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EXISTENCE RESULTS FOR A COUPLED SYSTEM OF THE SINGULAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, we establish the existence results for boundary value problems of the singular fractional differential systems. Our analysis relies on the well known Leray-Schauder nonlinear alternative theory. An example is presented to illustrate the main results.

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1. Introduction

Fractional differential equations have many applications in modeling of physical and chemical processes and in engineering and have been of great interest recently. In its turn, mathematical aspects of studies on fractional differential equations were discussed by many authors, see the text books [1,2], the survey papers [3,4] and papers [5-11] and the references therein.

The use of the Leray-Schauder nonlinear alternative theory in the study of the existence of solutions to boundary value problems for fractional differential equations with Caputo fractional derivatives has a rich and diverse history, see the paper [16] and the survey paper [3] and the references therein.

In [17], the authors investigated the existence of positive solutions for the singular fractional boundary value problem

$$D_0^\alpha u(t) + f(t, u(t), D_0^\mu u(t)) = 0, \;\; u(0) = u(1) = 0,$$

where $1 < \alpha < 2$, $0 < \mu < \alpha - 1$, D_0^α is the standard Riemann-Liouville fractional derivative, f is a positive Caratheodory function and $f(t, x, y)$ is singular at $x = 0$. By means of the fixed point theorem on the cones, the existence of positive solutions is obtained. The proofs are based on regularization and sequential techniques.

In [16], the authors studied the solvability of the following boundary value problem

$$\begin{cases} D_{0^+}^\alpha u(t) + f(t, v(t), D_{0^+}^{\beta-1} v(t)) = 0, \; t \in (0,1), 1 < \alpha < 2, \\ D_{0^+}^\beta u(t) + g(t, u(t), D_{0^+}^{\alpha-1} u(t)) = 0, \; t \in (0,1), 1 < \beta < 2, \\ u(0) = 0, \;\; u(1) = \gamma u(\eta), \; v(0) = 0, \;\; v(1) = \gamma u(\eta), \end{cases}$$

at the case where the homogeneous problem

$$\begin{cases} D_{0+}^{\alpha} u(t) = 0, \ t \in (0,1), 1 < \alpha < 2, \\ D_{0+}^{\beta} u(t) = 0, \ t \in (0,1), 1 < \beta < 2, \\ u(0) = 0, \ \ u(1) = \gamma u(\eta), \ v(0) = 0, \ \ v(1) = \gamma u(\eta), \end{cases}$$

has nontrivial solutions $(u(t), v(t)) = (c_1 t^{\alpha-1}, c_2 t^{\beta-1})$. The methods used in [16] is based upon coincidence degree theory.

In [12], the author studied the existence of solutions of the following boundary value problems for fractional differential equations with Riemann-Liouville fractional derivatives

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, v(t), D_{0+}^{p} v(t)) = 0, \ t \in (0,1), 1 < \alpha < 2, \\ D_{0+}^{\beta} u(t) + g(t, u(t), D_{0+}^{q} u(t)) = 0, \ t \in (0,1), 1 < \beta < 2, \\ u(0) = 0, \ \ u(1) = 0, \ v(0) = 0, \ \ v(1) = 0, \end{cases} \tag{1}$$

and in [13] the following boundary value problem for fractional differential equations was studied

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, v(t), D_{0+}^{p} v(t)) = 0, \ t \in (0,1), 1 < \alpha < 2, \\ D_{0+}^{\beta} u(t) + g(t, u(t), D_{0+}^{q} u(t)) = 0, \ t \in (0,1), 1 < \beta < 2, \\ u(0) = 0, \ \ u(1) = \gamma u(\eta), \ v(0) = 0, \ \ v(1) = \gamma u(\eta), \end{cases} \tag{2}$$

where $1 < \alpha, \beta < 2$, $p, q, \gamma > 0$, $0 < \eta < 1$, $\alpha - q \geq 1$, $\beta - p \geq 1$, $\gamma \eta^{\alpha-1} < 1$ and $\gamma \eta^{\beta-1} < 1$, D_{0+} is the standard Riemann-Liouville fractional derivative, $f, g : [0,1] \times R \times R \to R$ are given continuous functions.

The main conditions imposed on f, g in [13] are as follows:

(BA1) there exists a nonnegative function $a \in L(0,1)$ such that

$$|f(t,x,y)| \leq a(t) + \epsilon_1 |x|^{\rho_1} + \epsilon_2 |y|^{\rho_2}, \ \epsilon_1, \epsilon_2 > 0, \ \rho_1, \rho_2 \in (0,1),$$

(BA2) there exists a nonnegative function $b \in L(0,1)$ such that

$$|g(t,x,y)| \leq b(t) + \delta_1 |x|^{\sigma_1} + \delta_2 |y|^{\sigma_2}, \ \delta_1, \delta_2 > 0, \ \sigma_1, \sigma_2 \in (0,1).$$

It is noted that $D_{0+}^{\alpha} u(t) = 0$ implies that $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2}$ for some $c_1, c_2 \in R$. Hence the boundary condition $u(0) = 0$ implies that $c_2 = 0$. Then $u(t) = c_1 t^{\alpha-1}$ is bounded on $[0,1]$. Hence the solutions obtained in [12,13] and [16] are bounded solutions. If one replaces should be replaced $u(0) = 0$ by $\left[I_{0+}^{2-\alpha} u(t)\right]'\Big|_{t=0}$, then $c_1 = 0$. So $u(t) = c_2 t^{\alpha-2}$. It is easy to that $u(t)$ is unbounded on $(0,1]$ when $\alpha \in (1,2)$.

Motivated by this reson, in this paper, we discuss the existence of solutions to the non-local boundary value problem of the nonlinear fractional differential system

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, v(t), D_{0+}^{p}v(t)) = 0, \ t \in (0,1), 1 < \alpha < 2, \\ D_{0+}^{\beta}u(t) + g(t, u(t), D_{0+}^{q}u(t)) = 0, \ t \in (0,1), 1 < \beta < 2, \\ \left[I_{0+}^{2-\alpha}u(t)\right]'\Big|_{t=0} = 0\,, \\ \left[I_{0+}^{2-\alpha}v(t)\right]'\Big|_{t=0} = 0\,, \\ u(1) = ku(\eta), \\ v(1) = lv(\xi), \end{cases} \tag{3}$$

where D_{0+}^{α} (D_{0+}^{β}) is the Riemann-Liouville fractional derivative of order $\alpha(\beta)$,$1 < \alpha, \beta < 2$, $1 \geq p, q > 0$, $\alpha > q+1$ and $\beta > p+1$, $0 < \xi, \eta < 1$, $k, l \in R$, $k\eta^{\alpha-2} \neq 1$ and $l\xi^{\beta-2} \neq 1$, $f, g : (0,1) \times R^2 \to R$ are given continuous functions, f, g may be singular at $t = 0$ or $t = 1$.

A pair of functions $u, v : (0,1] \to R$ is called a solution of BVP(3) if both u, v are continuous on $(0,1]$ and all equations in (3) are satisfied.

The purpose of this paper is to establish the existence results for solutions BVP(3) by using the Leray-Schauder nonlinear alternative theory in Banach space. An example is presented to illustrate the main results.

2. Main results

For the convenience of the reader, we firstly present here the necessary definitions and fixed point theory that can be found in the literatures in [1,2] and [14].

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to R$ is given by

$$I_{0+}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds,$$

provided that the right-hand side exists.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to R$ is given by

$$D_{0+}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^{n+1}}{dt^{n+1}}\int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n - 1 < \alpha \leq n$, provided that the right-hand side is point-wise defined on $(0, \infty)$.

Definition 2.3. Let $m > 0, n > 0$. A function $F : (0,1) \times R \times R \to R$ is called a $(m,n)-$Caratheodory function if F is continuous and for each $r > 0$ there exists a function $\phi_r \in L(0,1)$ such that

$$|F((t, t^{m-2}x, t^{m-n-2}y)| \le \phi_r(t), \;\; t \in (0,1), \; |x| \le r, \; |y| \le r.$$

Lemma 2.1. *Let $n-1 < \alpha \le n$, $u \in C^0(0,1) \bigcap L^1(0,1)$. Then*

$$I^{\alpha}_{0+} D^{\alpha}_{0+} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \ldots + C_n t^{\alpha-n},$$

where $C_i \in R$, $i = 1, 2, \ldots n$.

Lemma 2.2. *The relations*

$$I^{\alpha}_{0+} I^{\beta}_{0+} \varphi = I^{\alpha+\beta}_{0+} \varphi, \; D^{\alpha}_{0+} I^{\alpha}_{0+} = \varphi$$

are valid in following case

$$Re\beta > 0, \quad Re(\alpha+\beta) > 0, \quad \varphi \in L_1(0,1).$$

Lemma 2.3. *Suppose that $k\eta^{\alpha-2} \ne 1$ and $h \in L^1(0,1)$. Then u satisfies*

$$\begin{cases} D^{\alpha} u(t) + h(t) = 0, 0 < t < 1, \\ \left[I_0^{2-\alpha} u(t) \right]' \Big|_{t=0} = 0, \\ u(1) = ku(\eta), \end{cases} \tag{4}$$

if and only if

$$u(t) = \int_0^1 G(t,s) h(s) ds, \tag{5}$$

where G is defined by

$$G(t,s) = \frac{1}{\Gamma(\alpha)(1 - k\eta^{\alpha-2})} \begin{cases} G_1(t,s), \; 0 < t \le \eta < 1, \\ G_2(t,s), \; \eta < t \le 1, \end{cases} \tag{6}$$

and

$$G_1(t,s) = \begin{cases} t^{\alpha-2}(1-s)^{\alpha-1} - kt^{\alpha-2}(\eta - s)^{\alpha-1} - (1 - k\eta^{\alpha-2})(t-s)^{\alpha-1}, \; 0 < s < t, \\ t^{\alpha-2}(1-s)^{\alpha-1} - kt^{\alpha-2}(\eta-s)^{\alpha-1}, t \le s \le \eta, \\ t^{\alpha-2}(1-s)^{\alpha-1}, \eta \le s \le 1, \end{cases}$$

$$G_2(t,s)=\begin{cases} t^{\alpha-2}(1-s)^{\alpha-1}-kt^{\alpha-2}(\eta-s)^{\alpha-1}-(1-k\eta^{\alpha-2})(t-s)^{\alpha-1}, 0<s\le\eta,\\ t^{\alpha-2}(1-s)^{\alpha-1}-(1-k\eta^{\alpha-2})(t-s)^{\alpha-1}, \eta\le s\le t,\\ t^{\alpha-2}(1-s)^{\alpha-1}, t\le s\le 1.\end{cases}$$

Proof. We may apply Lemma 2.1 to reduce BVP(4) to an equivalent integral equation

$$u(t)=-\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+c_1t^{\alpha-1}+c_2t^{\alpha-2} \tag{7}$$

for some $c_i\in R, i=1,2$. We get

$$[I_0^{2-\alpha}u(t)]'=-\int_0^t h(s)ds+c_1\Gamma(\alpha).$$

From $\left[I_0^{2-\alpha}u(t)\right]'\Big|_{t=0}=0$, we get $c_1=0$. Then $u(1)=ku(\eta)$ implies

$$-\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+c_2=k\left(-\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+c_2\eta^{\alpha-2}\right).$$

It follows that

$$c_2=\frac{1}{1-k\eta^{\alpha-2}}\left(\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds-k\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\right). \tag{8}$$

Therefore, the unique solution of BVP(4) is

$$\begin{aligned} u(t) &= -\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\\ &\quad+\frac{t^{\alpha-2}}{1-k\eta^{\alpha-2}}\left(\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds-k\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\right)\\ &= \frac{1}{\Gamma(\alpha)(1-k\eta^{\alpha-2})}\left[-\int_0^t(1-k\eta^{\alpha-2})(t-s)^{\alpha-1}h(s)ds\right.\\ &\qquad+\int_0^\eta\left(t^{\alpha-2}(1-s)^{\alpha-1}-kt^{\alpha-2}(\eta-s)^{\alpha-1}\right)h(s)ds\\ &\qquad\left.+\int_\eta^1 t^{\alpha-2}(1-s)^{\alpha-1}h(s)ds\right].\end{aligned}$$

Then (5) holds and $G(t,s)$ is defined by (6). Reciprocally, let u satisfy (5). Then

$$u(1)=ku(\eta),\qquad \left[I_0^{2-\alpha}u(t)\right]'\Big|_{t=0}=0,$$

furthermore, we have $D^\alpha u(t) = -h(t)$. The proof is complete.

Remark 2.1. It follows from (7) and (8) that

$$\begin{aligned} D^q_{0+}u(t) &= -\int_0^t \frac{(t-s)^{\alpha-q-1}}{\Gamma(\alpha-q)}h(s)ds \\ &\quad +\frac{t^{\alpha-q-2}}{1-k\eta^{\alpha-2}}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds \\ &\quad -\frac{kt^{\alpha-q-2}}{1-k\eta^{\alpha-2}}\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-q-1)}\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds \\ &= \int_0^1 K(t,s)h(s)ds, \end{aligned}$$

where

$$K(t,s) = \frac{1}{\alpha-1}\frac{1}{1-k\eta^{\alpha-2}}\frac{1}{\Gamma(\alpha-q)}\frac{1}{\Gamma(\alpha-q-1)}\begin{cases} K_1(t,s), \ 0<t\le\eta<1, \\ K_2(t,s), \ \eta<t\le 1, \end{cases} \tag{9}$$

and

$$K_1(t,s) = \begin{cases} \Gamma(\alpha-q)t^{\alpha-q-2}[(1-s)^{\alpha-1}-k(\eta-s)^{\alpha-1}] \\ \qquad -(1-k\eta^{\alpha-2})(\alpha-1)\Gamma(\alpha-q-1)(t-s)^{\alpha-1}, \ 0<s<t, \\ \Gamma(\alpha-q)t^{\alpha-q-2}[(1-s)^{\alpha-1}-k(\eta-s)^{\alpha-1}], t\le s\le\eta, \\ \Gamma(\alpha-q)t^{\alpha-q-2}(1-s)^{\alpha-1}, \eta\le s\le 1, \end{cases}$$

$$K_2(t,s) = \begin{cases} \Gamma(\alpha-q)t^{\alpha-q-2}[(1-s)^{\alpha-1}-k(\eta-s)^{\alpha-1}] \\ \qquad -(1-k\eta^{\alpha-2})(\alpha-1)\Gamma(\alpha-q-1)(t-s)^{\alpha-1}, 0<s\le\eta, \\ \Gamma(\alpha-q)t^{\alpha-q-2}(1-s)^{\alpha-1} \\ \qquad -(1-k\eta^{\alpha-2})(\alpha-1)\Gamma(\alpha-q-1)(t-s)^{\alpha-1}, \eta\le s\le t, \\ \Gamma(\alpha-q)t^{\alpha-q-2}(1-s)^{\alpha-1}, t\le s\le 1. \end{cases}$$

Remark 2.2. It is easy to see that

$$\begin{aligned} & t^{2-\alpha}|G_1(t,s)| \\ = & \begin{cases} \left|(1-s)^{\alpha-1}-k(\eta-s)^{\alpha-1}-(1-k\eta^{\alpha-2})t^{2-\alpha}(t-s)^{\alpha-1}\right|, \ 0<s<t, \\ \left|(1-s)^{\alpha-1}-k(\eta-s)^{\alpha-1}\right|, t\le s\le\eta, \\ |1-s|^{\alpha-1}, \eta\le s\le 1, \end{cases} \\ \le & (1+|k|\eta^{\alpha-1}+\eta|1-k\eta^{\alpha-2}|)(1-s)^{\alpha-1} \\ \le & (1+|k|\eta^{\alpha-1}+|1-k\eta^{\alpha-2}|)(1-s)^{\alpha-1} \end{aligned}$$

and

$$
\begin{aligned}
& t^{2-\alpha}|G_2(t,s)| \\
= & \begin{cases} \left|(1-s)^{\alpha-1} - k(\eta-s)^{\alpha-1} - (1-k\eta^{\alpha-2})t^{2-\alpha}(t-s)^{\alpha-1}\right|, \ 0 < s < \eta, \\ \left|(1-s)^{\alpha-1} - (1-k\eta^{\alpha-2})t^{2-\alpha}(t-s)^{\alpha-1}\right|, \eta \le s \le t, \\ |1-s|^{\alpha-1}, t \le s \le 1, \end{cases} \\
\le & (1+|k|\eta^{\alpha-1} + |1-k\eta^{\alpha-2}|)(1-s)^{\alpha-1}.
\end{aligned}
$$

Similarly, we can show that

$$
\begin{aligned}
& t^{2+q-\alpha}|K_1(t,s)| \\
= & \left| \begin{cases} \Gamma(\alpha-q)[(1-s)^{\alpha-1} - k(\eta-s)^{\alpha-1}] \\ -(1-k\eta^{\alpha-2})(\alpha-1)\Gamma(\alpha-q-1)t^{2+q-\alpha}(t-s)^{\alpha-1}, \ 0 < s < t, \\ \Gamma(\alpha-q)[(1-s)^{\alpha-1} - k(\eta-s)^{\alpha-1}], t \le s \le \eta, \\ \Gamma(\alpha-q)(1-s)^{\alpha-1}, \eta \le s \le 1, \end{cases} \right| \\
\le & [\Gamma(\alpha-q)(1+|k|\eta^{\alpha-1}) + |1-k\eta^{\alpha-2}|(\alpha-1)\Gamma(\alpha-q-1)](1-s)^{\alpha-1}
\end{aligned}
$$

and

$$
\begin{aligned}
& t^{2+q-\alpha}|K_2(t,s)| \\
= & \left| \begin{cases} \Gamma(\alpha-q)[(1-s)^{\alpha-1} - k(\eta-s)^{\alpha-1}] \\ -(1-k\eta^{\alpha-2})(\alpha-1)\Gamma(\alpha-q-1)t^{2+q-\alpha}(t-s)^{\alpha-1}, 0 < s \le \eta, \\ \Gamma(\alpha-q)(1-s)^{\alpha-1} \\ -(1-k\eta^{\alpha-2})(\alpha-1)\Gamma(\alpha-q-1)t^{2+q-\alpha}(t-s)^{\alpha-1}, \eta \le s \le t, \\ \Gamma(\alpha-q)(1-s)^{\alpha-1}, t \le s \le 1 \end{cases} \right| \\
\le & [\Gamma(\alpha-q)(1+|k|\eta^{\alpha-1}) + |1-k\eta^{\alpha-2}|(\alpha-1)\Gamma(\alpha-q-1)](1-s)^{\alpha-1}.
\end{aligned}
$$

Similarly to Lemma 2.3, we get the following Lemma.

Lemma 2.4. *Suppose that* $l\xi^{\beta-2} \neq 1$ *and* $h \in L^1(0,1)$. *Then* v *satisfies*

$$
\begin{cases} D^\beta v(t) + h(t) = 0, 0 < t < 1, \\ \left[I_0^{2-\beta} v(t)\right]'\Big|_{t=0} = 0, \\ v(1) = lv(\xi), \end{cases} \tag{10}
$$

if and only if

$$
v(t) = \int_0^1 H(t,s)h(s)ds, \tag{11}
$$

where G *is defined by*

$$
H(t,s) = \frac{1}{\Gamma(\beta)(1-l\xi^{\beta-2})} \begin{cases} H_1(t,s), \ 0 < t \le \xi < 1, \\ H_2(t,s), \ \xi < t \le 1, \end{cases} \tag{12}
$$

and

$$H_1(t,s)=\begin{cases} t^{\beta-2}(1-s)^{\beta-1}-lt^{\beta-2}(\xi-s)^{\beta-1}-(1-l\xi^{\beta-2})(t-s)^{\beta-1},\ 0<s<t,\\ t^{\beta-2}(1-s)^{\beta-1}-lt^{\beta-2}(\xi-s)^{\beta-1}, t\le s\le \xi,\\ t^{\beta-2}(1-s)^{\beta-1}, \xi\le s\le 1,\end{cases}$$

$$H_2(t,s)=\begin{cases} t^{\beta-2}(1-s)^{\beta-1}-lt^{\beta-2}(\xi-s)^{\beta-1}-(1-l\xi^{\beta-2})(t-s)^{\beta-1}, 0<s\le \xi,\\ t^{\beta-2}(1-s)^{\beta-1}-(1-l\xi^{\beta-2})(t-s)^{\beta-1}, \xi\le s\le t,\\ t^{\beta-2}(1-s)^{\beta-1}, t\le s\le 1,\end{cases}$$

Remark 2.3. It follows from (11) that

$$\begin{aligned} D_{0^+}^p v(t) &= -\int_0^t \frac{(t-s)^{\beta-p-1}}{\Gamma(\beta-p)}h(s)ds \\ &\quad +\frac{t^{\beta-p-2}}{1-l\xi^{\beta-2}}\frac{\Gamma(\beta-1)}{\Gamma(\beta-p-1)}\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}h(s)ds \\ &\quad -\frac{lt^{\beta-p-2}}{1-l\xi^{\beta-2}}\frac{\Gamma(\beta-1)}{\Gamma(\beta-p-1)}\int_0^\xi\frac{(\xi-s)^{\beta-1}}{\Gamma(\beta)}h(s)ds \\ &= \int_0^1 F(t,s)h(s)ds, \end{aligned}$$

where

$$F(t,s)=\frac{1}{\Gamma(\beta-p)}\frac{1}{\beta-1}\frac{1}{\Gamma(\beta-p-1)}\frac{1}{1-l\xi^{\beta-2}}\begin{cases} F_1(t,s),\ 0<t\le\xi<1,\\ F_2(t,s),\ \xi<t\le 1,\end{cases} \tag{13}$$

and

$$F_1(t,s)=\begin{cases} \Gamma(\beta-p)t^{\beta-p-2}[(1-s)^{\beta-1}-l(\xi-s)^{\beta-1}] \\ \qquad -(1-l\xi^{\beta-2})(\beta-1)\Gamma(\beta-p-1)(t-s)^{\beta-1},\ 0<s<t,\\ \Gamma(\beta-p)t^{\beta-p-2}[(1-s)^{\beta-1}-l(\xi-s)^{\beta-1}], t\le s\le\xi,\\ \Gamma(\beta-p)t^{\beta-p-2}(1-s)^{\beta-1}, \xi\le s\le 1,\end{cases}$$

$$F_2(t,s)=\begin{cases} \Gamma(\beta-p)t^{\beta-p-2}[(1-s)^{\beta-1}-l(\xi-s)^{\beta-1}] \\ \qquad -(1-l\xi^{\beta-2})(\beta-1)\Gamma(\beta-p-1)(t-s)^{\beta-1}, 0<s\le\xi,\\ \Gamma(\beta-p)t^{\beta-p-2}(1-s)^{\beta-1} \\ \qquad -(1-l\xi^{\beta-2})(\beta-1)\Gamma(\beta-p-1)(t-s)^{\beta-1}, \xi\le s\le t,\\ \Gamma(\beta-p)t^{\beta-p-2}(1-s)^{\beta-1}, t\le s\le 1.\end{cases}$$

Remark 2.4. It is easy to show that

$$t^{2-\beta}|H_1(t,s)| \le (1+|l|\xi^{\beta-1}+|1-l\xi^{\beta-2}|)(1-s)^{\beta-1},$$

$$\begin{aligned}
t^{2-\beta}|H_2(t,s)| &\leq (1+|l|\xi^{\beta-1}+|1-l\xi^{\beta-2}|)(1-s)^{\beta-1},\\
t^{2+p-\beta}|F_1(t,s)| &\leq [\Gamma(\beta-p)(1+|l|\xi^{\beta-1})\\
&\quad +|1-l\xi^{\beta-2}|(\beta-1)\Gamma(\beta-p-1)](1-s)^{\beta-1},\\
t^{2+p-\beta}|F_2(t,s)| &\leq [\Gamma(\beta-p)(1+|l|\xi^{\beta-1})\\
&\quad +|1-l\xi^{\beta-2}|(\beta-1)\Gamma(\beta-p-1)](1-s)^{\beta-1}.
\end{aligned}$$

Let $C(0,1]$ denote the space of all continuous functions defined on $(0,1]$. Let

$$X=\left\{u:(0,1]\to R \quad \begin{array}{c} u\in C(0,1] \text{ and } D_{0+}^q u\in C(0,1]\\ \text{there exist the limits}\\ \lim_{t\to 0} t^{2-\alpha}u(t),\\ \lim_{t\to 0} t^{2+q-\alpha}D_{0+}^q u(t)\end{array}\right\}$$

be the Banach space endowed with the norm

$$||u||_X=\max\left\{\sup_{t\in(0,1]} t^{2-\alpha}|u(t)|,\quad \sup_{t\in(0,1]} t^{2+q-\alpha}|D_{0+}^q u(t)|\right\} \text{ for } u\in X$$

and

$$Y=\left\{u:(0,1]\to R \quad \begin{array}{c} v\in\in C(0,1] \text{ and } D_{0+}^p v\in C(0,1]\\ \text{there exist the limits}\\ \lim_{t\to 0} t^{2-\beta}v(t)\\ \lim_{t\to 0} t^{2+p-\beta}D_{0+}^p v(t)\end{array}\right\}$$

be the Banach space endowed with the norm

$$||v||_Y=\max\left\{\sup_{t\in(0,1]} t^{2-\beta}|v(t)|,\quad \sup_{t\in(0,1]} t^{2+p-\beta}|D_{0+}^p v(t)|\right\} \text{ for } v\in Y.$$

Then

$X\times Y$ is a Banach space with the norm $||(u,v)||=\max\{||u||_X,\ \ ||v||_Y\}$.

Consider the coupled system of integral equatons

$$\begin{cases} u(t)=\int_0^1 G(t,s)f(s,v(s),D_{0+}^p v(s))ds,\\ v(t)=\int_0^1 H(t,s)g(s,u(s),D_{0+}^q u(s))ds.\end{cases} \tag{14}$$

Lemma 2.5. *Suppose that f is a $(\beta,p)-$Caratheodory function and g a $(\alpha,q)-$Caratheodory function. Then $(u,v)\in X\times Y$ is a solution of BVP(3) if and only if $(u,v)\in X\times Y$ is a solution of (14).*

Proof. The proof is immediate from Lemma 2.3 and 2.4. So we omit it.

Let us define an operator T on $X \times Y$ as

$$T(u,v)(t) = (T_1 v(t),\ \ T_2 u(t)),\ \ (u,v) \in X \times Y \tag{15}$$

where

$$\begin{aligned} T_1 v(t) &= \int_0^1 G(t,,s) f(s, v(s), D_{0^+}^p v(s)) ds, \\ T_2 u(t) &= \int_0^1 H(t,s) g(s, u(s), D_{0^+}^q u(s)) ds. \end{aligned}$$

In view of Lemma 2.5, the fixed point of the operator T coincides with the solution of BVP(3).

Let E_1 and E_2 be Banach spaces. Let us recall that an operator $T : E_1 \to E_2$ is called a completely continuous operator if it is continuous and maps each bounded subsets of E_1 into a relative compact subset in E_2.

Lemma 2.6. *Let $\Omega \subset X \times Y$. Then Ω is relative compact in $X \times Y$ if Ω satisfies the following conditions:*

(i) *Ω is uniformly bounded in $X \times Y$, i.e., there exists a constant $M > 0$ such that*

$$||(u,v)|| \leq M \text{ for all } (u,v) \in \Omega,$$

(ii) *Ω is an equicontinuous set, i.e., for each $\epsilon > 0$ there exists $\delta > 0$ such that, for all $t, \tau \in (0,1)$, $|t - \tau| < \delta$ implies*

$$|t^{2-\alpha} T_1 v(t) - \tau^{2-\alpha} T_1 v(\tau)| < \epsilon,\ |t^{2-\beta} T_2 u(t) - \tau^{2-\beta} T_2 u(\tau)| < \epsilon,$$

and

$$t^{2+q-\alpha} |D_{0^+}^q v(t) - \tau^{2+q-\alpha} D_{0^+}^q v(\tau)| < \epsilon,\ t^{2+p-\beta} |D_{0^+}^p u(t) - \tau^{2+p-\beta} D_{0^+}^p u(\tau)| < \epsilon.$$

Proof. The proof is standard and is omitted.

Lemma 2.7[14]Leray-Schauder Nonlinear Alternative. *Let E be a Banach space and Ω a bounded open subset of E with $0 \in \Omega$. Suppose $T : \overline{\Omega} \to E$ is a completely continuous operator. Then either there exist $x \in \partial\Omega$ and $\lambda \in (01,)$ such that $x = \lambda T x$ or there exists $x \in \overline{\Omega}$ such that $x = Tx$.*

Lemma 2.8. *Suppose that f is a $(\beta, p)-$Caratheodory function and g a $(\alpha, q)-$Caratheodory function. Then T is completely continuous.*

Proof. We first prove that $T : X \times Y \to X \times Y$ is well defined and T is continuous.

For $(u,v) \in X \times Y$, one has $u \in X$, $v \in Y$ and

$$T(u,v)(t) = \left(\int_0^1 G(t,s) f(s, v(s), D^p_{0+} v(s)) ds, \int_0^1 H(t,s) g(s, u(s), D^q_{0+} u(s)) ds \right).$$

There exists $r > 0$ such that

$$t^{2-\alpha}|u(t)|,\ t^{2+q-\alpha}|D^q_{0+}u(t)|,\ t^{2-\beta}|v(t)|,\ t^{2+p-\beta}|D^p_{0+}v(t)| \le r,\ t \in (0,1).$$

Since f is a $(\beta,p)-$Caratheodory function and g a $(\alpha,q)-$Caratheodory function, we know that there exist $\phi_r, \psi_r \in L^1(0,1)$ such that

$$|f(t, v(t), D^p_{0+}v(t))| \le \phi_r(t),\ |g(t, u(t), D^p_{0+}u(t))| \le \psi_r(t),\ t \in (0,1).$$

Hence

$$T_1 v(t) = \int_0^1 G(t,s) f(s, v(s), D^p_{0+} v(s)) ds$$

and

$$T_2 u(t) = \int_0^1 H(t,s) g(s, u(s), D^q_{0+} u(s)) ds$$

are continuous on $(0,1]$ and there exist the limits

$$\lim_{t\to 0} t^{2-\alpha} \int_0^1 G(t,s) f(s, v(s), D^p_{0+} v(s)) ds,$$
$$\lim_{t\to 0} t^{2-\beta} \int_0^1 H(t,s) g(s, u(s), D^q_{0+} u(s)) ds.$$

On the other hand, we have

$$\begin{aligned} D^p_{0+} T_1 v(t) &= \int_0^1 F(t,s) f(s, v(s), D^p_{0+} v(s)) ds, \\ D^q_{0+} T_2 u(t) &= \int_0^1 K(t,s) g(s, v(s), D^p_{0+} v(s)) ds. \end{aligned}$$

It is easy to see that $D^p_{0+}T_1 v$ and $D^q_{0+}T_2 u$ are continuous on $(0,1]$ and there exist the limits

$$\lim_{t\to 0} t^{2+q-\alpha} \int_0^1 F(t,s) f(s, v(s), D^p_{0+} v(s)) ds,$$
$$\lim_{t\to 0} t^{2+p-\beta} \int_0^1 K(t,s) g(s, u(s), D^q_{0+} u(s)) ds.$$

Hence $T(u,v) \in X \times Y$. Then T is well defined.

Suppose that $(u_n, v_n), (u_0, v_0) \in X \times Y$ with $(u_n, v_n) \to (u_0, v_0)$ as $n \to \infty$. Since that f is a $(\beta, p)-$Caratheodory function and g $(\alpha, q)-$Caratheodory function, we can prove that $T(u_n, v_n) \to T(u_0, v_0)$ as $n \to \infty$. The details are omitted.

Since that f is a $(\beta, p)-$Caratheodory function and g $(\alpha, q)-$Caratheodory function, we can prove that T maps bounded sets of $X \times Y$ to bounded sets. In fact, Ω is a bounded subset of $X \times Y$ implies that

$$||(u, v)|| \leq r \text{ for all } (u, v) \in \Omega.$$

Then there exist functions $\phi_r \in L^1(0, 1)$ and $\psi_r \in L^1(0, 1)$ such that

$$|f(t, v(t), D_{0+}^p v(t))| = \left| f(t, t^{\beta-2} t^{2-\beta} v(t), t^{\beta-p-2} t^{2+p-\beta} D_{0+}^p v(t)) \right| \leq \phi_r(t)$$

and

$$|g(t, u(t), D_{0+}^q u(t))| = \left| g(t, t^{\alpha-2} t^{2-\alpha} u(t), t^{\alpha-q-2} t^{2+q-\alpha} D_{0+}^q u(t)) \right| \leq \psi_r(t).$$

It is easy to see that

$$\begin{aligned} t^{2-\alpha}|T_1 v(t)| &\leq \left| \int_0^1 t^{2-\alpha} G(t,, s) f(s, v(s), D_{0+}^p v(s)) ds \right| \\ &\leq (1 + |k|\eta^{\alpha-1} + |1 - k\eta^{\alpha-2}|) \int_0^1 (1-s)^{\alpha-1} \phi_r(s) ds \end{aligned}$$

and

$$\begin{aligned} t^{2-\beta}|T_2 u(t)| &\leq \left| \int_0^1 t^{2-\beta} H(t,, s) g(s, v(s), D_{0+}^p v(s)) ds \right| \\ &\leq (1 + |k|\eta^{\alpha-1} + |1 - k\eta^{\alpha-2}|) \int_0^1 (1-s)^{\alpha-1} \phi_r(s) ds. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} t^{2+q-\alpha}|D_{0+}^q T_1 v(t)| &\leq \left| \int_0^1 t^{2+q-\alpha} H(t, s) f(s, v(s), D_{0+}^p v(s)) ds \right| \\ &\leq (1 + |l|\xi^{\beta-1} + |1 - l\xi^{\beta-2}|) \int_0^1 (1-s)^{\beta-1} \phi_r(s) ds \end{aligned}$$

and

$$\begin{aligned} t^{2+p-\beta}|D_{0+}^p T_2 u(t)| &\leq \left| \int_0^1 t^{2+p-\beta} F(t,, s) g(s, v(s), D_{0+}^p v(s)) ds \right| \\ &\leq [\Gamma(\beta - q)(1 + |l|\xi^{\beta-1}) + |1 - l\xi^{\beta-2}|(\beta - 1)\Gamma(\beta - p - 1)] \times \\ &\qquad \int_0^1 (1-s)^{\beta-1} \phi_r(s) ds. \end{aligned}$$

From above discussion, we know hat there exists a constant $M > 0$ such that

$$||T(u,v)|| \leq M \text{ for all } (u,v) \in \Omega.$$

Then T maps bounded sets of $X \times Y$ to bounded sets.

To prove that T maps Ω to relative compact subsets, one sees that if $t < \eta$, then

$$\begin{aligned}
& |t^{2-\alpha}T_1v(t) - \tau^{2-\alpha}T_1v(\tau)| \\
= & \left|\int_0^1 t^{\alpha-1}G(t,s)f(s,v(s),D_{0^+}^p v(s))ds - \int_0^1 \tau^{2-\alpha}G(\tau,s)f(s,v(s),D_{0^+}^p v(s))ds\right| \\
\leq & \int_0^t |1-k\eta^{\alpha-2}| \left|t^{2-\alpha}(t-s)^{\alpha-1} - \tau^{2-\alpha}(\tau-s)^{\alpha-1}\right| |f(s,v(s),D_{0^+}^p v(s))|ds \\
\leq & |1-k\eta^{\alpha-2}| \int_0^1 \left|t^{2-\alpha}(t-s)^{\alpha-1} - \tau^{2-\alpha}(\tau-s)^{\alpha-1}\right| \phi_r(s)ds
\end{aligned}$$

and if $t \geq \eta$, then

$$\begin{aligned}
& |t^{2-\alpha}T_1v(t) - \tau^{2-\alpha}T_1v(\tau)| \\
= & \left|\int_0^1 t^{\alpha-1}G(t,s)f(s,v(s),D_{0^+}^p v(s))ds - \int_0^1 \tau^{2-\alpha}G(\tau,s)f(s,v(s),D_{0^+}^p v(s))ds\right| \\
\leq & \int_0^\eta |1-k\eta^{\alpha-2}| \left|t^{2-\alpha}(t-s)^{\alpha-1} - \tau^{2-\alpha}(\tau-s)^{\alpha-1}\right| f(s,v(s),D_{0^+}^p v(s))ds \\
& + \int_\eta^t |1-k\eta^{\alpha-2}| \left|t^{2-\alpha}(t-s)^{\alpha-1} - \tau^{2-\alpha}(\tau-s)^{\alpha-1}\right| |f(s,v(s),D_{0^+}^p v(s))|ds \\
\leq & |1-k\eta^{\alpha-2}| \int_0^1 \left|t^{2-\alpha}(t-s)^{\alpha-1} - \tau^{2-\alpha}(\tau-s)^{\alpha-1}\right| \phi_r(s)ds.
\end{aligned}$$

Hence

$$|t^{2-\alpha}T_1v(t)-\tau^{2-\alpha}T_1v(\tau)| \leq |1-k\eta^{\alpha-2}| \int_0^1 \left|t^{2-\alpha}(t-s)^{\alpha-1} - \tau^{2-\alpha}(\tau-s)^{\alpha-1}\right| \phi_r(s)ds. \tag{16}$$

Furthermore, we have that

$$\begin{aligned}
& t^{2+p-\alpha}|D_{0^+}^p v(t) - \tau^{2+p-\alpha}D_{0^+}^p v(\tau)| \\
\leq & (1-l\xi^{\beta-2})(\beta-1)\Gamma(\beta-p-1) \times \\
& \int_0^1 \left|t^{2+p-\alpha}(t-s)^{\beta-1} - t^{2+p-\alpha}(t-s)^{\beta-1}\right| |f(s,v(s),D_{0^+}^p v(s))|ds \\
\leq & (1-l\xi^{\beta-2})(\beta-1)\Gamma(\beta-p-1) \times \\
& \int_0^1 \left|t^{2+p-\alpha}(t-s)^{\beta-1} - \tau^{2+p-\alpha}(\tau-s)^{\beta-1}\right| \phi_r(s)ds.
\end{aligned}$$

Analogously, it can be proved that

$$\begin{aligned} &|t^{2-\beta}T_2u(t)-\tau^{2-\beta}T_2u(\tau)| \\ &\leq |1-l\xi^{\beta-2}|\int_0^1 \left|t^{2-\beta}(t-s)^{\beta-1}-\tau^{2-\beta}(\tau-s)^{\beta-1}\right|\psi_r(s)ds, \end{aligned} \tag{17}$$

and

$$\begin{aligned} & t^{2+p-\alpha}|D^q_{0^+}u(t)-\tau^{2+p-\alpha}D^q_{0^+}u(\tau)| \\ \leq\ & (1-k\eta^{\alpha-2})(\alpha-1)\Gamma(\alpha-p-1)\times \\ & \int_0^1 \left|t^{2+p-\alpha}(t-s)^{\alpha-1}-t^{2+p-\alpha}(t-s)^{\alpha-1}\right||f(s,v(s),D^p_{0^+}v(s))|ds \\ \leq\ & (1-l\eta^{\alpha-2})(\alpha-1)\Gamma(\alpha-p-1)\times \\ & \int_0^1 \left|t^{2+p-\alpha}(t-s)^{\alpha-1}-\tau^{2+p-\alpha}(\tau-s)^{\alpha-1}\right|\psi_r(s)ds. \end{aligned}$$

It follows from Lemma 2.6 that $T\Omega$ is an equicontinuous set. Also it is uniformly bounded. Thus we conclude that T is completely continuous.

Lemma 2.9. *Suppose that* $m_i \geq 0 (i=1,2,3,4)$ *and* $l_i \geq 0 (i=1,2,3,4)$. *Then*
(i) *the inequality system*

$$\begin{cases} 0\leq x\leq m_1+m_2y+m_3y^{\rho_1}+m_4y^{\rho_2},\ 0<\rho_1,\rho_2<1, \\ 0\leq y\leq l_1+l_2x+l_3x^{\sigma_1}+l_4x^{\sigma_2},\ 0<\sigma_1,\sigma_2<1 \end{cases}$$

and $m_2l_2<1$ *imply that there exists a positive number* M_1 *depending only on* m_1, l_i *such that* $0\leq x\leq M_1$ *and* $0\leq y\leq M_1$*;*
(ii) *the inequality system*

$$\begin{cases} 0\leq x\leq m_2y+m_3y^{\rho_1}+m_4y^{\rho_2},\ \rho_1,\rho_2>1, \\ 0\leq y\leq l_2x+l_3x^{\sigma_1}+l_4x^{\sigma_2},\ \sigma_1,\sigma_2>1 \end{cases}$$

and $m_2l_2<1$ *imply that there exists a positive number* M_2 *depending only on* m_1, l_i *such that* $x\leq M_2$ *and* $y\leq M_2$.

Proof. **(i)** From the inequality system, we get

$$\begin{aligned} 0\leq x\ \leq\ & m_1+m_2[l_1+l_2x+l_3x^{\sigma_1}+l_4x^{\sigma_2}] \\ & +m_2[l_1+l_2x+l_3x^{\sigma_1}+l_4x^{\sigma_2}]^{\rho_1} \\ & +m_4[l_1+l_2x+l_3x^{\sigma_1}+l_4x^{\sigma_2}]^{\rho_2} \\ =\ & m_1+m_2l_1+m_2l_2x+m_2[l_3x^{\sigma_1}+l_4x^{\sigma_2}] \\ & +m_2[l_1+l_2x+l_3x^{\sigma_1}+l_4x^{\sigma_2}]^{\rho_1} \\ & +m_4[l_1+l_2x+l_3x^{\sigma_1}+l_4x^{\sigma_2}]^{\rho_2}. \end{aligned}$$

It follows that

$$\begin{aligned}0 \le 1 \quad \le \quad & m_2 l_2 + \frac{m_1 + m_2 l_1}{x} + \frac{m_2[l_3 x^{\sigma_1} + l_4 x^{\sigma_2}]}{x} \\ & + \frac{m_2[l_1 + l_2 x + l_3 x^{\sigma_1} + l_4 x^{\sigma_2}]^{\rho_1}}{x} \\ & + \frac{m_4[l_1 + l_2 x + l_3 x^{\sigma_1} + l_4 x^{\sigma_2}]^{\rho_2}}{x}.\end{aligned}$$

Since $m_2 l_2 < 1$ and $\rho_1, \rho_2, \sigma_1, \sigma_2 \in [0, 1)$, we get easily that there exists $M' > 0$ such that $x \le M'$.

Similarly we get that there exists $M'' > 0$ such that $y \le M''$. Then there exists a positive number M_1 depending only on m_i, l_i such that $0 \le x \le M_1$ and $0 \le y \le M_1$;

(ii) One sees that

$$\begin{aligned}0 \le x \quad \le \quad & m_2 l_2 x + m_2 l_3 x^{\sigma_1} + m_2 l_4 x^{\sigma_2} \\ & + m_3[l_2 x + l_3 x^{\sigma_1} + l_4 x^{\sigma_2}]^{\rho_1} \\ & + m_4[l_2 x + l_3 x^{\sigma_1} + l_4 x^{\sigma_2}]^{\rho_2}.\end{aligned}$$

If $x > 0$, then

$$\begin{aligned}1 \quad \le \quad & m_2 l_2 + \frac{m_2 l_3 x^{\sigma_1} + m_2 l_4 x^{\sigma_2}}{x} \\ & + \frac{m_3[l_2 x + l_3 x^{\sigma_1} + l_4 x^{\sigma_2}]^{\rho_1}}{x} \\ & + \frac{m_4[l_2 x + l_3 x^{\sigma_1} + l_4 x^{\sigma_2}]^{\rho_2}}{x}.\end{aligned}$$

Since $\rho_1, \rho_2, \sigma_1, \sigma_2 > 1$ and $m_2 l_2 < 1$, we know that there exists a positive number M' depending only on m_i, l_i such that $x \ge M'$. Similarly, we can show that that there exists a positive number M'' depending only on m_i, l_i such that $y \ge M''$. Hence there exists a positive number M_2 depending only on m_i, l_i such that $x \ge M_2$ and $y \ge M_2$. The proof is completed.

For the forthcoming analysis, we introduce the growth conditions on f and g as

(A) there exist nonnegative functions $a \in L^1(0, 1)$ and $\epsilon_i (i = 1, 2, 3, 4)$ defined on $(0, 1)$ such that

$$|f(t, x, y)| \le a(t) + \epsilon_1(t)|x| + \epsilon_2(t)|y| + \epsilon_3(t)|x|^{\rho_1} + \epsilon_4(t)|y|^{\rho_2}, 0 < \rho_1, \rho_2 < 1,$$

(B) there exist nonnegative functions $b \in L^1(0, 1)$ and $\delta_i (i = 1, 2, 3, 4)$ defined on $(0, 1)$ such that

$$|g(t, x, y)| \le b(t) + \delta_1(t)|x| + \delta_2(t)|y| + \delta_3(t)|x|^{\sigma_1} + \delta_4(t)|y|^{\sigma_2}, 0 < \sigma_1, \sigma_2 < 1.$$

(C) there exist nonnegative functions $\epsilon_i (i = 1, 2, 3, 4)$ defined on $(0, 1)$ such that

$$|f(t, x, y)| \leq \epsilon_1(t)|x| + \epsilon_2(t)|y| + \epsilon_3(t)|x|^{\rho_1} + \epsilon_4(t)|y|^{\rho_2}, \rho_1, \rho_2 > 1,$$

(D) there exist nonnegative functions $\delta_i (i = 1, 2, 3, 4)$ defined on $(0, 1)$ such that

$$|g(t, x, y)| \leq \delta_1(t)|x| + \delta_2(t)|y| + \delta_3(t)|x|^{\sigma_1} + \delta_4(t)|y|^{\sigma_2}, \sigma_1, \sigma_2 > 1.$$

Remark. The assumptions (A), (B), (C) and (D) generalize (BA1) and (BA2) supposed in [13].

Let us set the following notations for the convenience:

$$\begin{aligned}
\Lambda_1 &= \frac{1 + |k|\eta^{\alpha-1} + |1 - k\eta^{\alpha-2}|}{\Gamma(\alpha)|1 - k\eta^{\alpha-2}|}, \\
\Lambda_2 &= \frac{\Gamma(\alpha - q)(1 + |k|\eta^{\alpha-1}) + |1 - k\eta^{\alpha-2}|(\alpha - 1)\Gamma(\alpha - q - 1)}{\Gamma(\alpha - q)(\alpha - 1)\Gamma(\alpha - q - 1)|1 - k\eta^{\alpha-2}|}, \\
\Pi_1 &= \frac{1 + |l|\xi^{\beta-1} + |1 - k\xi^{\beta-2}|}{\Gamma(\beta)|1 - l\xi^{\beta-2}|}, \\
\Pi_2 &= \frac{\Gamma(\beta - p)(1 + |l|\xi^{\beta-1}) + |1 - l\xi^{\beta-2}|(\beta - 1)\Gamma(\beta - p - 1)}{\Gamma(\beta - p)(\beta - 1)\Gamma(\beta - p - 1)|1 - l\xi^{\beta-2}|}.
\end{aligned}$$

Theorem 2.1. *Suppose that f is a $(\beta, p)-$Caratheodory function and g $(\alpha, q)-$Caratheodory function, (A) and (B) hold. Then BVP(3) has at least one solution if*

$$\begin{aligned}
&\max\{\Lambda_1, \Lambda_2\} \max\{\Pi_1, \Pi_2\} \times \\
&\left(\int_0^1 (1 - s)^{\alpha-1} s^{\alpha-2} \epsilon_1(s) ds + \int_0^1 (1 - s)^{\alpha-1} s^{\alpha-q-2} \epsilon_2(s) ds\right) \times \\
&\left(\int_0^1 (1 - s)^{\beta-1} s^{\beta-2} \delta_1(s) ds + \int_0^1 (1 - s)^{\beta-1} s^{\beta-p-2} \delta_2(s) ds\right) < 1.
\end{aligned}$$

Proof. Consider the set

$$\Omega_1 = \{(u, v) \in X \times Y : (u, v) = \lambda T(u, v) \text{ for some } \lambda \in (0, 1)\}.$$

We first prove that Ω_1 is bounded. For $(u, v) \in \Omega_1$, we see that $(u, v) = \lambda T(u, v)$. Then $u = \lambda T_1 v$ and $v = \lambda T_2 u$.

We obtain by Remark 2.1 and (A) that

$$
\begin{aligned}
& |t^{2-\alpha}T_1v(t)| \\
= \ & t^{2-\alpha}\left|\int_0^1 G(t,,s)f(s,v(s),D_{0+}^p v(s))ds\right| \\
\leq \ & \Lambda_1\int_0^1 (1-s)^{\alpha-1}|f(s,v(s),D_{0+}^p v(s))|ds \\
\leq \ & \Lambda_1\int_0^1 (1-s)^{\alpha-1}a(s)ds+\Lambda_1\int_0^1 (1-s)^{\alpha-1}\epsilon_1(s)|v(s)|ds \\
& +\Lambda_1\int_0^1 (1-s)^{\alpha-1}\epsilon_2(s)|D_{0+}^p v(s)|ds+\Lambda_1\int_0^1 (1-s)^{\alpha-1}\epsilon_3(s)|v(s)|^{\rho_1}ds \\
& +\Lambda_1\int_0^1 (1-s)^{\alpha-1}\epsilon_4(s)|D_{0+}^p v(s)|^{\rho_2}ds \\
= \ & \Lambda_1\left(\int_0^1 (1-s)^{\alpha-1}a(s)ds+\int_0^1 (1-s)^{\alpha-1}\epsilon_1(s)s^{\alpha-2}s^{2-\alpha}|v(s)|ds\right. \\
& +\int_0^1 (1-s)^{\alpha-1}\epsilon_2(s)s^{\alpha-q-2}s^{2+q-\alpha}|D_{0+}^p v(s)|ds \\
& +\int_0^1 (1-s)^{\alpha-1}\epsilon_3(s)s^{(\alpha-2)\rho_1}s^{(2-\alpha)\rho_1}|v(s)|^{\rho_1}ds \\
& \left.+\int_0^1 (1-s)^{\alpha-1}\epsilon_4(s)s^{\rho_2(\alpha-q-2)}s^{\rho_2(2+q-\alpha)}|D_{0+}^p v(s)|^{\rho_2}ds\right) \\
\leq \ & \Lambda_1\int_0^1 (1-s)^{\alpha-1}a(s)ds \\
& +\Lambda_1\left[\int_0^1 (1-s)^{\alpha-1}s^{\alpha-2}\epsilon_1(s)ds\sup_{t\in(0,1]}t^{2-\alpha}|v(t)|\right. \\
& +\int_0^1 (1-s)^{\alpha-1}s^{\alpha-q-2}\epsilon_2(s)ds\sup_{t\in(0,1]}t^{2+q-\alpha}|D_{0+}^p v(t)| \\
& +\int_0^1 (1-s)^{\alpha-1}s^{(\alpha-2)\rho_1}\epsilon_3(s)ds\left(\sup_{t\in(0,1]}t^{2-\alpha}|v(t)|\right)^{\rho_1} \\
& \left.+\int_0^1 (1-s)^{\alpha-1}s^{\rho_2(\alpha-q-2)}\epsilon_4(s)ds\left(\sup_{t\in(0,1]}t^{2+q-\alpha}|D_{0+}^p v(t)|\right)^{\rho_2}\right] \\
\leq \ & \Lambda_1\int_0^1 (1-s)^{\alpha-1}a(s)ds \\
& +\Lambda_1\left[\int_0^1 (1-s)^{\alpha-1}s^{\alpha-2}d\epsilon_1(s)s||v||_Y\right. \\
& +\int_0^1 (1-s)^{\alpha-1}s^{\alpha-q-2}\epsilon_2(s)ds||v||_Y
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^1 (1-s)^{\alpha-1}s^{(\alpha-2)\rho_1}\epsilon_3(s)ds||v||_Y^{\rho_1}\\
&+\int_0^1 (1-s)^{\alpha-1}s^{\rho_2(\alpha-q-2)}\epsilon_4(s)ds||v||_Y^{\rho_2}\Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
& |t^{2+q-\alpha}D_{0^+}^q T_1 v(t)|\\
= \ & t^{2+q-\alpha}\left|\int_0^1 K(t,,s)f(s,v(s),D_{0^+}^p v(s))ds\right|\\
\leq \ & \Lambda_2\int_0^1 (1-s)^{\alpha-1}|f(s,v(s),D_{0^+}^p v(s))|ds\\
\leq \ & \Lambda_2\int_0^1 (1-s)^{\alpha-1}a(s)ds\\
& +\Lambda_2\Bigg[\int_0^1 (1-s)^{\alpha-1}s^{\alpha-2}\epsilon_1(s)ds||v||_Y\\
& +\int_0^1 (1-s)^{\alpha-1}s^{\alpha-q-2}\epsilon_2(s)ds||v||_Y\\
& +\int_0^1 (1-s)^{\alpha-1}s^{(\alpha-2)\rho_1}\epsilon_3(s)ds||v||_Y^{\rho_1}\\
& +\int_0^1 (1-s)^{\alpha-1}s^{\rho_2(\alpha-q-2)}\epsilon_4(s)ds||v||_Y^{\rho_2}\Bigg].
\end{aligned}
$$

Thus

$$
\begin{aligned}
||T_1 v||_X &= \max\left\{\sup_{t\in(0,1]} t^{2-\alpha}|T_1 v(t)|,\ \sup_{t\in(0,1]} t^{2+q-\alpha}|D_{0^+}^q T_1 v(t)|\right\}\\
&\leq \max\{\Lambda_1,\Lambda_2\}\Bigg[\int_0^1 (1-s)^{\alpha-1}a(s)ds\\
&\quad +\left(\int_0^1 (1-s)^{\alpha-1}s^{\alpha-2}\epsilon_1(s)ds+\int_0^1 (1-s)^{\alpha-1}s^{\alpha-q-2}\epsilon_2(s)ds\right)||v||_Y\\
&\quad +\int_0^1 (1-s)^{\alpha-1}s^{(\alpha-2)\rho_1}\epsilon_3(s)ds||v||_Y^{\rho_1}\\
&\quad +\int_0^1 (1-s)^{\alpha-1}s^{\rho_2(\alpha-q-2)}\epsilon_4(s)ds||v||_Y^{\rho_2}\Bigg].
\end{aligned}
$$

Since $||u||_X=\lambda||T_1 v||_X\leq ||T_1 v||_X$, we get

$$
||u||_X \leq \max\{\Lambda_1,\Lambda_2\}\left[\int_0^1 (1-s)^{\alpha-1}a(s)ds\right.
$$

$$+\left(\int_0^1 (1-s)^{\alpha-1}s^{\alpha-2}\epsilon_1(s)ds+\int_0^1(1-s)^{\alpha-1}s^{\alpha-q-2}\epsilon_2(s)ds\right)||v||_Y$$
$$+\int_0^1 (1-s)^{\alpha-1}s^{(\alpha-2)\rho_1}\epsilon_3(s)ds||v||_Y^{\rho_1}$$
$$\left.+\int_0^1 (1-s)^{\alpha-1}s^{\rho_2(\alpha-q-2)}\epsilon_4(s)ds||v||_Y^{\rho_2}\right].$$

Similarly, it can be shown that

$$\begin{aligned}||T_2u||_Y &= \max\left\{\sup_{t\in(0,1]} t^{2-\beta}|T_1u(t)|,\ \sup_{t\in(0,1]} t^{2+p-\beta}|D_{0^+}^pT_1v(t)|\right\}\\ &\le \max\{\Pi_1,\Pi_2\}\left[\int_0^1(1-s)^{\alpha-1}b(s)ds\right.\\ &\quad+\left(\int_0^1(1-s)^{\beta-1}s^{\beta-2}\delta_1(s)ds+\int_0^1(1-s)^{\beta-1}s^{\beta-p-2}\delta_2(s)ds\right)||u||_X\\ &\quad+\int_0^1(1-s)^{\beta-1}s^{(\beta-2)\sigma_1}\delta_3(s)ds||u||_X^{\sigma_1}\\ &\quad\left.+\int_0^1(1-s)^{\beta-1}s^{\sigma_2(\beta-p-2)}\delta_4(s)ds||u||_X^{\sigma_2}\right].\end{aligned}$$

Then $||v||_Y=\lambda||T_2u||_Y\le||T_2u||_Y$ implies that

$$\begin{aligned}||v||_Y &\le \max\{\Pi_1,\Pi_2\}\left[\int_0^1(1-s)^{\alpha-1}b(s)ds\right.\\ &\quad+\left(\int_0^1(1-s)^{\beta-1}s^{\beta-2}\delta_1(s)ds+\int_0^1(1-s)^{\beta-1}s^{\beta-p-2}\delta_2(s)ds\right)||u||_X\\ &\quad+\int_0^1(1-s)^{\beta-1}s^{(\beta-2)\sigma_1}\delta_3(s)ds||u||_X^{\sigma_1}\\ &\quad\left.+\int_0^1(1-s)^{\beta-1}s^{\sigma_2(\beta-p-2)}\delta_4(s)ds||u||_X^{\sigma_2}\right].\end{aligned}$$

From Lemma 2.9(i) and above discussions, we see that there exists positive number $M_1>0$ such that

$$||u||_X\le M_1,\ \ ||v||_Y\le M_1.$$

Hence $||(u,v)||\le M_1$ for all $(u,v)\in\Omega_1$.

Choose $\Omega=\{(u,v)\in X\times Y: ||(u,v)||<M_1+1\}$. It is easy to find that

$$(u,v)\ne\lambda T(u,v) \text{ for all } (u,v)\in\partial\Omega \text{ and } \lambda\in(0,1).$$

Then Lemma 2.7 implies that there is at least one $(u,v)\in\overline{\Omega}$ such that $(u,v)=T(u,v)$. Hence (u,v) is a solution of BVP(3). The proof is completed.

Theorem 2.2. *Suppose that f is a $(\beta,p)-$Caratheodory function and g a $(\alpha,q)-$ Caratheodory function, (C) and (D) hold. Then BVP(3) has at least one solution if*

$$\max\{\Lambda_1,\Lambda_2\}\max\{\Pi_1,\Pi_2\}\times \\ \left(\int_0^1(1-s)^{\alpha-1}s^{\alpha-2}\epsilon_1(s)ds+\int_0^1(1-s)^{\alpha-1}s^{\alpha-q-2}\epsilon_2(s)ds\right)\times \\ \left(\int_0^1(1-s)^{\beta-1}s^{\beta-2}\delta_1(s)ds+\int_0^1(1-s)^{\beta-1}s^{\beta-p-2}\delta_2(s)ds\right)<1.$$

Proof. Consider the set

$$\Omega_1=\{(u,v)\in X\times Y:(u,v)=\lambda T(u,v)\text{ for some }\lambda\in(0,1)\}.$$

We first prove that Ω_1 is bounded. For $(u,v)\in\Omega_1$, we see that $(u,v)=\lambda T(u,v)$.

Similarly to the proof of Theorem 2.1, we get

$$||u||_X\le\max\{\Lambda_1,\Lambda_2\}\times \\ \left[\left(\int_0^1(1-s)^{\alpha-1}s^{\alpha-2}\epsilon_1(s)ds+\int_0^1(1-s)^{\alpha-1}s^{\alpha-q-2}\epsilon_2(s)ds\right)||v||_Y \right.\\ \left.+\int_0^1(1-s)^{\alpha-1}s^{(\alpha-2)\rho_1}\epsilon_3(s)ds||v||_Y^{\rho_1}+\int_0^1(1-s)^{\alpha-1}s^{\rho_2(\alpha-q-2)}\epsilon_4(s)ds||v||_Y^{\rho_2}\right],$$

and

$$||v||_Y\le\max\{\Pi_1,\Pi_2\}\times \\ \left[\left(\int_0^1(1-s)^{\beta-1}s^{\beta-2}\delta_1(s)ds+\int_0^1(1-s)^{\beta-1}s^{\beta-p-2}\delta_2(s)ds\right)||u||_X \right.\\ \left.+\int_0^1(1-s)^{\beta-1}s^{(\beta-2)\sigma_1}\delta_3(s)ds||u||_X^{\sigma_1}+\int_0^1(1-s)^{\beta-1}s^{\sigma_2(\beta-p-2)}\delta_4(s)ds||u||_X^{\sigma_2}\right].$$

Since $\rho_1,\rho_2>1$, from Lemma 2.9(ii), we know that there exists a constant $M_2>0$ such that $||u||_X\le M_2$ and $||v||_Y>M_2$. Hence $||(u,v)||>M_2$ for all $(u,v)\in\Omega_1$.

Choose $\Omega=\left\{(u,v)\in X\times Y:||(u,v)||<\frac{1}{2}M_2\right\}$. It is easy to find that

$$(u,v)\ne\lambda T(u,v)\text{ for all }(u,v)\in\partial\Omega\text{ and }\lambda\in(0,1).$$

Then Lemma 2.7 implies that there is at least one $(u,v)\in\overline{\Omega}$ such that $(u,v)=T(u,v)$. Hence (u,v) is a solution of BVP(3). The proof is completed.

Corollary 2.1. *Suppose that*

(A1) *there exist nonnegative functions* $a \in L^1(0,1)$ *and* $\epsilon_i(i=1,2)$ *such that*

$$|f(t,x,y)| \le a(t) + \epsilon_1(t)|x|^{\rho_1} + \epsilon_2(t)|y|^{\rho_2}, 0 < \rho_1, \rho_2 < 1,$$

(B1) *there exist nonnegative functions* $b \in L^1(0,1)$ *and* $\delta_i(i=1,2)$ *such that*

$$|g(t,x,y)| \le b(t) + \delta_1(t)|x|^{\sigma_1} + \delta_2(t)|y|^{\sigma_2}, 0 < \sigma_1, \sigma_2 < 1.$$

Then BVP(3) has at least one solution.
Proof. It follows from Theorem 2.1 and the proof is omitted.

Corollary 2.2. *Suppose that*

(C1) *there exist nonnegative functions* $\epsilon_i(i=1,2)$ *such that*

$$|f(t,x,y)| \le \epsilon_1(t)|x|^{\rho_1} + \epsilon_2(t)|y|^{\rho_2}, \rho_1, \rho_2 > 1,$$

(D1) *there exist nonnegative function* $\delta_i(i=1,2)$ *such that*

$$|g(t,x,y)| \le \delta_1(t)|x|^{\sigma_1} + \delta_2(t)|y|^{\sigma_2}, \sigma_1, \sigma_2 > 1.$$

Then BVP(3) has at least one solution.
Proof. It follows from Theorem 2.2 and the proof is omitted.

3. An example

In this section, we give an example to illustrate Theorem 2.1.
Example 3.1. Consider the three-point boundary value problem of the form

$$\begin{cases} D_{0^+}^{\frac{3}{2}}u(t) = a_1 + \lambda t^{-\frac{1}{10}}v(t) + \lambda t D_{0^+}^{\frac{1}{10}}v(t) \\ \qquad + \left(t-\frac{2}{3}\right)^4 \left[(v(t))^{\frac{1}{2}} + (D_{0^+}^{\frac{1}{10}}v(t))^{\frac{1}{2}}\right],\ t \in (0,1), \\ D_{0^+}^{\frac{7}{5}}u(t) = a_2 + \mu t^{-\frac{1}{4}}u(t) + \mu t D_{0^+}^{\frac{1}{10}}u(t) \\ \qquad + \left(t-\frac{2}{5}\right)^6 \left[(u(t))^{\frac{1}{2}} + (D_{0^+}^{\frac{1}{10}}u(t))^{\frac{1}{2}}\right],\ t \in (0,1), \\ \left[I_{0^+}^{\frac{1}{2}}u(t)\right]'\Big|_{t=0} = 0\,, \\ \left[I_{0^+}^{\frac{3}{5}}v(t)\right]'\Big|_{t=0} = 0\,, \\ u(1) = 0, \\ v(1) = 0, \end{cases} \tag{18}$$

where $\rho_i, \sigma_i \in (0,1)(i=1,2)$ and a_1, a_2 are constants different from 0.

Corresponding to BVP(1), we have $\alpha = \frac{3}{2}, \beta = \frac{7}{5}$, $p = q = \frac{1}{10}$, $\rho_1 = \rho_2 = \sigma_1 = \sigma_2 = 0$, $k == l = 0$ and

$$f(t,x,y) = a_1 + \lambda t^{-\frac{1}{10}} x + \lambda t y + \left(t - \frac{2}{3}\right)^4 \left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right),$$

and

$$g(t,x,y) = a_2 + \mu t^{-\frac{1}{4}} x + \mu t y + \left(t - \frac{2}{5}\right)^6 \left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right).$$

It is easy to see that

$$\begin{aligned} f\left(t, t^{\frac{7}{5}-2}x, t^{\frac{7}{5}-\frac{1}{10}-2}y\right) &= f\left(t, t^{-\frac{3}{5}}x, t^{-\frac{7}{10}}y\right) \\ &= a_1 + \lambda t^{-\frac{7}{10}}x + \lambda t^{\frac{3}{10}}y + \left(t - \frac{2}{3}\right)^4 \left[t^{-\frac{3}{10}}x^{\frac{1}{2}} + t^{-\frac{7}{20}}y^{\frac{1}{2}}\right], \\ g\left(t, t^{\frac{3}{2}-2}x, t^{\frac{3}{2}-\frac{1}{10}-2}y\right) &= a_2 + \mu t^{-\frac{3}{4}}x + \mu t^{\frac{2}{5}}y + \left(t - \frac{2}{5}\right)^6 \left[t^{-\frac{3}{8}}x^{\frac{1}{2}} + t^{\frac{1}{5}}y^{\frac{1}{2}}\right]. \end{aligned}$$

So f is a $\left(\frac{7}{5}, \frac{1}{10}\right)$ −Caratheodory function and g $\left(\frac{3}{2}, \frac{1}{10}\right)$ −Caratheodory function.

By computation, we know that

$$\begin{aligned} \Lambda_1 &= \frac{1 + |k|\eta^{\alpha-1} + |1 - k\eta^{\alpha-2}|}{\Gamma(\alpha)|1 - k\eta^{\alpha-2}|} = \frac{2}{\Gamma(3/2)}, \\ \Lambda_2 &= \frac{\Gamma(\alpha-q)(1 + |k|\eta^{\alpha-1}) + |1 - k\eta^{\alpha-2}|(\alpha-1)\Gamma(\alpha-q-1)}{\Gamma(\alpha-q)(\alpha-1)\Gamma(\alpha-q-1)|1 - k\eta^{\alpha-2}|} \\ &= \frac{9}{5\Gamma(17/5)}, \\ \Pi_1 &= \frac{1 + |l|\xi^{\beta-1} + |1 - k\xi^{\beta-2}|}{\Gamma(\beta)|1 - l\xi^{\beta-2}|} = \frac{2}{\Gamma(3/2)}, \\ \Pi_2 &= \frac{\Gamma(\beta-p)(1 + |l|\xi^{\beta-1}) + |1 - l\xi^{\beta-2}|(\beta-1)\Gamma(\beta-p-1)}{\Gamma(\beta-p)(\beta-1)\Gamma(\beta-p-1)|1 - l\xi^{\beta-2}|} \\ &= \frac{7}{4\Gamma(13/10)}. \end{aligned}$$

Then Theorem 2.1 implies that BVP(18) has at least one solution if

$$\frac{4}{[\Gamma(3/2)]^2} \left(\lambda \int_0^1 (1-s)^{\frac{1}{2}} s^{-\frac{3}{5}} ds + \lambda \int_0^1 (1-s)^{\frac{1}{2}} s^{\frac{2}{5}} ds\right) \times \left(\mu \int_0^1 (1-s)^{\frac{2}{5}} s^{-\frac{7}{20}} ds + \mu \int_0^1 (1-s)^{\frac{2}{5}} s^{\frac{13}{20}} ds\right) < 1.$$

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