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A NEW GENERALIZED AL-OBOUDI DIFFERENTIAL OPERATOR DEFINED BY WRIGHT'S GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract. In this paper, firstly we define the generalization of the generalized Al-Oboudi differential operator from some well-known operators on the class $A(p,n)$ of analytic functions in the unit disc $U = \{z \in C :| z |< 1\}$. Then we also define new classes of analytic and p-valently starlike and convex functions with complex order by means of this new general differential operator. Our main purpose is to determine coefficient bounds for functions in certain subclasses of this classes.Relevant connections of some results obtained with those in earlier works are also provided.

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1. Introduction

Let $A(p,n)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k, (p,n \in N = \{1,2,...\}) \tag{1}$$

which are analytic and p-valent in the open unit disc $U = \{z \in C :| z |< 1\}$. In particular, we set $A(p,1) = A(p), A(1,1) = A(1) = A$. Now, following the earlier investigations by Goodman [11], Ruscheweyh [20] and Altïntas et al. [2] (see also [4,3]), we define the (n,w) -neighborhood of a function $f(z) \in A(p,n)$, by

$$N^p_{n,w}(f) = \left\{ q : q(z) \in A(p,n), q(z) = z^p - \sum_{k=n+p}^{\infty} c_k z^k and \sum_{k=n+p}^{\infty} k|a_k - c_k| \le w \right\}. \tag{2}$$

In particular, if

$$h(z) = z^p \qquad (p \in N), \tag{3}$$

we immediately have

$$N^p_{n,w}(h) = \left\{ q : q(z) \in A(p,n), q(z) = z^p - \sum_{k=n+p}^{\infty} c_k z^k and \sum_{k=n+p}^{\infty} k|c_k| \leq w \right\}. \quad (4)$$

For positive real parameters $\alpha_1, A_1, ..., \alpha_u, A_u$ and $\beta_1, B_1, ..., \beta_v, B_v (u, v \in N = 1, 2, 3, ...)$ such that

$$1 + \sum_{j=1}^{v} B_j - \sum_{j=1}^{u} A_j \geq 0$$

in [26]. The Wright's generalized hypergeometric function

$$\begin{aligned} &{}_u\psi_v[(\alpha_1, A_1), ..., (\alpha_u, A_u); (\beta_1, B_1), ..., (\beta_v, B_v); z] \\ &= {}_u\psi_v[(\alpha_j, A_j)_{1,u}; (\beta_j, B_j)_{1,v}; z] \end{aligned}$$

is defined by

$$\begin{aligned} &{}_u\psi_v[(\alpha_j, A_j)_{1,u}; (\beta_j, B_j)_{1,v}; z] \\ &= \sum_{k=0}^{\infty} \left(\prod_{j=1}^{u} \Gamma(\alpha_j + kA_j) \right) \left(\prod_{j=1}^{v} \Gamma(\beta_j + kB_j) \right)^{-1} \frac{z^k}{k!}, \qquad z \in U. \end{aligned}$$

If $A_j = 1 (j = 1, 2, ..., u)$ and $B_j = 1 (j = 1, 2, ..., v)$, we have

$$\begin{aligned} \Omega\, {}_u\psi_v[(\alpha_j, 1)_{1,u}; (\beta_j, 1)_{1,v}; z] &= {}_uF_v(\alpha_1, ..., \alpha_u; \beta_1, ..., \beta_v; z) \\ &= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k ... (\alpha_u)_k}{(\beta_1)_k ... (\beta_v)_k} \frac{z^k}{k!} \end{aligned} \quad (5)$$

$(u \leq v + 1; u, v \in N; z \in U)$ is the generalized hypergeometric function (see for details [10]) where (α_k) is the Pochhammer symbol and

$$\Omega = \left(\prod_{j=1}^{u} \Gamma(\alpha_j) \right)^{-1} \left(\prod_{j=1}^{v} \Gamma(\beta_j) \right).$$

By using the generalized hypergeometric function, Dziok and Srivastava introduced the linear operator in [10]. Also, Dziok and Raina extended the linear operator by using Wright's generalized hypergeometric function in [9]. Firstly in [12], the authors defined the function ${}_u\phi_v$ as follows:

$${}_u\phi_v[(\alpha_j, A_j)_{1,u}; (\beta_j, B_j)_{1,v}; z] = \Omega z^p\, {}_u\psi_v[(\alpha_j, A_j)_{1,u}; (\beta_j, B_j)_{1,v}; z].$$

$$_u\phi_v[(\alpha_j, A_j)_{1,u}; (\beta_j, B_j)_{1,v}; z] = \Omega z^p \, _u\psi_v[(\alpha_j, A_j)_{1,u}; (\beta_j, B_j)_{1,v}; z].$$

Let $\Theta[(\alpha_j, A_j)_{1,u}; (\beta_j, B_j)_{1,v}] : A(p,n) \to A(p,n)$ be a linear operator defined by

$$\Theta[(\alpha_j, A_j)_{1,u}; (\beta_j, B_j)_{1,v}]f(z) = \, _u\phi_v[(\alpha_j, A_j)_{1,u}; (\beta_j, B_j)_{1,v}; z] * f(z).$$

We observe that, for $f(z)$ of the form (1), we have

$$\Theta[(\alpha_j, A_j)_{1,u}; (\beta_j, B_j)_{1,v}]f(z) = z^p + \sum_{k=n+p}^{\infty} \rho_k a_k z^k \tag{6}$$

where

$$\rho_k = \frac{\Omega\Gamma(\alpha_1 + A_1(k-p))...\Gamma(\alpha_u + A_u(k-p))}{(k-p)!\Gamma(\beta_1 + B_1(k-p))...\Gamma(\beta_v + B_v(k-p))}.$$

For convenience, we use the linear operator

$$\Theta[\alpha_1]f(z) = \Theta[(\alpha_1, A_1), ..., (\alpha_u, A_u); (\beta_1, B_1), ..., (\beta_v, B_v)]f(z).$$

Indeed, by setting $A_j = 1 (j = 1, ..., u)$, $B_j = 1 (j = 1, ..., v)$ and $p = 1$ the linear operator $\Theta[\alpha_1]$, leads immediately to the Dziok-Srivastava operator [10] which contains, as its further special cases, such other linear operators of Geometric Function Theory as the Hohlov operator, the Carlson-Shaffer operator [7], the Ruscheweyh derivative operator [19], the generalized Bernardi-Libera-Livingston operator, the fractional derivative operator [24]. See also [10] and [9] in which comprehensive details of various other operators are given.

Motivated by the earlier works of [10,13,14,22,23,25] we introduce a new subclass of p-valent functions with negative coefficients and discuss some interesting properties of this generalized function class. Now we define the generalization of the generalized Al-Oboudi differential operator $D^{m,\gamma}_{\lambda,l,p,\delta,\beta}$ as follows:

$$D^0 f(z) = f(z)$$

$$D^{1,\gamma}_{\lambda,l,p,\delta,\beta} f(z) = \frac{p - p(\lambda-\delta)(\beta-\gamma) + l}{p+l}\Theta f(z) + \frac{(\lambda-\delta)(\beta-\gamma)}{p+l} z(\Theta f(z))'$$

$$D^{2,\gamma}_{\lambda,l,p,\delta,\beta} f(z) = D^{\gamma}_{\lambda,l,p,\delta,\beta}(D^{1,\gamma}_{\lambda,l,p,\delta,\beta} f(z)),$$

$$\vdots$$

$$D^{m,\gamma}_{\lambda,l,p,\delta,\beta} f(z) = D^{\gamma}_{\lambda,l,p,\delta,\beta}(D^{m-1,\gamma}_{\lambda,l,p,\delta,\beta} f(z)), \tag{7}$$

where $\beta, \lambda, \delta, \gamma, l \geq 0, \lambda > \delta, \beta > \gamma, m \in N_0 = N \cup \{0\}$.

If $f(z)$ is given by (1), then , by (6) and (7), we see that

$$D^{m,\gamma}_{\lambda,l,p,\delta,\beta}f(z) = z^p + \sum_{k=n+p}^{\infty} \sigma_k^m a_k z^k,$$

where

$$\sigma_k = \left[\frac{\Omega\Gamma(\alpha_1 + A_1(k-p))...\Gamma(\alpha_u + A_u(k-p))}{(k-p)!\Gamma(\beta_1 + B_1(k-p))...\Gamma(\beta_v + B_v(k-p))} \frac{p + (\lambda-\delta)(\beta-\gamma)(k-p) + l}{p+l} \right].$$

Remark 1.1 Let be $n = 1, p = 1, l = 0$.Then

(i)For the function $\Theta[\alpha_1]f(z) = \Theta[(1,1),(1,1),(1,1);(1,1),(1,1)]f(z)$ we have $D^k_{\alpha,\beta,\lambda,\delta}$ which was introduced [18].

(ii)For $\gamma = 0, \beta = 1, \delta = 0$ and the function $\Theta[\alpha_1]f(z) = \Theta[(1,1),(1,1),(1,1);(1,1),(1,1)]f(z)$ we have $I^m(\lambda, l)$ and D^m_λ which were studied in [16,1].

(iii)For $\gamma = 0$ and the function $\Theta[\alpha_1]f(z) = \Theta[(1,1),(1,1),(1,1);(1,1),(1,1)]f(z)$ we have $D^k_{\alpha,\beta,\lambda}$ which is generalized differential operator [8].

(iv)For $\gamma = 0, \beta = 1, \delta = 0$ and the function

$\Theta[\alpha_1]f(z) = \Theta\left[\left(1, -1 + \frac{k-\alpha}{k-p}\right), \left(1, 1 + \frac{1}{k-p}\right), (1,1); \left(1, 1 + \frac{1-\alpha}{k-p}\right), \left(1, -1 + \frac{k}{k-p}\right)\right] f(z)$
we have $D^{n,\gamma}_\lambda$ which was studied in [5].

(v)For $\gamma = 0, \lambda = 1, \beta = 1, \delta = 0$ and the function

$\Theta[\alpha_1]f(z) = \Theta[(1,1),(1,1),(1,1);(1,1),(1,1)]f(z)$ we have D^m which is Salagean differential operator [21].

Remark 1.2
(i)For $\gamma = 0, \beta = 1, \delta = 0$ and the function

$\Theta[\alpha_1]f(z) = \Theta\left[\left(1, -1 + \frac{k-\alpha}{k-p}\right), \left(1, 1 + \frac{p}{k-p}\right), (1,1); \left(1, 1 + \frac{p-\alpha}{k-p}\right), \left(1, -1 + \frac{k}{k-p}\right)\right] f(z)$
we have $D^{m,\alpha}_{\lambda,l,p}$ which was generalized Al-Oboudi differential operator [6].

(ii)If we set $n = 1$ and $p = 1$, for $m = 1, \delta = \lambda = 1$ or $\beta = \gamma = 1$ and the function $\Theta[\alpha_1]f(z) = \Theta\left[\left(1, -1 + \frac{k-\alpha}{k-1}\right), \left(1, 1 + \frac{1}{k-1}\right), (1,1); \left(1, 1 + \frac{1-\alpha}{k-1}\right), \left(1, -1 + \frac{k}{k-1}\right)\right] f(z)$
we have Ω^α which is Owa-Srivastava fractional differential operator [17].

Now we define following new classes of p-valently starlike and convex functions with complex order ,respectively, as follows:

Let $S^{m,\gamma,p}_{\lambda,l,\delta,\beta}(n,b)$ be the class of functions $f(z)\in A(p,n)$ satisfying

$$Re\left\{1+\frac{1}{b}\left(\frac{1}{p}\frac{z(D^{m,\gamma}_{\lambda,l,p,\delta,\beta}f(z))'}{D^{m,\gamma}_{\lambda,l,p,\delta,\beta}f(z)}-1\right)\right\}>0 \tag{8}$$

where $z\in U, b\in\mathcal{C}-\{0\}$ and $D^{m,\gamma}_{\lambda,l,p,\delta,\beta}$ is the generalization of the generalized Al-Oboudi differential operator.

Let $K^{m,\gamma,p}_{\lambda,l,\delta,\beta}(n,b)$ be the class of functions $f(z)\in A(p,n)$ satisfying

$$Re\left\{1-\frac{1}{b}+\frac{1}{bp}\left(1+\frac{z(D^{m,\gamma}_{\lambda,l,p,\delta,\beta}f(z))''}{\left(D^{m,\gamma}_{\lambda,l,p,\delta,\beta}f(z)\right)'}\right)\right\}>0,$$

where $z\in U, b\in\mathcal{C}-\{0\}$ and $D^{m,\gamma}_{\lambda,l,p,\delta,\beta}$ is the generalization of the generalized Al-Oboudi differential operator.

We note that $f(z)\in K^{m,\gamma,p}_{\lambda,l,\delta,\beta}(n,b)$ if and only if $\frac{zf'(z)}{p}\in S^{m,\gamma,p}_{\lambda,l,\delta,\beta}(n,b)$.

Remark 1.3 We also have

(i) $S^{0,\gamma,1}_{\lambda,0,\delta,\beta}(1,b)\equiv S^*(b)$ defined by Nasr and Aouf in [15].

(ii) $S^{0,\gamma,1}_{\lambda,0,\delta,\beta}(1,1)\equiv S^*$ which is familiar class of starlike functions in U.

2.Coefficient Inequalities

Theorem 2.1*Let the function $f(z)$ be defined by (1). Then $f(z)\in S^{m,\gamma,p}_{\lambda,l,\delta,\beta}(n,b)$ if the inequality*

$$\sum_{k=n+p}^{\infty}\left[|b|p+(k-p)\right]\sigma_k^m a_k\le|b|p \tag{9}$$

is satisfied where

$$\sigma_k=\left[\frac{\Omega\Gamma(\alpha_1+A_1(k-p))...\Gamma(\alpha_u+A_u(k-p))}{(k-p)!\Gamma(\beta_1+B_1(k-p))...\Gamma(\beta_v+B_v(k-p))}\frac{p+(\lambda-\delta)(\beta-\gamma)(k-p)+l}{p+l}\right]$$

for $u\le v+1\,;u,v\in N\,;\lambda,\delta,\gamma,l\ge0\,;\lambda>\delta\,;\beta>\gamma$ and $z\in U$.

Proof. In view of (8), we need to prove that $Re\{w\}\ge0$, where

$$w=\frac{bpD^{m,\gamma}_{\lambda,l,p,\delta,\beta}f(z)+z(D^{m,\gamma}_{\lambda,l,p,\delta,\beta}f(z))'-pD^{m,\gamma}_{\lambda,l,p,\delta,\beta}f(z)}{bpD^{m,\gamma}_{\lambda,l,p,\delta,\beta}f(z)}$$

$$= \frac{bpz^p + \sum_{k=n+p}^{\infty}(bp+k-p)\sigma_k^m a_k z^k}{bpz^p + \sum_{k=n+p}^{\infty} bp\sigma_k^m a_k z^k}.$$

Using the fact that $Re\{w\} \geq 0$ if and only if $|1+w| \geq |1-w|$, it sufficies to show that $|1+w| - |1-w| \geq 0$.Therefore, we obtain

$$= \left| bpz^p + \sum_{k=n+p}^{\infty} bp\sigma_k^m a_k z^k + bpz^p + \sum_{k=n+p}^{\infty} (bp+k-p)\sigma_k^m a_k z^k \right|$$
$$- \left| bpz^p + \sum_{k=n+p}^{\infty} bp\sigma_k^m a_k z^k - bpz^p - \sum_{k=n+p}^{\infty} (bp+k-p)\sigma_k^m a_k z^k \right|$$
$$\geq 2|b|p|z|^p - \sum_{k=n+p}^{\infty} (2|b|p + 2k - 2p)\sigma_k^m a_k |z|^k$$
$$= |b|p - \sum_{k=n+p}^{\infty} [|b|p + (k-p)]\sigma_k^m a_k \geq 0.$$

By hypothesis, last expression is nonnegative. Thus the proof is complete. Further let $\overline{S}_{\lambda,l,\delta,\beta}^{m,\gamma,p}(n,b) = S_{\lambda,l,\delta,\beta}^{m,\gamma,p}(n,b) \cap T(p)$, where

$$T(p) := \left\{ f \in A(p,n); f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, a_k \geq 0, z \in U, p, n \in N \right\}. \quad (10)$$

Our next theorem gives a necessary and sufficient condition for function of the form (10) to be in the class $\overline{S}_{\lambda,l,\delta,\beta}^{m,\gamma,p}(n,b)$.

Theorem 2.2$f(z) \in \overline{S}_{\lambda,l,\delta,\beta}^{m,\gamma,p}(n,b)$ *if and only if the inequality*

$$\sum_{k=n+p}^{\infty} [|b|p + (k-p)]\sigma_k^m a_k \leq |b|p \quad (11)$$

holds.

Proof. Since $\overline{S}_{\lambda,l,\delta,\beta}^{m,\gamma,p}(n,b) \subset S_{\lambda,l,\delta,\beta}^{m,\gamma,p}(n,b)$,we only need to prove the " only if " part of the Theorem 2.2. For functions $f(z)$ of the form (10), we notice that the condition

$$Re\left\{1 + \frac{1}{b}\left(\frac{1}{p}\frac{z(D_{\lambda,l,p,\delta,\beta}^{m,\gamma}f(z))'}{D_{\lambda,l,p,\delta,\beta}^{m,\gamma}f(z)} - 1\right)\right\} > 0 \qquad z \in U,$$

implies that

$$\left|\frac{1}{b}\left(\frac{1}{p}\frac{z(D^{m,\gamma}_{\lambda,l,p,\delta,\beta}f(z))'}{D^{m,\gamma}_{\lambda,l,p,\delta,\beta}f(z)}-1\right)\right|\leq 1$$

and so

$$\left|\frac{-\sum_{k=n+p}^{\infty}(k-p)\sigma_k^m a_k z^k}{pz^p-\sum_{k=n+p}^{\infty}p\sigma_k^m a_k z^k}\right|\leq |b|$$

Thus putting $z=r,(0\leq r<1)$, we obtain

$$\frac{\sum_{k=n+p}^{\infty}(k-p)\sigma_k^m a_k r^{k-1}}{p-\sum_{k=n+p}^{\infty}p\sigma_k^m a_k r^{k-1}}\leq |b|, \qquad z\in U. \tag{12}$$

Hence, we observe that the expression in the denominator on the left-hand side of (12) is positive for $z=1$ and also for all r, $(0\leq r<1)$. Thus, by letting $r\to 1^-$ through real values, (12) leads us to the desired assertion (9) of Theorem 2.1.

Corollary 2.1*If $f(z)$ of the form (10) is in $\overline{S}^{m,\gamma,p}_{\lambda,l,\delta,\beta}(n,b)$, then*

$$a_k\leq \frac{|b|p}{[|b|p+(k-p)]\sigma_k^m}, \qquad k=n+p,n+p+1,.... \tag{13}$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z^p-\frac{|b|p}{[|b|p+(k-p)]\sigma_k^m}z^k, \qquad k=n+p,n+p+1,.... \tag{14}$$

3.Distortion Bounds and Extreme Points

In this section we obtain distortion bounds and extreme points for the class $\overline{S}^{m,\gamma,p}_{\lambda,l,\delta,\beta}(n,b)$.

Theorem 3.1*Let $f(z)$ be in the class $\overline{S}^{m,\gamma,p}_{\lambda,l,\delta,\beta}(n,b)$ for $|z|=r\leq 1$, then*

$$r^p-r^{n+p}\frac{|b|p}{[|b|p+n]\sigma_{n+p}^m}\leq |f(z)|\leq r^p+r^{n+p}\frac{|b|p}{[|b|p+n]\sigma_{n+p}^m} \tag{15}$$

$$pr^{p-1}-r^{n+p-1}\frac{|b|p}{[|b|p+n]\sigma_{n+p}^m}\leq |f'(z)|\leq pr^{p-1}+r^{n+p-1}\frac{|b|p}{[|b|p+n]\sigma_{n+p}^m}. \tag{16}$$

The bounds in (15) and (16) are sharp since the equalities are attained by the function

$$f(z)=z^p-\frac{|b|p}{[|b|p+n]\sigma_{n+p}^m}z^{n+p}. \tag{17}$$

Proof. In the view of Theorem 2.1, we have

$$\sum_{k=n+p}^{\infty} a_k \leq \frac{|b|p}{[|b|p+n]\sigma_{n+p}^m}. \tag{18}$$

Using (10) and (18), we obtain

$$|z|^p - |z|^{n+p} \sum_{k=n+p}^{\infty} a_k \leq |f(z)| \leq |z|^p + |z|^{n+p} \sum_{k=n+p}^{\infty} a_k \tag{19}$$

$$r^p - r^{n+p} \frac{|b|p}{[|b|p+n]\sigma_{n+p}^m} \leq |f(z)| \leq r^p + r^{n+p} \frac{|b|p}{[|b|p+n]\sigma_{n+p}^m}.$$

Hence (15) follows from (19). Also, we obtain

$$|f'(z)| \leq pr^{p-1} + r^{n+p-1} \sum_{k=n+p}^{\infty} ka_k \leq pr^{p-1} + r^{n+p-1} \frac{|b|p}{[|b|p+n]\sigma_{n+p}^m}.$$

Similarly, we can prove the left hand inequality given in (16) which completes the proof of the theorem.

Now we obtain extreme points of the class $\overline{S}_{\lambda,l,\delta,\beta}^{m,\gamma,p}(n,b)$.

Theorem 3.2 *Let $f_p(z) = z^p$ and*

$$f_n(z) = z^p - \frac{|b|p}{[|b|p+(k-p)]\sigma_k^m} z^k, \qquad k = n+p, n+p+1, ...,. \tag{20}$$

Then $f(z)$ is in the class $\overline{S}_{\lambda,l,\delta,\beta}^{m,\gamma,p}(n,b)$ if and only if it can be expressed in the form

$$f(z) = \mu_p z^p + \sum_{k=n+p}^{\infty} \mu_k f_k(z) \tag{21}$$

where $\mu_k \geq 0$ and $\mu_p + \sum_{k=n+p}^{\infty} \mu_k = 1$.

Proof. Suppose $f(z)$ can be written as in (21). Then

$$f(z) = \mu_p z^p - \sum_{k=n+p}^{\infty} \mu_k \left[z^p - \frac{|b|p}{[|b|p+(k-p)]\sigma_k^m} z^k \right] = z^p - \sum_{k=n+p}^{\infty} \mu_k \frac{|b|p}{[|b|p+(k-p)]\sigma_k^m} z^k.$$

Now,

$$\sum_{k=n+p}^{\infty} \frac{[|b|p+(k-p)]\sigma_k^m}{|b|p} \mu_k \frac{|b|p}{[|b|p+(k-p)]\sigma_k^m} = \sum_{k=n+p}^{\infty} \mu_k = 1-\mu_p \leq 1.$$

Thus $f(z) \in \overline{S}_{\lambda,l,\delta,\beta}^{m,\gamma,p}(n,b)$. Conversely, let us have $f(z) \in \overline{S}_{\lambda,l,\delta,\beta}^{m,\gamma,p}(n,b)$. Then by using (13), we set $\mu_k = \frac{[|b|p+(k-p)]\sigma_k^m}{|b|p} a_k$, $k \geq n+p$ and $\mu_p = 1 - \sum_{k=n+p}^{\infty} \mu_k$. Then we have (21) and hence this completes the proof of Theorem 3.2.

4.Radii of Starlikeness, Convexity and Close-to-convexity

Theorem 4.1*Let the function $f(z)$ defined by (10) be in the class $\overline{S}_{\lambda,l,\delta,\beta}^{m,\gamma,p}(n,b)$. Then*

(i) $f(z)$ is p-valently close-to-convex of order $\vartheta (0 \leq \vartheta < p)$ in the disc $|z| < r_1$, where

$$r_1 = \underbrace{inf}_{k} \left[\left(\frac{p-\vartheta}{k} \right) \frac{[|b|p+(k-p)]\sigma_k^m}{|b|p} \right]^{\frac{1}{k-p}}. \tag{22}$$

$(k \geq n+p; k,n \in N)$.

(ii) $f(z)$ is p-valently starlike of order $\vartheta (0 \leq \vartheta < p)$ in the disc $|z| < r_2$, where

$$r_2 = \underbrace{inf}_{k} \left[\left(\frac{p-\vartheta}{k-\vartheta} \right) \frac{[|b|p+(k-p)]\sigma_k^m}{|b|p} \right]^{\frac{1}{k-p}}. \tag{23}$$

$(k \geq n+p; k,n \in N)$.

(iii) $f(z)$ is p-valently convex of order $\vartheta (0 \leq \vartheta < p)$ in the disc $|z| < r_3$, where

$$r_3 = \underbrace{inf}_{k} \left[\frac{p}{k} \left(\frac{p-\vartheta}{k-\vartheta} \right) \frac{[|b|p+(k-p)]\sigma_k^m}{|b|p} \right]^{\frac{1}{k-p}}. \tag{24}$$

$(k \geq n+p; k,n \in N)$.Each of these results is sharp for the function $f(z)$ given by (14).

Proof. It is sufficient to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \vartheta \qquad (|z| < r_1; 0 \leq \vartheta < p; p \in N), \tag{25}$$

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \vartheta \qquad (|z| < r_2; 0 \leq \vartheta < p; p \in N), \tag{26}$$

and that

$$\left|1 + \frac{zf''(z)}{f'(z)} - p\right| \leq p - \vartheta \qquad (|z| < r_3; 0 \leq \vartheta < p; p \in N), \tag{27}$$

for a function $f(z) \in \overline{S}_{\lambda,l,\delta,\beta}^{m,\gamma,p}(n,b)$, where r_1 ,r_2 , and r_3 are defined by (22),(23) and (24), respectively. The details involved are fairly straightforward and may be omitted.

5.Inclusion Results

The following result gives being closed under convex combinations of the class $\overline{S}_{\lambda,l,\delta,\beta}^{m,\gamma,p}(n,b)$.

Theorem 5.1 *The family $\overline{S}_{\lambda,l,\delta,\beta}^{m,\gamma,p}(n,b)$ is closed under convex combinations.*

Proof. Let $f_i \in \overline{S}_{\lambda,l,\delta,\beta}^{m,\gamma,p}(n,b)$, $(i = 1, 2, ...)$, where $f_i(z) = z^p - \sum_{k=n+p}^{\infty} |a_{i_k}|z^k$. The convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z^p - \sum_{k=n+p}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{i_k}|\right) z^k$$

provided that $\sum_{i=1}^{\infty} t_i = 1$, $(0 \leq t_i \leq 1)$. Applying the inequality (11) of Theorem 2.2 we obtain

$$\sum_{k=n+p}^{\infty} \frac{[|b|p + (k-p)]\sigma_k^m}{|b|p} \left(\sum_{i=1}^{\infty} t_i |a_{i_k}|\right) = \sum_{i=1}^{\infty} t_i \left(\sum_{k=n+p}^{\infty} \frac{[|b|p + (k-p)]\sigma_k^m}{|b|p} |a_{i_k}|\right) \leq \sum_{i=1}^{\infty} t_i = 1$$

and therefore $\sum_{i=1}^{\infty} t_i f_i \in \overline{S}_{\lambda,l,\delta,\beta}^{m,\gamma,p}(n,b)$.

6.Neighborhood for class $\overline{S}_{\lambda,l,\delta,\beta}^{m,\gamma,p}(n,b)$

In this section, we obtain inclusion involving $N_{n,w}^p$ for functions in the class $\overline{S}_{\lambda,l,\delta,\beta}^{m,\gamma,p}(n,b)$.

Theorem 6.1 *If*

$$\delta := \frac{|b|p[n+p]}{[|b|p + n]\sigma_{n+p}^m}; |b| < 1 \tag{28}$$

then

$$\overline{S}^{m,\gamma,p}_{\lambda,l,\delta,\beta}(n,b) \subset N^p_{n,w}(h). \tag{29}$$

Proof.

Let$f(z) \in \overline{S}^{m,\gamma,p}_{\lambda,l,\delta,\beta}(n,b)$.Then, in view of the assertion (11) of Theorem 2.2, we have

$$[|b|p+n]\sigma^m_{n+p}\sum_{k=n+p}^{\infty} a_k \leq |b|p, \qquad |b| \leq 1$$

which yields

$$\sum_{k=n+p}^{\infty} a_k \leq \frac{|b|p}{[|b|p+n]\sigma^m_{n+p}}. \tag{30}$$

Applying the assertion (11) of Theorem 2.2 again, in conjunction with (30), we obtain

$$\sigma^m_{n+p}\sum_{k=n+p}^{\infty} ka_k \leq |b|p + \sigma^m_{n+p}(p-|b|p)\sum_{k=n+p}^{\infty} a_k.$$

Hence

$$\sum_{k=n+p}^{\infty} ka_k = \frac{|b|p[n+p]}{[|b|p+n]\sigma^m_{n+p}} =: \delta \tag{31}$$

which, by virtue of (8), establishes the inclusion relation (29) of Theorem 6.1.

Remark 6.1 In view of the relationship (5) the linear operator (6) and by setting $A_j = 1, (j = 1, 2, ..., t)$ and $B_j = 1, (j = 1, 2, ..., v)$ specific choices of parameters v, t, α_1, β_1 the various results presented in this paper would provide interesting extensions and generalizations of p-valent function classes. The details involved in the derivations of such specializations of the results presented here are fairly straightforward.

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