deduce that inequality *(2.18.1.0)* implies that

$$n \leq 49 + 9o_{n,1} - 37 \qquad (2.18.1.5)$$

[since $n \equiv 0 \ mod(37)$ and $49+9o_{n,1} \equiv 0 \ mod(37)$ and $n < 49+9o_{n,1}$]. Inequality *(2.18.1.5)* clearly says that

$$n \leq 12 + 9o_{n,1} \qquad (2.18.1.6).$$

That being so, using inequalities *(2.18.0.)* and *(2.18.1.6)*, then we immediately deduce that $n = 12 + 9o_{n,1}$. Observation.2.18.1 follows.

*Observation.*2.18.2. *Look at* $o_{n,1}$ [recall that $o_{n,1}$ is a remarkable element]; *then* n *is a minimum counter-example to Theorem* 2.10 *and* n *is of type* 37 *and* $n \geq 11 + 185F_5^{F_5}$ *and* $n - 37$ *is of type* 37 *and* $o_{n-37,1}$ *exists.*

Clearly n is a minimum counter-example to Theorem 2.10 and is of type 37 [since $o_{n,1}$ is a remarkable element]; $n \geq 11 + 185F_5^{F_5}$ and $o_{n-37,1}$ exists and $n - 37$ is of type 37 [by recalling that $o_{n,1}$ is a remarkable element and by using property (2.16.0) of Proposition 2.16]. Observation.2.18.2 follows.

Observation 2.18.3. Look at $o_{n,1}$ [recall that $o_{n,1}$ is a remarkable element]; *then* $o_{n,1} \leq n - 63$.

This observation is immediate, by recalling that $o_{n,1}$ is a remarkable element and by using property (2.16.3) of Proposition 2.16.

Observation 2.18.4.*Look at* $o_{n,1}$ *[recall that* $o_{n,1}$ *is a remarkable element] and consider* $o_{n-37,1}$ *[this consideration gets sense, since* $o_{n-37,1}$ *exists (by using Observation.*2.18.2*)]. Then* $o_{n,1} = o_{n-37,1}$.

Indeed this Observation is also immediate, by recalling that $o_{n,1}$ is a remarkable element and by using property (2.16.2) of Proposition 2.16.

Observation 2.18.5.*Look at* $o_{n,1}$ *[recall that* $o_{n,1}$ *is a remarkable element] and consider* $o_{n-37,1}$. *Now let* $\phi(o_{n,1})$ *[see definitions* 2.0 *for the equation of* $\phi(o_{n,1})$*] and via* $\phi(o_{n,1})$*, consider* $\phi(o_{n-37,1})$*; then*

$$R[Fix[\phi(o_{n-37,1})]] + I[Fix[\phi(o_{n-37,1})]] \not\equiv 74mod(185)$$

[we recall (see Recall 2.1*) that* $I[Fix[\phi(o_{n-37,1})]]$ *is the imaginary part of* $Fix[\phi(o_{n-37,1})]$ *and* $R[Fix[\phi(o_{n-37,1})]]$ *is the real part of* $Fix[\phi(o_{n-37,1})]$. $Fix[\phi(o_{n-37,1})]$ *is known, via definitions* 2.6 *].*

Indeed, look at $\phi(o_{n-37,1})$; recalling that $o_{n,1}$ is a remarkable element, then, using Proposition 2.17, it becomes immediate to deduce that $R[Fix[\phi(o_{n-37,1})]] + I[Fix[\phi(o_{n-37,1})]] \not\equiv 74 mod(185)$. Observation 2.18.5 follows.

Observation 2.18.6. *Look at* $o_{n,1}$ *[recall that* $o_{n,1}$ *is a remarkable element] and put*

$$q(n) = i(9o_{n,1} - 25)^2(o_{n,1}^3 - o_{n,1} + 1) + 48877(9o_{n,1} - 25).$$

Then $R[Fix[q(n)]] + I[Fix[q(n)]] \not\equiv 74mod(185)$.

Indeed, look at $\phi(o_{n-37,1})$; observing (via definitions 2.0) that $\phi(o_{n,1}) = in^2(o_{n,1}^3 - o_{n,1} + 1) + 48877n$, then using the previous equation, we immediately deduce that

$$\phi(o_{n-37,1}) = i(n - 37)^2(o_{n-37,1}^3 - o_{n-37,1} + 1) + 48877(n - 37) \qquad (2.18.6.0).$$

Observing (by using Observation.2.18.4) that $o_{n,1} = o_{n-37,1}$, then it becomes trivial to deduce that equation *(2.18.6.0)* is of the form

$$\phi(o_{n-37,1}) = i(n - 37)^2(o_{n,1}^3 - o_{n,1} + 1) + 48877(n - 37) \qquad (2.18.6.1).$$

Now noticing (by Observation.2.18.1) that $n = 12+9o_{n,1}$, clearly $n-37 = 9o_{n,1}-25$, and it becomes immmediate to deduce that equation *(2.18.6.1)* is of the form

$$\phi(o_{n-37,1}) = i(9o_{n,1} - 25)^2(o_{n,1}^3 - o_{n,1} + 1) + 48877(9o_{n,1} - 25) \qquad (2.18.6.2).$$

That being so, look at the equation $q(n)$ (we recall that the equation of $q(n)$ is given in the begining of statement of this Observation.2.18.6); then using equation *(2.18.6.2)*, it becomes trivial to see that

$$\phi(o_{n-37,1}) = q(n) \qquad (2.18.6.3).$$

Observe (by Observation.2.18.5) that

$$R[Fix[\phi(o_{n-37,1})]] + I[Fix[\phi(o_{n-37,1})]] \not\equiv 74 \ mod(185) \qquad (2.18.6.4).$$

Now using *(2.18.6.4)* and *(2.18.6.3)*, then it becomes trivial to deduce that $R[Fix[q(n)]] + I[Fix[q(n)]] \not\equiv 74 \ mod(185)$. Observation.2.18.6 follows.

Observation.2.18.7.*Look at* $q(n)$ *defined in Observation* 2.18.6. *Then* $R[q(n)] + I[q(n)] \not\equiv 74 \ mod(185)$.

Indeed, consider $q(n)$ defined in Observation.2.18.6 and look at $o_{n,1}$ (recall that $o_{n,1}$ is a remarkable element) ; noticing (by using Observation.2.18.2) that n is of type 37, and observing (by Observation.2.18.4) that $o_{n,1} = o_{n-37,1}$, clearly all the hypotheses of Remark 2.9 are satisfied; so the conclusion of Remark 2.9 is also satisfied; consequently

$$q(n) \ \ is\, n - conform \ and\, Fix[q(n)] = q(n) \qquad (2.18.7.0).$$

(2.18.7.0) clearly implies that

$$Fix[q(n)] = q(n) \qquad (2.18.7.1).$$

That being so, observe (by Observation.2.18.6) that

$$R[Fix[q(n)]] + I[Fix[q(n)]] \ \not\equiv \ 74 \ mod(185) \qquad (2.18.7.2).$$

Now using *(2.18.7.1)* and *(2.18.7.2)*, then it becomes trivial to deduce that $R[q(n)] + I[q(n)] \ \not\equiv \ 74 \ mod(185)$. Observation.2.18.7 follows.

These simple observations made, consider $q(n)$ defined in Observation.2.18.6 and look at $o_{n,1}$; observing (by using Observation.2.18.2) that $n \geq 11 + 185F_5^{F_5}$, then it becomes trivial to deduce that all the hypotheses of Proposition 2.12 are satisfied; so the conclusion of Proposition 2.12 is also satisfied; in particular property (2.12.1) of Proposition 2.12 is satisfied; consequently $R[q(n)] + I[q(n)] \ \equiv \ 74 \ mod(185)$ and the previous congruence clearly contradicts
Observation.2.18.7. Proposition 2.18 follows. □

The previous simple Propositions made, we now prove simply Theorem 2.10.

Proof of Theorem 2.10. Otherwise [we reason by reduction to absurd], let $o_{n,1}$ be a remarkable element [such a $o_{n,1}$ exists, by using Proposition 2.14 and definition 2.15]. Fix once and for all $o_{n,1}$ [$o_{n,1}$ is fixed once and for all; so $o_{n,1}$ does not move anymore]. We observe the following.

Observation.2.10.*i*. *n is a minimum counter-example to Theorem* 2.10 *and n is of type* 37.

Indeed, this observation is immediate, by recalling that $o_{n,1}$ is a remarkable element.

Observation 2.10.ii.$\phi(o_{n,1})$ *is not n-conform or* $R[Fix[\phi(o_{n,1})]] + I[Fix[\phi(o_{n,1})]] \equiv 74 \ mod(185)$ *or* $n \geq 12 + 9o_{n,1}$ *[we recall (see Recall* 2.1*) that* $I[Fix[\phi(o_{n,1})]]$ *is the imaginary part of* $Fix[\phi(o_{n,1})]$ *and* $R[Fix[\phi(o_{n,1})]]$ *is the real part of* $Fix[\phi(o_{n,1})]$. $\phi(o_{n,1})$ *is given in definitions* 2.0 *and* $Fix[\phi(o_{n,1})]$ *is known via definitions* 2.6 *].*

Indeed, this observation is also immediate, by recalling that $o_{n,1}$ is a remarkable element.

*Observation 2.10.iii. None of properties (*2.10.0*) and (*2.10.1*) of Theorem* 2.10 *is satisfied by n.*

Indeed, observing [by Observation.2.10.*i*] that n is a minimum counter-example to

Theorem 2.10, then in particular n is a counter-example to Theorem 2.10, and clearly none of properties (2.10.0) and (2.10.1) of Theorem 2.10 is satisfied by n.

Observation 2.10.iv. $n \geq 11 + 185F_5^{F_5}$ *and* $o_{n-37,1}$ *exists and* $o_{n,1} = o_{n-37,1}$ *and* $o_{n,1} \leq n - 63$ *and* $n < 12 + 9o_{n,1}$.

Indeed $n \geq 11 + 185F_5^{F_5}$ and $o_{n-37,1}$ exists [by noticing that $o_{n,1}$ is a remarkable element and by using property (2.16.0) of Proposition 2.16]; $o_{n,1} = o_{n-37,1}$ [by noticing that $o_{n,1}$ is a remarkable element and by using property (2.16.2) of Proposition 2.16]; $o_{n,1} \leq n - 63$ [by noticing that $o_{n,1}$ is a remarkable element and by using property (2.16.3) of Proposition 2.16]; and $n < 12 + 9o_{n,1}$ [by noticing that $o_{n,1}$ is a remarkable element and by using Proposition 2.18]. Observation 2.10.*iv* follows.

Observation.2.10.*v*. *Consider* $o_{n,1}$ *and look at* $\phi(o_{n,1})$ *(see definitions* 2.0 *for the equation of* $\phi(o_{n,1})$*). Then* $\phi(o_{n,1})$ *is* n *conform (see definition* 2.2 *for the meaning of* n*-conform).*

Indeed, observing (by using Observation 2.10.*i*) then n is of type 37 and noticing (by using Observation 2.10.*iv*) that $o_{n,1} = o_{n-37,1}$, then it becomes trivial to deduce that all the hypotheses of assertion 2.3.1 of Remark 2.3 are satisfied; so the conclusion of assertion 2.3.1 of Remark 2.3 is also satisfied; therefore $\phi(o_{n,1})$ is n-conform. Observation 2.10.*v* follows.

Observation.2.10.*vi*.*Consider* $o_{n,1}$ *and look at* $\phi(o_{n,1})$. *Then*

$$R[Fix[\phi(o_{n,1})]] + I[Fix[\phi(o_{n,1})]] \equiv \ 74 \ mod(185).$$

Otherwise [we reason by reduction to absurd], clearly

$$R[Fix[\phi(o_{n,1})]] + I[Fix[\phi(o_{n,1})]] \not\equiv \ 74 \ mod(185) \qquad (2.10.vi.0).$$

Now observe [by using Observation.2.10.*iv*] that

$$n < 12 + 9o_{n,1} \qquad (2.10.vi.1),$$

and Remark [by using Observation.2.10.*v*] that

$$\phi(o_{n,1}) \ is \ n - conform \qquad (2.10.vi.2).$$

That being so, using *(2.10.vi.2)* and *(2.10.vi.0)* and *(2.10.vi.1)*, then it becomes trivial to deduce that $\phi(o_{n,1})$ is n-conform and $R[Fix[\phi(o_{n,1})]] + I[Fix[\phi(o_{n,1})]] \not\equiv$ 74 $mod(185)$ and $n < 12 + 9o_{n,1}$; this contradicts Observation.2.10.*ii*. Observation 2.10.*vi* follows.

Observation 2.10.vii.*Let* $o_{n,1}$ *[recall that* $o_{n,1}$ *is a remarkable element] and look at* $\phi(o_{n,1})$*; now, via* $\phi(o_{n,1})$*, consider* $\phi(o_{n-37,1})$ *[this consideration gets sense,* *since* $o_{n-37,1}$ *exists, by using Observation* 2.10.*iv]. Then* $R[Fix[\phi(o_{n-37,1})]] + I[Fix[\phi(o_{n-37,1})]] \not\equiv 74mod(185)$.

Indeed, this Observation is immediate, by recalling that $o_{n,1}$ is a remarkable element and by using Proposition 2.17.

Observation 2.10.viii.*Let* $o_{n,1}$ *and look at* $\phi(o_{n,1})$*; now, via* $\phi(o_{n,1})$*, consider* $\phi(o_{n-37,1})$ *[this consideration gets sense, since* $o_{n-37,1}$ *exists, by using Observation* 2.10.*iv]. Now put*

$$Z'(n) = i(-74n + 1369)(o_{n,1}^3 - o_{n,1} + 1) - 1808449.$$

Then $\phi(o_{n-37,1}) = \phi(o_{n,1}) + Z'(n)$ *and* $Z'(n)$ *is* n-*conform and*
$Fix[Z'(n)] = 1369i(o_{n,1}^3 - o_{n,1} + 1) - 1808449$.
Indeed, observing (by using Observation 2.10.*i*) then n is of type 37 and noticing (by using Observation 2.10.*iv*) that $o_{n,1} = o_{n-37,1}$, then it becomes trivial to deduce that all the hypotheses of Proposition 2.11 are satisfied; so the conclusion of Proposition 2.11 is also satisfied; so the two properties (2.11.0) and (2.11.1) of Proposition 2.11 are simultaneously satisfied; therefore $\phi(o_{n-37,1}) = \phi(o_{n,1}) + Z'(n)$ (by property (2.11.0) of Proposition 2.11) and, $Z'(n)$ is n-conform and $Fix[Z'(n)] = 1369i(o_{n,1}^3 - o_{n,1} + 1) - 1808449$ (by property (2.11.1) of Proposition 2.11). Observation 2.10.*viii* follows.

Observation 2.10.ix. Let $o_{n,1}$ and look at $\phi(o_{n,1})$. Now let $Z'(n)$ introduced in Observation.2.10.*viii*.
Then $R[Fix[\phi(o_{n,1}) + Z'(n)]] + I[Fix[\phi(o_{n,1}) + Z'(n)]] \not\equiv 74\ mod(185)$.
Indeed, observing (by using Observation.2.10.*viii*) that $\phi(o_{n-37,1}) = \phi(o_{n,1}) + Z'(n)$ and noticing (by using Observation.2.10.*vii*) that
$R[Fix[\phi(o_{n-37,1})]] + I[Fix[\phi(o_{n-37,1})]] \not\equiv 74\,mod(185)$, then it becomes trivial to deduce that $R[Fix[\phi(o_{n,1}) + Z'(n)]] + I[Fix[\phi(o_{n,1}) + Z'(n)]] \not\equiv 74\,mod(185)$ .Observation 2.10.*ix* follows.

Observation 2.10.x. Let $o_{n,1}$ and look at $\phi(o_{n,1})$. Now let $Z'(n)$ introduced in Observation.2.10.*viii*. Then $\phi(o_{n,1}) + Z'(n)$ is n-conform and $Fix[\phi(o_{n,1}) + Z'(n)] = Fix[\phi(o_{n,1})] + Fix[Z'(n)]$.

Indeed, observing (by Observation 2.10.*v*) that $\phi(o_{n,1})$ is n-conform and noticing (by using Observation 2.10.*viii*) that $Z'(n)$ is n-conform, then, by using property (2.7.1) of Remark 2.7, it becomes trivial to deduce that $\phi(o_{n,1}) + Z'(n)$ is n-conform and $Fix[\phi(o_{n,1}) + Z'(n)] = Fix[\phi(o_{n,1})] + Fix[Z'(n)]$. Observation.2.10.$x$ follows.

Observation 2.10.xi.*Let* $o_{n,1}$ *and look at* $\phi(o_{n,1})$*. Now let* $Z'(n)$ *introduced in Observation.*2.10.*viii. Then* $R[Fix[\phi(o_{n,1})]] + R[Fix[Z'(n)]] + I[Fix[\phi(o_{n,1})]] + I[Fix[[Z'(n)]] \not\equiv 74mod(185)$

Indeed, observing (by using Observation.2.10.x) that $Fix[\phi(o_{n,1}) + Z'(n)] =$

$Fix[\phi(o_{n,1})] + Fix[Z'(n)]$, then it becomes trivial to deduce that

$$R[Fix[\phi(o_{n,1}) + Z'(n)]] = R[Fix[\phi(o_{n,1})] + Fix[Z'(n)]\ ]$$

and

$$I[\ Fix[\phi(o_{n,1}) + Z'(n)]\ ] = I[\ Fix[\phi(o_{n,1})] + Fix[Z'(n)]\ ] \qquad (b.).$$

Since it is trivial to see that $R[\ Fix[\phi(o_{n,1})] + Fix[Z'(n)]\ ] = R[Fix[\phi(o_{n,1})]] + R[Fix[Z'(n)]]$ and $I[\ Fix[\phi(o_{n,1})] + Fix[Z'(n)]\ ] = I[Fix[\phi(o_{n,1})]] + I[Fix[Z'(n)]]$ (because $R[Fix[\phi(o_{n,1})] + Fix[Z'(n)]]$ is the real part of $Fix[\phi(o_{n,1})] + Fix[Z'(n)]$ and $I[Fix[\phi(o_{n,1})] + Fix[Z'(n)]]$ is the imaginary part of $Fix[\phi(o_{n,1})] + Fix[Z'(n)]$), then, using the previous two trivial equalities, it becomes immediate to deduce that (b.) clearly says that

$$R[Fix[\phi(o_{n,1}) + Z'(n)]] = R[Fix[\phi(o_{n,1})]] + R[Fix[Z'(n)]]$$

and

$$I[Fix[\phi(o_{n,1}) + Z'(n)]] = I[Fix[\phi(o_{n,1})]] + I[Fix[Z'(n)]] \qquad (c.).$$

That being so, observe (by using Observation.2.10.*ix*) that

$$R[Fix[\phi(o_{n,1}) + Z'(n)]] + I[Fix[\phi(o_{n,1}) + Z'(n)]] \not\equiv 74 mod(185) \qquad (2.10.xi.1).$$

Now using the two equalities of (c.), then it becomes trivial to see that *(2.10.xi.1)* clearly says that $R[Fix[\phi(o_{n,1})]] + R[Fix[Z'(n)]] + I[Fix[\phi(o_{n,1})]] + I[Fix[[Z'(n)]] \not\equiv 74\ mod(185)$.

Observation.2.10.*xi* follows.

Observation.2.10.*xii*.*Let* $Z'(n)$ *introduced in Observation.*2.10.*viii and let* $o_{n,1}$. *Now put*

$$Z(n) = 1369i(o_{n,1}^3 - o_{n,1} + 1) - 1808449.$$

Then $R[Fix[\phi(o_{n,1})]] + R[Z(n)] + I[Fix[\phi(o_{n,1})]] + I[Z(n)] \not\equiv 74\ mod(185)$.

Indeed look at $Z(n)$ and $Z'(n)$, observing [by using Observation.2.10.*viii*] that
$Fix[Z'(n)] = 1369i(o_{n,1}^3 - o_{n,1} + 1) - 1808449$, then it becomes trivial to deduce that

$$Fix[Z'(n)] = Z(n) \qquad (2.10.xii.0).$$

Observe [by Observation.2.10.*xi*] that

$$R[Fix[\phi(o_{n,1})]] + R[Fix[Z'(n)]] + I[Fix[\phi(o_{n,1})]] + I[Fix[[Z'(n)]] \not\equiv 74 mod(185) \qquad (2.10.xii.1).$$

That being so, using equality *(2.10.xii.0)*, then it becomes trivial to deduce that *(2.10.xii.1)* clearly says that $R[Fix[\phi(o_{n,1})]] + R[Z(n)] + I[Fix[\phi(o_{n,1})]] + I[Z(n)] \not\equiv$

$74mod(185)$. Observation.2.10.xii follows.
Observation.2.10.$xiii$**.** Let $Z(n)$ introduced in Observation.2.10.xii and let $o_{n,1}$. Then $R[Z(n)] + I[Z(n)] \equiv 0\ mod(185)$.
Indeed let $Z(n)$ introduced in Observation 2.10.xii; observing (by using Observation.2.10.iv) that $n \geq 11{+}185F_5^{F_5}$, then it becomes immediate to deduce that all the hypotheses of Example.3 of Recall 2.1 are satisfied; so the conclusion of Example.3 of Recall 2.1 is also satisfied; consequently $R[Z(n)]{+}I[Z(n)] \equiv 0\ mod(185)$. Observation.2.10.$xiii$ follows.

These thirdteen simple observations made, Let $o_{n,1}$ and look at $\phi(o_{n,1})$. Now let $Z(n)$ introduced in Observation.2.10.xii; then, by applying Observation.2.10.xii, we have

$$R[Fix[\phi(o_{n,1})]] + R[Z(n)] + I[Fix[\phi(o_{n,1})]] + I[Z(n)] \not\equiv 74mod(185) \quad (2.10.xiv).$$

It is trivial to see that *(2.10.xiv)* clearly says that

$$R[Fix[\phi(o_{n,1})]] + I[Fix[\phi(o_{n,1})]] + R[Z(n)] + I[Z(n)] \not\equiv 74mod(185) \quad (2.10.xv).$$

Now observe (by using Observation.2.10.$xiii$) that

$$R[Z(n)] + I[Z(n)] \equiv 0\ mod(185) \quad (2.10.xvi).$$

That being so, using congruence *(2.10.xvi)*, then it becomes trivial to deduce that *(2.10.xv)* implies that $R[Fix[\phi(o_{n,1})]] + I[Fix[\phi(o_{n,1})]] \not\equiv 74mod(185)$ and this contradicts Observation 2.10.vi. Theorem 2.10 follows.□

Theorem 2.10 *immediately implies that there are infinitely many Fermat composite numbers and the Fermat composite conjecture is only an obvious special case of the Goldbach conjecture.*

Corollary 2.19. *Let n be of type* 37 *and let $o_{n,1}$; then $12 + 9o_{n,1} > n$.*
Proof. Observe [by using Theorem 2.10] that at least one of the following two properties *(2.19.0)* and *(2.19.1)* is satisfied by n.

$$o_{n,1} \geq n + 11 \quad (2.19.0);$$

$\phi(o_{n,1})$ is n-conform and

$$R[Fix[\phi(o_{n,1})]] + I[Fix[\phi(o_{n,1})]] \not\equiv 74mod(185) and n < 12 + 9o_{n,1} \quad (2.19.1).$$

Using *(2.19.0)* and *(2.19.1)* and all the previous, then it becomes trivial to deduce that

$$o_{n,1} \geq n + 11 \ or \ n < 12 + 9o_{n,1} \quad (2.19.2).$$

Now noticing [by using property (1.1.0) of Proposition 1.1] that $o_{n,1} > 10+185F_5^{F_5}$ for every integer $n \geq 2$ (in particular, for every n of type 37, we have $o_{n,1} > 10 + 185F_5^{F_5}$), then using *(2.19.2)*, it becomes immediate to deduce that in all the cases, we have $12 + 9o_{n,1} > n$. □

Corollary 2.20. *Let n be of type* 37 *and let $o_{n,1}$; then $10o_{n,1} > n$.*

Proof. Indeed look at the quantities $12+9o_{n,1}$ and $10o_{n,1}$; noticing [by using property (1.1.0) of Proposition 1.1] that $o_{n,1} > 10+185F_5^{F_5}$ for every integer $n \geq 2$, then it becomes immediate to deduce that

$$10o_{n,1} > 12 + 9o_{n,1},\ where\ n\ is\ of\ type\ 37 \qquad (2.20.0).$$

That being so, observing [by Corollary 2.19] that

$$12 + 9o_{n,1} > n,\ where\ n\ is\ of\ type\ 37 \qquad (2.20.1),$$

then, using properties *(2.20.0)* and *(2.20.1)*, it becomes immediate to deduce that

$$10o_{n,1} > n,\ where\ n\ is\ of\ type\ 37.$$

Corollary 2.20 follows. □

Theorem 2.21. *The following two property are simulaneously satisfied.*

*(*2.21.0*). The Fermat composite conjecture is an obvious special case of the Goldbach conjecture.*

*(*2.21.1*). There are infinitely many Fermat composite numbers .*

Proof. (2.21.0). To simply prove property (2.21.0), we observe the following.

Observation 2.21.0.1. For every n of type 37, we have $30o_{n,1} > g''_{n+1}$.

Indeed observing (by Corollary 2.20) that for every n of type 37, we have $10o_{n,1} > n$, then, using the previous inequality, it becomes trivial to deduce that for every n of type 37, we have $30o_{n,1} > 3n > 2n+4 > g''_{n+1}$; consequently $30o_{n,1} > g''_{n+1}$.

This Observation made, recalling that n is of type 37, then (by using the definition of type 37),

$$n\ is\ clearly\ of\ the\ form\ n = 37k\ where\ k\ is\ an\ integer\ \geq 3 \qquad (2.21.0.2).$$

Now using *(2.21.0.2)* and Observation 2.21.0.1 and property 1.2.4 of Proposition 1.2, it becomes trivial to deduce that the Fermat composite conjecture is an obvious special case of the Goldbach conjecture. Property (2.21.0) follows.

(2.21.1). Observing (by Corollary 2.20) that for every n of type 37, we have $10o_{n,1} > n$, then we immediately deduce that,

$$for\ every\ n\ of\ type\ 37,\ we\ have\ 30o_{n,1} > n \qquad (2.21.1.0).$$

Now using *(2.21.1.0)* and property (1.2.3) of Proposition 1.2 [note (by the definition of type 37) that n is of type 37 if $n = 37k$, where k is an integer ≥ 3], we immediately deduce that there are infinitely many Fermat composite numbers. Property (2.21.1) follows and Theorem 2.21 immediately follows.□

Remark 2.22. Via equation $\phi(o_{n,1})$ given in definitions 2.0, then, Theorem 2.10 clearly shows that the Fermat composite conjecture that we have solved was only related to the Goldbach conjecture and was also related to elementary complex analysis mixting with elementary arithmetic calculus and elementary arithmetic congruences. Consequently, the Fermat composite conjecture that we have solved, was only simple combinatorial number theory problem.

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