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SUBORDINATION RESULTS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS DEFINED BY A DIFFERENTIAL OPERATOR

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Abstract. By using subordination theorems for analytic functions we derive several subordination results for certain classes of analytic functions defined by the differential operator of fractional power introduced by Rabha W. Ibrahim and Maslina Darus.

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1. Introduction and Preliminaries

Let $\mathcal{H}$ be the class of functions analytic in the open uit disk $\mathcal{U} = \{z : |z| < 1\}$ and let for $n \in \mathbb{N}$ and $a \in \mathbb{C}$, $\mathcal{H}(a, n)$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots$. Let

$$\mathcal{A}_n = \{f \in \mathcal{H}, f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots\}$$

and let $\mathcal{A}_1 = \mathcal{A}$.

Let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ which are univalent in the unit disk $\mathcal{U}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Suppose f and g are analytic in $\mathcal{U}$. We say that the function f is subordinate to g and write $f \prec g$ if there exists a Schwarz function $\omega(z)$, analytic in $\mathcal{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$ for $z \in \mathcal{U}$. In particular, if the function g is univalent in $\mathcal{U}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

In [5], R.W. Ibrahim and Darus introduced the class A^+_α and A^-_α which are defined as follows:

The class A_α^+ consists of normalized analytic functions $F(z)$ in $\mathcal{U}$ which are of the form $F(z) = z + \sum_{n=2}^{\infty} a_{n,\alpha} z^{n+\alpha-1}$ where $a_{0,\alpha} = 0$, $a_{1,1} = 1$ and the class A_α^- consists of normalized analytic functions $F(z)$ in $\mathcal{U}$ of the form $F(z) = z - \sum_{n=2}^{\infty} a_{n,\alpha} z^{n+\alpha-1}$, $a_{n,\alpha} \geq 0$; $n = 2, 3, \ldots$ where $\alpha \geq 1$ takes its values from the relation $\alpha = \dfrac{n+m}{m}$, $m \in \mathbb{N}$.

Also the authors [5] introduced the differential operator $D_{\alpha,\lambda}^k$ for $f(z) \in A_\alpha^+$ [5] which is defined as follows:

$$\begin{aligned}
D_{\alpha,\lambda}^0 F(z) &= F(z) = z + \sum_{n=2}^{\infty} a_{n,\alpha} z^{n+\alpha-1}, \quad \alpha \geq 1, \lambda < \alpha \\
D_{\alpha,\lambda}^1 F(z) &= (\lambda - \alpha + 1)F(z) + (\alpha - \lambda) z F'(z) \\
&= z + \sum_{n=2}^{\infty} [(\alpha - \lambda)(n + \alpha - 2) + 1] a_{n,\alpha} z^{n+\alpha-1} \\
&\;\;\vdots \\
D_{\alpha,\lambda}^k F(z) &= D(D^{k-1} F(z)) = z + \sum_{n=2}^{\infty} [(\alpha - \lambda)(n + \alpha - 2) + 1] a_{n,\alpha} z^{n+\alpha-1}.
\end{aligned}$$

Ali et al. [1] used the results obtained by Bulboacă [3] and gave the sufficient conditions for certain normalized analytic functions $f \in \mathcal{A}$ to satisfy
$q_1(z) \prec \dfrac{zf'(z)}{f(z)} \prec q_2(z)$ where q_1 and q_2 are given univalent functions in $\mathcal{U}$ with $q_1(0) = 1$ and $q_2(0) = 1$.

The purpose of this paper is to apply a method based on the differential subordination in order to derive sufficient conditions for $F \in A_\alpha^+$ and $F \in A_\alpha^-$ to satisfy

$$\left(D_{\alpha,\lambda}^k F(z)\right)' \left(\frac{z}{D_{\alpha,\lambda}^k F(z)}\right)^{1+r} \prec q(z) \quad \text{for} \quad 0 \leq r \leq 1 \quad \text{and for every} \quad z \in \mathcal{U}.$$

where q is a given univalent function in $\mathcal{U}$ with $q(z) \neq 0$.

In order to prove our subordination results, we make use of the following known results.

Theorem 1. *[7] Let the function q be univalent in the open unit disk $\mathcal{U}$ and θ and ϕ be analytic in a domain D containing $q(\mathcal{U})$ with $\phi(\omega) \neq 0$ when $\omega \in q(\mathcal{U})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$.*
Suppose that

1. *Q is starlike univalent in $\mathcal{U}$, and*

2. *$Re\left\{\dfrac{zh'(z)}{Q(z)}\right\} > 0$ for $z \in \mathcal{U}$.*

If $\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$, then $p(z) \prec q(z)$ and q is the best dominant.

Definition 1. *[7] Denote by Q the set of all functions f that are analytic and injective on $\overline{\mathcal{U}}\backslash E(f)$, where $E(f) = \{\zeta \in \partial\, \mathcal{U} : \lim\limits_{z\to\zeta} f(z) = \infty\}$ and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathcal{U}\backslash E(f)$.*

Theorem 2. *[9] Let $q(z)$ be convex univalent in the unit disk $\mathcal{U}$ and ψ and $\delta \in \mathbb{C}$ with $\Re\left\{1 + \dfrac{zq''(z)}{q'(z)} + \dfrac{\psi}{\delta}\right\} > 0$. If $p(z)$ is analytic in $\mathcal{U}$ and $\psi p(z) + \delta zp'(z) < \psi q(z) + \delta zq'(z)$, then $p(z) \prec q(z)$ and q is the best dominant.*

Theorem 3. *[3] Let the function q be univalent in the open unit disk $\mathcal{U}$ and v and ϕ be analytic in a domain D containing $q(\mathcal{U})$. Suppose that*

1. *$\Re\left\{\dfrac{v'(q(z))}{\phi(q(z))}\right\} > 0$ for $z \in \mathcal{U}$ and*

2. *$zq'(z)\phi(q(z))$ is starlike univalent in $\mathcal{U}$.*

If $p \in \mathcal{H}[q(0), 1] \cap Q$, with $P(\mathcal{U}) \subseteq D$, and $v(p(z)) + zp'(z)\phi(p(z))$ is univalent in $\mathcal{U}$ and $v(q(z)) + zq'(z)\phi(q(z)) \prec v(p(z)) + zp'(z)\phi(p(z))$ then $q(z) \prec p(z)$ and q is the best subordinant.

2. Subordination and Superordination between Analytic Functions

Theorem 4. *Let the function $q(z)$ be analytic and univalent in $\mathcal{U}$ such that $q(z) \neq 0 \ \ \forall\ z \in \mathcal{U}$ and let δ be a non-zero complex number. Suppose that $\dfrac{zq'(z)}{q(z)}$ is starlike univalent in $\mathcal{U}$ and*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{q(z)}{\delta}(b + 2cq(z) + 3dq^2(z))\right\} > 0, \quad b, c, d \in \mathbb{C}. \qquad (1)$$

Let

$$\Phi^k_{\alpha,\lambda}(\delta,\gamma,F)(z) = a + b(D^k_{\alpha,\lambda}F(z))'\left(\frac{z}{D^k_{\alpha,\lambda}F(z)}\right)^{1+\gamma}$$
$$+ c\left[(D^k_{\alpha,\lambda}F(z))'\left(\frac{z}{D^k_{\alpha,\lambda}F(z)}\right)^{1+\gamma}\right]^2 + d\left[(D^k_{\alpha,\lambda}F(z))'\left(\frac{z}{D^k_{\alpha,\lambda}F(z)}\right)^{1+\gamma}\right]^3$$
$$+ \delta\left[\frac{z(D^k_{\alpha,\lambda}F(z))''}{(D^k_{\alpha,\lambda}F(z))'} + (1+\gamma)\left(1 - \frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}\right)\right] \qquad (2)$$

If $F \in \mathcal{A}^+_\alpha$ *satisfies the subordination:*

$$\Phi^k_{\alpha,\lambda}(\delta,\gamma,F)(z) \prec a + bq(z) + cq^2(z) + dq^3(z) + \delta\frac{zq'(z)}{q(z)}$$

then for $0 \le \gamma \le 1$,

$$(D^k_{\alpha,\lambda}F(z))'\left(\frac{z}{D^k_{\alpha,\lambda}F(z)}\right)^{1+\gamma} \prec q(z) \qquad (3)$$

and q *is the best dominant.*

Proof. Let the function p be defined by

$$p(z) := (D^k_{\alpha,\lambda}F(z))'\left(\frac{z}{D^k_{\alpha,\lambda}F(z)}\right)^{1+\gamma}, \quad z \in \mathcal{U}, z \neq 0, F \in \mathcal{A}^+_\alpha, D^k_{\alpha,\lambda}F(z) \neq 0.$$

By a straight forward computation, we have,

$$\frac{zp'(z)}{p(z)} = \frac{z(D^k_{\alpha,\lambda}F(z))''}{(D^k_{\alpha,\lambda}F(z))'} + (1+\gamma)\left(1 - \frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}\right)$$

By setting $\theta(w) := a + bw + cw^2 + dw^3$ and $\phi(w) = \dfrac{\delta}{w}$, $a \neq 0$ it can be verified that θ is analytic in $\mathbb{C}$ and ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Also by letting

$$Q(z) = zq'(z)\phi(q(z)) = \delta\frac{zq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z)$$
$$= a + bq(z) + cq^2(z) + dq^3(z) + \delta\frac{zq'(z)}{q(z)}$$

We find that $Q(z)$ is starlike univalent in $\mathcal{U}$ and that

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{b}{\delta}q(z)+\frac{2c}{\delta}q^2(z)+\frac{3d}{\delta}q^3(z)\right\} > 0.$$

Now

$$\begin{aligned}
&\theta(p(z))+zp'(z)\phi(p(z)) = a+bp(z)+cp^2(z)+dp^3(z)+\delta\frac{zp'(z)}{p(z)}\\
&= a+b(D^k_{\alpha,\lambda}F(z))'\left(\frac{z}{D^k_{\alpha,\lambda}F(z)}\right)^{1+\gamma}+c\left[(D^k_{\alpha,\lambda}F(z))'\left(\frac{z}{D^k_{\alpha,\lambda}F(z)}\right)^{1+\gamma}\right]^2\\
&+d\left[(D^k_{\alpha,\lambda}F(z))'\left(\frac{z}{D^k_{\alpha,\lambda}F(z)}\right)^{1+\gamma}\right]^3\\
&+\delta\left[\frac{z(D^k_{\alpha,\lambda}F(z))''}{(D^k_{\alpha,\lambda}F(z))'}+(1+\gamma)\left(1-\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}\right)\right]\\
&\prec a+bq(z)+cq^2(z)+dq^3(z)+\delta\frac{zq'(z)}{q(z)}
\end{aligned}$$

Assertion (3) of the theorem follows by an application of Theorem 1.

For the choices $q(z)=\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$, $q(z)=\left(\frac{1+z}{1-z}\right)^{\mu}$, $\mu\neq 0$ and $q(z)=e^{\mu Az}$, $\mu\neq 0$, in Theorem 4, we get the following results.

Corollary 1. *Let δ be a non-complex number. Assume that (1) holds and q is convex univalent in $\mathcal{U}$. If $F\in A^+_\alpha$ and*

$$\begin{aligned}
\Phi^k_{\alpha,\lambda}(\delta,\gamma,F)(z) &\prec a+b\frac{1+Az}{1+Bz}+c\left[\frac{1+Az}{1+Bz}\right]^2\\
&+d\left[\frac{1+Az}{1+Bz}\right]^3+\delta\frac{z(A-B)}{(1+Az)(1+Bz)},
\end{aligned}$$

where $\Phi^k_{\alpha,\lambda}(\delta,\gamma,F)$ is defined as in (2), then for $0\le\gamma\le 1$, $\left(D^k_{\alpha,\lambda}F(z)\right)'\left(\frac{z}{D^k_{\alpha,\lambda}F(z)}\right)^{1+\gamma}\prec\frac{1+Az}{1+Bz}$, $-1\le B<A\le 1$ and $\frac{1+Az}{1+Bz}$ is the best dominant.

Corollary 2. *Let δ be a non-zero complex number. Assume that (1) holds and q is convex univalent in $\mathcal{U}$. If $F \in A_\alpha^+$ and*

$$\Phi_{\alpha,\lambda}^k(\delta,\gamma,F)(z) \prec a + b\left(\frac{1+z}{1-z}\right)^{\mu} + c\left(\frac{1+z}{1-z}\right)^{2\mu} + d\left(\frac{1+z}{1-z}\right)^{3\mu} + \frac{2\delta\mu z}{(1-z^2)}, \quad \text{for } z \in \mathcal{U}, \mu \neq 0$$

then

$$\left(D_{\alpha,\lambda}^k F(z)\right)'\left(\frac{z}{D_{\alpha,\lambda}^k F(z)}\right)^{1+\gamma} \prec \left(\frac{1+z}{1-z}\right)^{\mu}$$

and $q(z) = \left(\frac{1+z}{1-z}\right)^{\mu}$ *is the best dominant.*

Corollary 3. *Assume that (1) holds and q is convex univalent in $\mathcal{U}$. If $F \in A_\alpha^+$ and*

$$\Phi_{\alpha,\lambda}^k(\delta,\gamma,F)(z) \prec a + be^{\mu Az} + ce^{2\mu Az} + de^{3\mu Az} + \mu\delta Az, \quad \text{for } z \in \mathcal{U}, \mu \neq 0$$

then

$$\left(D_{\alpha,\lambda}^k F(z)\right)'\left(\frac{z}{D_{\alpha,\lambda}^k F(z)}\right)^{1+\gamma} \prec e^{\mu Az}$$

and $q(z) = e^{\mu Az}$ *is the best dominant.*

Remark 1. Letting $r = 0$ and $k = 0$ in Theorem 4, $\left(D_{\alpha,\lambda}^k F(z)\right)'\left(\frac{z}{D_{\alpha,\lambda}^k F(z)}\right)^{1+\gamma} \prec q(z)$ reduces to the result in [4] which is $\frac{z(F(z))'}{F(z)} \prec q(z)$, $z \in \mathcal{U}$, and $F(z) \neq 0$ and $q(z)$ is the best dominant.

Theorem 5. *Let the function q be convex univalent in the unit disk $\mathcal{U}$ such that*

$$Re\left\{1 + \frac{zq''(z)}{q'(z)} + \frac{1}{\delta}\right\} > 0, \quad \delta \neq 0 \tag{4}$$

Suppose that $\left(D_{\alpha,\lambda}^k F(z)\right)'\left(\frac{z}{D_{\alpha,\lambda}^k F(z)}\right)^{1+\gamma}$ *is analytic in the disk $\mathcal{U}$. If*

$F \in A_\alpha^-$ satisfies the subordination

$$\psi_{\alpha,\lambda}^k(\delta,\gamma,F)(z) = \left(D_{\alpha,\lambda}^k F(z)\right)' \left(\frac{z}{D_{\alpha,\lambda}^k F(z)}\right)^{1+\gamma} + \delta \left(D_{\alpha,\lambda}^k F(z)\right)' \left(\frac{z}{D_{\alpha,\lambda}^k F(z)}\right)^{1+\gamma}$$
$$\left\{ \frac{z(D_{\alpha,\lambda}^k F(z))''}{(D_{\alpha,\lambda}^k F(z))'} + (1+\gamma)\left[1 - \frac{z(D_{\alpha,\lambda}^k F(z))'}{D_{\alpha,\lambda}^k F(z)}\right]\right\} \prec q(z) + \delta z q'(z).$$

Then $\left(D_{\alpha,\lambda}^k F(z)\right)' \left(\frac{z}{D_{\alpha,\lambda}^k F(z)}\right)^{1+\gamma} \prec q(z)$ $(z \in \mathcal{U}, D_{\alpha,\lambda}^k F(z) \neq 0)$ *and q is the best dominant.*

Proof. Let the function p be defined by

$$p(z) = \left(D_{\alpha,\lambda}^k F(z)\right)' \left(\frac{z}{D_{\alpha,\lambda}^k F(z)}\right)^{1+\gamma}, \quad D_{\alpha,\lambda}^k F(z) \neq 0, z \in \mathcal{U}.$$

By setting $\psi = 1$, it can be observed that

$$p(z) + \delta z p'(z) = \left(D_{\alpha,\lambda}^k F(z)\right)' \left(\frac{z}{D_{\alpha,\lambda}^k F(z)}\right)^{1+\gamma}$$
$$+ \delta \left(D_{\alpha,\lambda}^k F(z)\right)' \left(\frac{z}{D_{\alpha,\lambda}^k F(z)}\right)^{1+\gamma} \left\{ \frac{z(D_{\alpha,\lambda}^k F(z))''}{(D_{\alpha,\lambda}^k F(z))'} + (1+\gamma)\left[1 - \frac{z(D_{\alpha,\lambda}^k F(z))'}{D_{\alpha,\lambda}^k F(z)}\right]\right\}$$
$$\prec q(z) + \delta z q'(z).$$

Assertion of Theorem 5 follows by an application of Theorem 2.

Corollary 4. *Assume that (4) holds and q is convex univalent in $\mathcal{U}$. If $F \in A_\alpha^-$ and*

$$\psi_{\alpha,\lambda}^k(\delta,\gamma,F)(z) \prec \frac{1+Az}{1+Bz} + \frac{\delta z(A-B)}{(1+Bz)^2}$$

then

$$\left(D_{\alpha,\lambda}^k F(z)\right)' \left(\frac{z}{D_{\alpha,\lambda}^k F(z)}\right)^{1+\gamma} \prec \frac{1+Az}{1+Bz}, \quad -1 \leq B < A \leq 1$$

and $q(z) = \dfrac{1+Az}{1+Bz}$ *is the best dominant.*

Corollary 5. *Assume that (4) holds and q is convex univalent in* $\mathcal{U}$*. If* $F \in A_\alpha^-$ *and*

$$\psi_{\alpha,\lambda}^k(\delta,\gamma,F)(z) \prec \left(\frac{1+z}{1-z}\right)^\mu + \frac{2\mu\delta z}{(1-z^2)}\left(\frac{1+z}{1-z}\right)^{\mu-1}$$

for $z \in \mathcal{U}$*,* $\mu \neq 0$ *then*

$$\left(D_{\alpha,\lambda}^k F(z)\right)'\left(\frac{z}{D_{\alpha,\lambda}^k F(z)}\right)^{1+\gamma} \prec \left(\frac{1+z}{1-z}\right)^\mu$$

and $q(z) = \left(\frac{1+z}{1-z}\right)^\mu$ *is the best dominant.*

Corollary 6. *Assume that (4) holds and q is convex univalent in* $\mathcal{U}$*. If* $F \in A_\alpha^-$ *and* $\psi_{\alpha,\lambda}^k(\delta,\gamma,F)(z) \prec e^{\mu Az} + \delta\mu Aze^{\mu Az}$ *for* $z \in \mathcal{U}$*,* $\mu \neq 0$ *then* $\left(D_{\alpha,\lambda}^k F(z)\right)'\left(\frac{z}{D_{\alpha,\lambda}^k F(z)}\right)^{1+\gamma} \prec e^{\mu Az}$ *and* $q(z) = e^{\mu Az}$ *is the best dominant.*

Remark 2. By taking $k = 0$ and $\gamma = 0$ in Theorem 5 we obtain the result found in [4].

Theorem 6. *Let* δ *be a non-zero complex number and let q be analytic and univalent in* $\mathcal{U}$ *such that* $q(z) \neq 0$ *and* $\frac{zq'(z)}{q(z)}$ *starlike univalent in* $\mathcal{U}$*. Further, let us assume that*

$$Re\left\{\frac{bq(z) + 2cq^2(z) + 3dq^3(z)}{\delta}\right\} > 0. \qquad (5)$$

If $F \in A_\alpha^+$*,*

$$0 \neq \left(D_{\alpha,\lambda}^k F(z)\right)'\left(\frac{z}{D_{\alpha,\lambda}^k F(z)}\right)^{1+\gamma} \in \mathcal{H}[q(0),1] \cap Q$$

and $\Phi_{\alpha,\lambda}^k(\delta,\gamma,F)$ *is univalent in* $\mathcal{U}$*, then*

$$q(z) + \delta\frac{zq'(z)}{q(z)} \prec \Phi_{\alpha,\lambda}^k(\delta,\gamma,F)$$

implies

$$q(z) \prec (D_{\alpha,\lambda}^k F(z))'\left(\frac{z}{D_{\alpha,\lambda}^k F(z)}\right)^{1+\gamma}$$

and q is the best subordinant where $\Phi_{\alpha,\lambda}^k(\delta,\gamma,F)$ *is defined as in (2).*

Proof. By setting $v(w) := a + bw + cw^2 + dw^3$ and $\phi(w) := \dfrac{\delta}{w}$ it can be easily verified that v is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and $\phi(w) \neq 0$ $(w \in \mathbb{C}\backslash\{0\})$.

By the hypothesis of Theorem 3, $zq'(z)\phi(q(z))$ is starlike univalent and

$$Re\left\{\frac{v'(q(z))}{\phi(q(z))}\right\} = Re\left\{\frac{bq(z) + 2cq^2(z) + 3dq^3(z)}{\delta}\right\} > 0.$$

Then the assertion of this theorem follows by an application of Theorem 3.

Combining Theorem 4 and 6, we get the following theorem.

Theorem 7. *Let δ be a non-zero complex number and let q_1 and q_2 be univalent in $\mathcal{U}$ such that $q_1(z) \neq 0$ and $q_2(z) \neq 0$, $\forall\ z \in \mathcal{U}$ with $\dfrac{zq_1'(z)}{q_1(z)}$ and $\dfrac{zq_2'(z)}{q_2(z)}$ being starlike univalent. Suppose that q_1 satisfies (5) and q_2 satisfies (1). If $F \in A_\alpha^+$, $\left(D_{\alpha,\lambda}^k F(z)\right)' \left(\dfrac{z}{D_{\alpha,\lambda}^k F(z)}\right)^{1+\gamma} \in \mathcal{H}[q(0), 1] \cap Q$, and $\Phi_{\alpha,\lambda}^k(\delta, \gamma, F)$ is univalent in $\mathcal{U}$, then*

$$q_1(z) + \frac{\delta z q_1'(z)}{q_1(z)} \prec \Phi_{\alpha,\lambda}^k(\delta, \gamma, F) \prec q_2(z) + \frac{\delta z q_2'(z)}{q_2(z)}$$

implies $q_1(z) \prec \left(D_{\alpha,\lambda}^k F(z)\right)' \left(\dfrac{z}{D_{\alpha,\lambda}^k F(z)}\right)^{1+\gamma} \prec q_2(z)$ and q_1 and q_2 are the best subordinant and the dominant respectively.

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