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# ON SOME TOPOLOGICAL PROPERTIES OF NEW TYPE OF DIFFERENCE SEQUENCE SPACES

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Abstract. In this paper, we define the sequence spaces $\ell_\infty(\Delta_{m,v})$, $c(\Delta_{m,v})$ and $c_0(\Delta_{m,v})$ $(m \in \mathbb{N})$. Also we give some topological properties and inclusion relations of these sequence spaces.

*Keywords*: Difference sequence spaces, Solid space, Symmetric space, Completeness.

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## 1. Introduction

Let $\ell_\infty$, $c$ and $c_0$ be the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|$$

where $k \in \mathbb{N} = \{1, 2, ...\}$, the set of positive integers.
The difference sequence spaces

$$X(\Delta) = \{x = (x_k) : \Delta x \in X\}$$

first defined by Kızmaz [1], where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$ and $X$ is any of the sets $\ell_\infty$, $c$ and $c_0$, and showed that these are Banach spaces with norm

$$\|x\|_\Delta = |x_1| + \sup_k |\Delta x_k| .$$

Then Çolak [2] defined the sequence spaces $\Delta_v(X) = \{x = (x_k) : \Delta_v x \in X\}$, where $\Delta_v x = (\Delta_v x_k) = (v_k x_k - v_{k+1} x_{k+1})$ and is any sequence space, and investigated some topological properties of this spaces.

Tripathy and Esi [3] defined the new type of difference sequence spaces

$$Z(\Delta_m) = \{x = (x_k) : \Delta_m x \in Z\}$$

for $Z = \ell_\infty$, $c$ and $c_0$, where $m \in \mathbb{N}$ be fixed, $\Delta_m x = (\Delta_m x_k) = (x_k - x_{k+m})$ for all $k \in \mathbb{N}$ and showed that these are Banach spaces with norm

$$\|x\|_{\Delta_m} = \sum_{r=1}^{m} |x_r| + \sup_k |\Delta_m x_k| .$$

**Definition 1.1.**[4] Let $X$ be a sequence space. Then $X$ is called:
**(i)** Solid (or normal), if $(\alpha_k x_k) \in X$ whenever $(x_k) \in X$ for all sequences $(\alpha_k)$ of scalar with $|\alpha_k| \leq 1$.

**(ii)** Monotone provided $X$ contains the canonical preimages of all its stepspace.

**(iii)** Symmetric if $(x_k) \in X$ implies $\left(x_{\pi(k)}\right) \in X$, where $\pi(k)$ is a permutation of $\mathbb{N}$.

**(iv)** A sequence algebra if $(x_k)$, $(y_k) \in X$ implies $(x_k y_k) \in X$.

**(v)** Convergence free if $(y_k) \in X$ whenever $(x_k) \in X$ and $y_k = \theta$ whenever $x_k = \theta$.

**(vi)** For $r > 0$, nonempty subset $V$ of linear space is said to be absolutely $r$-convex if $x, y \in V$ and $|\lambda|^r + |\mu|^r \leq 1$ together imply that $\lambda x + \mu x \in V$. A linear topological space $X$ is said to be $r$-convex if every neighborhood of $\theta \in X$ contains as absolutely $r$-convex neighborhood of $\theta \in X$ (see for instance [5]).

## 2. Main Results

Let $v = (v_k)$ be any fixed sequence of nonzero complex numbers. Now we define

$$\begin{aligned} \ell_\infty(\Delta_{m,v}) &= \{x = (x_k) : \Delta_{m,v} x \in \ell_\infty\}, \\ c(\Delta_{m,v}) &= \{x = (x_k) : \Delta_{m,v} x \in c\}, \\ c_0(\Delta_{m,v}) &= \{x = (x_k) : \Delta_{m,v} x \in c_0\} \end{aligned}$$

where $m \in \mathbb{N}$ be fixed, $\Delta_{m,v} x = (\Delta_{m,v} x_k) = (v_k x_k - v_{k+m} x_{k+m})$ for all $k \in \mathbb{N}$.

If we take $(v_k) = (1, 1, ...)$, then we obtain $\ell_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$.

**Theorem 2.1.** *The sequence spaces $\ell_\infty(\Delta_{m,v})$, $c(\Delta_{m,v})$ and $c_0(\Delta_{m,v})$ are normed linear spaces, normed by*

$$\|x\| = \sum_{r=1}^{m} |v_r x_r| + \sup_k |\Delta_{m,v} x_k| . \tag{2.1}$$

*Proof.* We shall prove only for $\ell_\infty(\Delta_{m,v})$. The other cases can be proved similarly. Let $\alpha$, $\beta$ be scalars and $(x_k)$, $(y_k) \in \ell_\infty(\Delta_{m,v})$. Then

$$\sup_k |\Delta_{m,v} x_k| < \infty \text{ and } \sup_k |\Delta_{m,v} y_k| < \infty. \tag{2.2}$$

Hence

$$\sup_k |\Delta_{m,v}(\alpha x_k + \beta y_k)| \leq |\alpha| \sup_k |\Delta_{m,v} x_k| + |\beta| \sup_k |\Delta_{m,v} y_k| < \infty$$

by (2.2). Hence $\ell_\infty(\Delta_{m,v})$ is a linear space.

Next for $x = \theta$, we have $\|\theta\| = 0$. Conversely, let $\|x\| = 0$. Then $\|x\| = \sum_{r=1}^{m} |v_r x_r| + \sup_k |\Delta_{m,v} x_k| = 0$. Since $v_r \neq 0$ for $\forall r \in \mathbb{N}$, we have $x_r = 0$ for $r = 1, 2, ..., m$ and $|\Delta_{m,v} x_k| = 0$ for all $k \in \mathbb{N}$. Consider $k = 1$ i.e. $|\Delta_{m,v} x_1| = 0 \Rightarrow |v_1 x_1 - v_{1+m} x_{1+m}| = 0 \Rightarrow x_{1+m} = 0$ $(v_{1+m} \neq 0)$, since $x_1 = 0$ $(v_1 \neq 0)$. Proceeding in this way we have $x_k = 0$, for all $k \in \mathbb{N}$. After then we write

$$\begin{aligned} \|x+y\| &= \sum_{r=1}^{m} |v_r(x_r + y_r)| + \sup_k |\Delta_{m,v}(x_k + y_k)| \\ &\leq \left( \sum_{r=1}^{m} |v_r x_r| + \sup_k |\Delta_{m,v} x_k| \right) + \left( \sum_{r=1}^{m} |v_r y_r| + \sup_k |\Delta_{m,v} y_k| \right) \\ &= \|x\| + \|y\|. \end{aligned}$$

Finally

$$\begin{aligned} \|\lambda x\| &= \sum_{r=1}^{m} |\lambda v_r x_r| + \sup_k |\Delta_{m,v}(\lambda x_k)| \\ &= |\lambda| \, \|x\|. \end{aligned}$$

Hence $\|.\|$ is a norm on the sequence spaces $\ell_\infty(\Delta_{m,v})$, $c(\Delta_{m,v})$ and $c_0(\Delta_{m,v})$.

This completes the proof.

**Theorem 2.2.** *The sequence spaces $\ell_\infty(\Delta_{m,v})$, $c(\Delta_{m,v})$ and $c_0(\Delta_{m,v})$ are Banach spaces under the norm (2.1).*

*Proof.* Let $(x^s)$ be a Cauchy sequence in $\ell_\infty(\Delta_{m,v})$, where $x^s = (x_i^s) = (x_1^s, x_2^s, ...) \in \ell_\infty(\Delta_{m,v})$, for each $s \in \mathbb{N}$. Then

$$\left\| x^s - x^t \right\| = \sum_{r=1}^{m} \left| v_r(x_r^s - x_r^t) \right| + \sup_k \left| \Delta_{m,v} x_k^s - \Delta_{m,v} x_k^t \right| \to 0$$

as $s, t \to \infty$. Hence for given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\left\|x^s - x^t\right\| = \sum_{r=1}^{m} \left|v_r(x_r^s - x_r^t)\right| + \sup_k \left|\Delta_{m,v} x_k^s - \Delta_{m,v} x_k^t\right| < \varepsilon \tag{2.3}$$

for all $s, t \geq n_0$. Hence we obtain $\left|v_k(x_k^s - x_k^t)\right| < \varepsilon$ and since $v_k \neq 0$ for all $k \in \mathbb{N}$ we have $\left|x_k^s - x_k^t\right| < \varepsilon$ for all $s, t \geq n_0$ and $k = 1, 2, ..., m$. Therefore $(x_k^t)$ is a Cauchy sequence in $\mathbb{C}$ for each $k \in \mathbb{N}$. Since $\mathbb{C}$ is a complete space, $(x_k^t)$ is a convergent in $\mathbb{C}$ for $k = 1, 2, ..., m$. Let $\lim_{t\to\infty} x_k^t = x_k$ say for each $k \in \mathbb{N}$.

From (2.3) we have

$$\left|\Delta_{m,v} x_k^s - \Delta_{m,v} x_k^t\right| < \varepsilon$$

for all $s, t \geq n_0$ and all $k \in \mathbb{N}$. Hence $(\Delta_{m,v} x_k^t)$ is a Cauchy sequence in $\mathbb{C}$ for all $k \in \mathbb{N}$. Thus $(\Delta_{m,v} x_k^t)$ is a convergent in $\mathbb{C}$ and let $\lim_{t\to\infty} \Delta_{m,v} x_k^t = y_k$ say for each $k \in \mathbb{N}$.

Then we have

$$\lim_{t\to\infty} \sum_{r=1}^{m} \left|v_r(x_r^s - x_r^t)\right| = \sum_{r=1}^{m} \left|v_r(x_r^s - x_r)\right| < \varepsilon$$

$s \geq n_0$, and

$$\lim_{t\to\infty} \left|v_k x_k^s - v_k x_k^t - (v_{k+m} x_{k+m}^s - v_{k+m} x_{k+m}^t)\right| = \left|v_k x_k^s - v_k x_k - (v_{k+m} x_{k+m}^s - v_{k+m} x_{k+m})\right| < \varepsilon$$

for all $k \in \mathbb{N}$ and $s \geq n_0$. Hence for all $s \geq n_0$, we have

$$\sup_k \left|\Delta_{m,v} x_k^s - \Delta_{m,v} x_k\right| < \varepsilon.$$

Thus we obtain by

$$\sum_{r=1}^{m} \left|v_r(x_r^s - x_r)\right| + \sup_k \left|\Delta_{m,v} x_k^s - \Delta_{m,v} x_k\right| < 2\varepsilon$$

and $(x^s - x) \in \ell_\infty(\Delta_{m,v})$ for all $s \geq n_0$. Since $\ell_\infty(\Delta_{m,v})$ is a linear space, we have $x = x^s - (x^s - x) \in \ell_\infty(\Delta_{m,v})$, for all $s \geq n_0$. Therefore $\ell_\infty(\Delta_{m,v})$ is complete.

It can be shown that $c(\Delta_{m,v})$ and $c_0(\Delta_{m,v})$ are closed subspaces of $\ell_\infty(\Delta_{m,v})$. Therefore these sequence spaces are Banach spaces with norm (2.1).

**Theorem 2.3.** *The sequence spaces $\ell_\infty(\Delta_{m,v})$, $c(\Delta_{m,v})$ and $c_0(\Delta_{m,v})$ are BK-spaces with the same norm as in (2.1).*

*Proof.* These sequence spaces showed to be Banach space in Theorem 2.2. Now let

$$\|x^n - x\| \to 0$$

as $n \to \infty$. Then

$$|x_k^n - x_k| \to 0 \ (n \to \infty)$$

for $k \leq m$ and

$$\|\Delta_{m,v}(x_k^n - x_k)\| \to 0 \ (n \to \infty)$$

for all $k \in \mathbb{N}$. Here also we obtain $|x_k^n - x_k| \to 0$ $(n \to \infty)$ for all $k \in \mathbb{N}$. Hence sequence spaces $\ell_\infty(\Delta_{m,v})$, $c(\Delta_{m,v})$ and $c_0(\Delta_{m,v})$ are BK-spaces.

**Theorem 2.4.** **(i)** *$X(\Delta) \subset X(\Delta_{m,v})$, for $X = \ell_\infty$, $c$, $c_0$ and the inclusions are strict.*

**(ii)** *$c_0(\Delta_{m,v}) \subset c(\Delta_{m,v}) \subset \ell_\infty(\Delta_{m,v})$ and the inclusions are strict.*

*Proof.* **(i)** The proof is obtain for $m = 1$ and $v_k = 1$ for all $k \in \mathbb{N}$.

To show the inclusions are strict consider the following example.

**Example 1.** Let $m = 2$, $v_k = 1$ for all $k \in \mathbb{N}$ and consider the sequence $(x_k)$ defined by $x_k = 1$ for $k$ odd and $x_k = 0$ for $k$ even. Then the sequence $(x_k)$ belongs to $c_0(\Delta_{m,v})$ but does not belong to $c_0(\Delta)$.

Let $m = 1$, $v_k = \frac{1}{k}$ for all $k \in \mathbb{N}$ and $x = (k^2)$. Then the sequence $(x_k)$ belongs to $c(\Delta_{m,v}) \subset \ell_\infty(\Delta_{m,v})$ but does not belong to $c(\Delta) \subset \ell_\infty(\Delta)$.

**(ii)** The inclusion $c_0(\Delta_{m,v}) \subset c(\Delta_{m,v})$ is obvious. Now let $x \in c(\Delta_{m,v})$. Since $\Delta_{m,v}(x_k) \in c \subset \ell_\infty$, we obtain $x \in \ell_\infty(\Delta_{m,v})$. Thus $c(\Delta_{m,v}) \subset \ell_\infty(\Delta_{m,v})$.

To show the inclusions are strict consider the following example.

**Example 2.** Let $m = 1$, $v_k = 1$ for all $k \in \mathbb{N}$ and $x_k = k$. Then the sequence $(x_k)$ belongs to $c(\Delta_{m,v})$ but does not belong to $c_0(\Delta_{m,v})$.

Let $m = 1$, $v_k = 1$ for all $k \in \mathbb{N}$ and consider the sequence $(x_k)$ defined by $x_k = 1$ for $k$ odd and $x_k = 0$ for $k$ even. Then the sequence $(x_k)$ belongs to $\ell_\infty(\Delta_{m,v})$ but does not belong to $c(\Delta_{m,v})$.

**Theorem 2.5.** *The sequence spaces $c(\Delta_{m,v})$ and $c_0(\Delta_{m,v})$ are closed subsets in $\ell_\infty(\Delta_{m,v})$.*

*Proof.* Since $c \subset \ell_\infty$, then $c(\Delta_{m,v}) \subset \ell_\infty(\Delta_{m,v})$ by Theorem 2.4 (ii). Now we show that $\overline{c(\Delta_{m,v})} = \overline{c}(\Delta_{m,v})$, where $\overline{c(\Delta_{m,v})}$, the closure of $c(\Delta_{m,v})$ and $\overline{c}$, the closure of $c$. Let $x \in \overline{c(\Delta_{m,v})}$, then there exists a sequence $(x^n)$ in $c(\Delta_{m,v})$ such that

$$\|x^n - x\| \to 0 \ (n \to \infty)$$

in $c(\Delta_{m,v})$, and so

$$\sum_{r=1}^{m} |v_r(x^n - x_r)| + \sup_k |\Delta_{m,v}x_k^n - \Delta_{m,v}x_k| \to \infty \ (n \to \infty)$$

in $c$. Thus $\Delta_{m,v}x \in \bar{c}$. Hence $x \in \bar{c}(\Delta_{m,v})$. Conversely if $x \in \overline{c(\Delta_{m,v})}$, then $\Delta_{m,v}(x) \in \bar{c}$. Since $c$ is closed, $x \in c(\Delta_{m,v}) \subset \overline{c(\Delta_{m,v})}$. Hence $x \in \overline{c(\Delta_{m,v})}$. This completes the proof.

The proof of $c_0(\Delta_{m,v})$ is similar to that of $c(\Delta_{m,v})$.

**Theorem 2.6.** *The sequence spaces $c(\Delta_{m,v})$ and $c_0(\Delta_{m,v})$ are separable spaces.*

*Proof.* The proof is similar to that of Theorem 2.5.

**Theorem 2.7.** *The sequence spaces $c(\Delta_{m,v})$ and $c_0(\Delta_{m,v})$ are nowhere dense subsets of $\ell_\infty(\Delta_{m,v})$.*

*Proof.* Suppose that $\overset{o}{\bar{c}} = \varnothing$, but $\overset{o}{\overline{c(\Delta_{m,v})}} \neq \varnothing$. Then $\bar{c}$ contains no neighborhood and $B(a) \subset \overline{c(\Delta_{m,v})}$, where $B(a)$ is a neighborhood of center $a$ and radius $r$. Hence

$$a \in B(a) \subset \overline{c(\Delta_{m,v})} = \bar{c}(\Delta_{m,v}).$$

This implies that $\Delta_{m,v}(a) \in \bar{c}$. So

$$B(\Delta_{m,v}(a)) \cap c \neq \varnothing.$$

This contradicts to $\overset{o}{\bar{c}} = \varnothing$. Hence $\overset{o}{\overline{c(\Delta_{m,v})}} = \varnothing$. The proof of $c_0(\Delta_{m,v})$ is similar to that of $c(\Delta_{m,v})$.

The proofs of the following theorems are obtained by using the same technique of Tripathy and Esi [3], therefore we give it without proof.

**Theorem 2.8.** *The sequence spaces $\ell_\infty(\Delta_{m,v})$, $c(\Delta_{m,v})$ and $c_0(\Delta_{m,v})$ are not solid, not monotone and not convergence free.*

**Theorem 2.9.** *The sequence spaces $\ell_\infty(\Delta_{m,v})$, $c(\Delta_{m,v})$ and $c_0(\Delta_{m,v})$ are not symmetric for $m > 1$.*

**Theorem 2.10.** *The sequence spaces $\ell_\infty(\Delta_{m,v})$, $c(\Delta_{m,v})$ and $c_0(\Delta_{m,v})$ are not sequence algebra.*

*Proof.* The proof follows from the following examples.

**Example 3.** Let $m = 1$, $v_k = 1$ for all $k \in \mathbb{N}$, $x_k = k$ and $y_k = k$ for all $k \in \mathbb{N}$. Then $x, y \in c(\Delta) \subset \ell_\infty(\Delta)$, but $(x.y) \notin c(\Delta) \subset \ell_\infty(\Delta)$.

Let $m = 1$, $v_k = \frac{1}{k}$ for all $k \in \mathbb{N}$, $x_k = k$ and $y_k = k$ for all $k \in \mathbb{N}$. Then $x, y \in c_0(\Delta_{m,v})$, but $(x.y) \notin c_0(\Delta_{m,v})$.

**Theorem 2.11.** *The sequence spaces $\ell_\infty(\Delta_{m,v})$, $c(\Delta_{m,v})$ and $c_0(\Delta_{m,v})$ are 1-convex.*

*Proof.* If $0 < \delta < 1$, then $V = \{x = (x_k) : \|x\| \leq \delta\}$ is an absolutely 1-convex set, for $x, y \in V$ and $|\lambda| + |\mu| \leq 1$, then

$$\begin{aligned} \|\lambda x + \mu y\| &= \sum_{r=1}^{m} |v_r(\lambda x_r + \mu y_r)| + \sup_k |\Delta_{m,v}(\lambda x_k + \mu y_k)| \\ &\leq |\lambda| \, \|x\| + |\mu| \, \|y\| \leq \delta(|\lambda| + |\mu|) \leq \delta. \end{aligned}$$

This completes the proof.

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