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SANDWICH-TYPE THEOREMS OF SOME SUBCLASSES OF MULTIVALENT FUNCTIONS INVOLVING DZIOK-SRIVASTAVA OPERATOR[1]

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Abstract. In the present paper subordination and superordination results of some subclasses of multivalent functions associated with Dziok-Srivastava operator and defined in the open unit disc are investigated. Differential Sandwich-type theorem for the above classes are also presented. Relevant connections of the results, which are presented in this paper, with various other known results are also pointed out.

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1. Introduction and Definitions

Let $\mathcal{H}$ be the class of functions analytic in the *open* unit disk

$$\mathcal{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

and $\mathcal{H}[a,p]$ $(p \in \mathbb{N} := \{1,2,3,\cdots\})$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \cdots .$$

Let $\mathcal{A}_p(\subset \mathcal{H})$ be the class of all analytic functions given by the power series

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (z \in \mathcal{U}). \tag{1}$$

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Recalling the principle of subordination between the analytic functions, i.e. we say f is *subordinate* to g, (or g is *superordinate* to f) written as $f \prec g$ in $\mathcal{U}$ or $f(z) \prec g(z) \quad (z \in \mathcal{U})$, if there exists a function ω, analytic in $\mathcal{U}$ satisfying the conditions of the Schwarz lemma (i.e. $\omega(0) = 0$ and $|\omega(z)| < 1$) such that $f(z) = g(\omega(z))$ $(z \in \mathcal{U})$. It follows that

$$f(z) \prec g(z)\ (z \in \mathcal{U}) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathcal{U}) \subset g(\mathcal{U}).$$

In particular, if g is univalent in $\mathcal{U}$, then the reverse implication also holds (cf.[14]).

Definition 1. Let $p, h \in \mathcal{H}$ and let $\varphi(r, s, t; z) : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$. If $p(z)$ and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if $p(z)$ satisfies the second order superordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \quad (z \in \mathcal{U}) \tag{2}$$

then $p(z)$ is a solution of the differential superordination (2).

An analytic function q is called a subordinant of the differential superordination, or more precisely a *subordinant* if $q \prec p$, for all p satisfying (2). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$, for all subordinants q of (2) is said to be *best subordinant.* Note that the best subordinant is unique upto a rotation of $\mathcal{U}$. Recently Miller and Mocanu[15] have obtained conditions on h, q and φ for which the following implication holds:

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z) \quad (z \in \mathcal{U}).$$

Motivated by the results due to Bulboaca (cf.[3] and [4]), Ali et al.[1] obtained some sufficient conditions for the class of analytic functions which satisfy

$$q_1(z) \prec zf'(z)/f(z) \prec q_2(z), \quad (z \in \mathcal{U})$$

where q_1, q_2 are univalent in $\mathcal{U}$ with $q_1(0) = 1 = q_2(0)$.

Definition 2. (*[15], Definition 2,p.817; also see [14], Defintion 2.2b, p.21*) Let Q be the set of all functions f that are analytic and injective on $\overline{\mathcal{U}} \setminus E(f)$, where

$$E(f) := \left\{\zeta : \zeta \in \partial\mathcal{U} \text{ and } \lim_{z \to \zeta} f(z) = \infty\right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathcal{U} \setminus E(f)$.

For functions $f_j(z) \in \mathcal{A}_p$, given by

$$f_j(z) = z^p + \sum_{n=p+1}^{\infty} a_{n,j} z^n \quad (j = 1, 2; p \in \mathbb{N}; z \in \mathcal{U}),$$

the *Hadamard product* (or *convolution*) of $f_1(z)$ and $f_2(z)$ be defined by

$$f_1(z) * f_2(z) = z^p + \sum_{n=p+1}^{\infty} a_{n,1}a_{n,2}z^n = f_2(z) * f_1(z) \quad (p \in \mathbb{N}; z \in \mathcal{U}),$$

A *generalized hypergeometric functions* ${}_lF_m$ with l numerator parameters $\alpha_j \in \mathbb{C}$ $(j = 1, 2, \cdots, l)$ and m denominator parameters $\beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-$ $(\mathbb{Z}_0^- := \{0, -1, -2, \cdots\}; j = 1, 2, \cdots, m)$ is defined by infinite series

$$\begin{aligned} {}_lF_m(z) := {}_lF_m(\alpha_1, \alpha_2, \cdots, \alpha_l; \beta_1, \beta_2, \cdots, \beta_m; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{z^n}{n!} \\ (l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ l < m + 1 \text{ and } z \in \mathbb{C};\ l = m + 1 \\ \text{and } z \in \mathcal{U};\ l = m + 1,\ z \in \partial\mathcal{U}, \text{ and } \Re(w) > 0), \end{aligned} \quad (3)$$

where an empty product is interpreted as 1 and

$$w := \sum_{j=1}^{m} \beta_j - \sum_{j=1}^{l} \alpha_j$$

and $(\lambda)_n$ is the Pochhammer symbol (or the *shifted factorial*) defined, in terms of the Gamma function Γ, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0), \\ \lambda(\lambda + 1)(\lambda + 2) \cdots (\lambda + n - 1) & (n \in \mathbb{N}). \end{cases}$$

Corresponding to a function $h_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m; z)$ defined by

$$h_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m; z) := z^p {}_lF_m$$

Dziok and Srivastava[8] (also see [9], [13] and [24]) considered the linear operator

$$H_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m) : \mathcal{A}_p \longrightarrow \mathcal{A}_p$$

defined by the following Hadamard product (or convolution):

$$\begin{aligned} H_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m)f(z) &:= [z^p\ {}_lF_m(z)] * f(z) \quad (4) \\ &= z^p + \sum_{n=p+1}^{\infty} \frac{(\alpha_1)_{n-p} \cdots (\alpha_l)_{n-p}}{(\beta_1)_{n-p} \cdots (\beta_m)_{n-p}} \frac{a_n z^n}{(n-p)!} \\ &\quad (l \leq m + 1;\ l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; n \in \mathbb{N};\ z \in \mathcal{U}). \end{aligned}$$

To make the notation simple, we write

$$H_p^{l,m}(\alpha_1) := H_p(\alpha_1, \cdots, \alpha_l; \beta_1, \cdots, \beta_m).$$

It easily follows from (4) that

$$z\big(H_p^{l,m}(\alpha_1)f(z)\big)' = \alpha_1 H_p^{l,m}(\alpha_1+1)f(z) - (\alpha_1 - p)H_p^{l,m}(\alpha_1)f(z) \quad (f \in \mathcal{A}_p). \quad (5)$$

It should be remarked that the linear operator $H_p^{l,m}(\alpha_1)$ is a generalization of many other linear operators considered earlier viz. the Hohlov operator[11], the linear operator studied by Goel and Sohi[10], Ruscheweyh derivative operator[20], the linear operator studied by Saitoh[22], Carlson-Shaffer linear operator[5], generalized Bernadi-Libera-Livingston integral operator[7], Owa-Srivastava operator[18, 19], the operator studied by Liu and Noor[12] and Cho-Kwon-Srivastava operator[6].

Recently using the operator $H_p^{l,m}(\alpha_1)$ various subordination and superordination results has been carried out in different contexts see ([2], [16] and [23]). The main object of the present sequel to the aforementioned work is to investigate subordination and superordination results of some subclasses of analytic multivalent functions involving $H_p^{l,m}(\alpha_1)$ in different settings. Together with these results, differential sandwich type theorems as an interesting consequences. Our results include various known results studied earlier for particular change in parameters.

2. Preliminaries

To establish our main results, we need the following:

Lemma 1.([14], Theorem 3.4h, p.132) *Let q be univalent in the open unit disk $\mathcal{U}$ and θ and ϕ be analytic in a domain D containing $q(\mathcal{U})$ with $\phi(w) \neq 0$ when $w \in q(\mathcal{U})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*

1. *Q is starlike in $\mathcal{U}$, and*
2. *$\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for $z \in \mathcal{U}$.*

If p is analytic in $\mathcal{U}$, with $p(0) = q(0)$, $p(\mathcal{U}) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$$

then $p \prec q$ and q is the best dominant.

Lemma 2.([14], Corollary 3.4h.1, p.135) *Let q be univalent in the open unit disk $\mathcal{U}$ and ϕ be analytic in a domain D containing $q(\mathcal{U})$. If $zq'(z).\phi(q(z))$ is starlike and*

$$zp'(z)\phi(p(z)) \prec zq'(z)\phi(q(z))$$

then $p \prec q$ *and* q *is the best dominant.*

Lemma 3.[21] *Let* q *be univalent convex in the open unit disk* $\mathcal{U}$ *and* $\psi, \gamma \in \mathbb{C}$ *with* $\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\{0, -\Re(\psi/\gamma)\}$. *If* $p(z)$ *is analytic and*

$$\psi p(z) + \gamma z p'(z) \prec \psi q(z) + \gamma z q'(z),$$

then $p \prec q$ *and* q *is the best dominant.*

Lemma 4.([4]) *Let* q *be univalent in the open unit disk* $\mathcal{U}$ *and* ϑ *and* φ *be analytic in a domain* D *containing* $q(\mathcal{U})$. *Suppose that*

1. $\Re\left(\frac{z\vartheta'(q(z))}{\varphi(q(z))}\right) > 0$ *for* $z \in \mathcal{U}$, *and*
2. $zq'(z)\varphi(q(z))$ *is starlike univalent in* $\mathcal{U}$.

If $p \in \mathcal{H}[q(0), 1] \cap Q$, *with* $p(\mathcal{U}) \subseteq D$, *and* $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ *is univalent in* $\mathcal{U}$ *and*

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \quad (z \in \mathcal{U})$$

then $q \prec p$ *and* q *is the best subordinant.*

Lemma 5.([15], Theorem 8, p.822) *Let* q *be univalent convex in the open unit disk* $\mathcal{U}$ *and* $\gamma \in \mathbb{C}$, *with* $\Re(\gamma) > 0$. *If* $p \in \mathcal{H}[q(0), 1] \cap Q$, *and* $p(z) + \gamma z p'(z)$ *is univalent in* $\mathcal{U}$ *then*

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z), \quad (z \in \mathcal{U})$$

then $q \prec p$ *and* q *is the best subordinant.*

3.Subordination Results Involving Dziok-Srivastava operator

We have the following subordination results:

Theorem 1. *Let* $\tau > 0$ *and the function* $f \in \mathcal{A}_p$ *satisfying the subordination conditions:*

$$(1-\tau)\frac{H_p^{l,m}(\alpha_1)f(z)}{z^p} + \tau\frac{H_p^{l,m}(\alpha_1+1)f(z)}{z^p} \prec q(z) + \frac{\tau z q'(z)}{\alpha_1} \quad (z \in \mathcal{U}), \tag{6}$$

where $H_p^{l,m}(\alpha_1)f(z)$ *is defined by (4) and* q *is univalent in* $\mathcal{U}$ *with*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\Re\left(\frac{\alpha_1}{\tau}\right)\right\}, \tag{7}$$

then

$$\frac{H_p^{l,m}(\alpha_1)f(z)}{z^p} \prec q(z) \quad (z \in \mathcal{U}), \tag{8}$$

and q is the best dominant.

Proof. Let the function p be defined by

$$p(z) := \frac{H_p^{l,m}(\alpha_1)f(z)}{z^p}.$$

Differentiation followed by applications of the identity (5) yields

$$p(z) + \frac{zp'(z)}{\alpha_1} = \frac{H_p^{l,m}(\alpha_1+1)f(z)}{z^p}$$

Therefore, in light of the hypothesis (6), we have

$$p(z) + \frac{\tau zp'(z)}{\alpha_1} \prec q(z) + \frac{\tau zq'(z)}{\alpha_1}$$

Taking $\gamma = \frac{\tau}{\alpha_1}$ and $\psi = 1$ in Lemma 3, the assertion of the Theorem 1 follows. This completes the proof of Theorem 1.

By taking $q(z) = \frac{1+Az}{1+Bz} \quad -1 \leq B < A \leq 1$ and $q(z) = \left(\frac{1+z}{1-z}\right)$, in Theorem 1, we get the following:

Corollary 1. *Let $\tau > 0$ and the function $f \in \mathcal{A}_p$ satisfying the subordination conditions:*

$$(1-\tau)\frac{H_p^{l,m}(\alpha_1)f(z)}{z^p} + \tau\frac{H_p^{l,m}(\alpha_1+1)f(z)}{z^p} \prec \frac{1+Az}{1+Bz} + \frac{\tau(A-B)z}{\alpha_1(1+Bz)^2},$$

where $H_p^{l,m}(\alpha_1)f(z)$ is defined by (4) and $\frac{1+Az}{1+Bz}$ is univalent in $\mathcal{U}$ with

$$\Re\left(\frac{1-Bz}{1+Bz}\right) > \max\left\{0, -\Re\left(\frac{\alpha_1}{\tau}\right)\right\},$$

then

$$\frac{H_p^{l,m}(\alpha_1)f(z)}{z^p} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}),$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Corollary 2. *Let $\tau > 0$ and the function $f \in \mathcal{A}_p$ satisfying the subordination conditions*

$$(1-\tau)\frac{H_p^{l,m}(\alpha_1)f(z)}{z^p} + \tau\frac{H_p^{l,m}(\alpha_1+1)f(z)}{z^p} \prec \left(\frac{1+z}{1-z}\right) + \frac{\tau}{\alpha_1}\frac{2z}{(1-z)^2} \quad (z \in \mathcal{U}),$$

where $H_p^{l,m}(\alpha_1)f(z)$ is defined by (4) and $\left(\frac{1+z}{1-z}\right)$ is univalent in $\mathcal{U}$ with

$$\Re\left(\frac{1+z}{1-z}\right) > \max\left\{0, -\Re\left(\frac{\alpha_1}{\tau}\right)\right\},$$

then

$$\frac{H_p^{l,m}(\alpha_1)f(z)}{z^p} \prec \left(\frac{1+z}{1-z}\right) \quad (z \in \mathcal{U}),$$

and $\left(\frac{1+z}{1-z}\right)$ is the best dominant.

Theorem 2. *If the function $f \in \mathcal{A}_p$ satisfying the subordination condition:*

$$1 + \gamma\eta\left[\frac{\lambda z(H_p^{l,m}(\alpha_1)f(z))' + \delta z(H_p^{l,m}(\alpha_1+1)f(z))'}{\lambda H_p^{l,m}(\alpha_1)f(z) + \delta H_p^{l,m}(\alpha_1+1)f(z)} - p\right] \prec 1 + \gamma\frac{zq'(z)}{q(z)}, \tag{9}$$

where $H_p^{l,m}(\alpha_1)f(z)$ is defined by (4), $\lambda, \delta, \eta, \gamma \in \mathbb{C}$ with $\gamma, \eta, \lambda + \delta \neq 0$ and q be univalent in $\mathcal{U}$ with

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0, \tag{10}$$

then

$$\left[\frac{\lambda H_p^{l,m}(\alpha_1)f(z) + \delta H_p^{l,m}(\alpha_1+1)f(z)}{(\lambda+\delta)z^p}\right]^\eta \prec q(z) \tag{11}$$

and q is the best dominant.

Proof. Let the function $p(z)$ be defined by

$$p(z) := \left[\frac{\lambda H_p^{l,m}(\alpha_1)f(z) + \delta H_p^{l,m}(\alpha_1+1)f(z)}{(\lambda+\delta)z^p}\right]^\eta$$

Taking logarithmic differentiation, we get

$$\frac{zp'(z)}{p(z)} = \eta\left[\frac{\lambda z(H_p^{l,m}(\alpha_1)f(z))' + \delta z(H_p^{l,m}(\alpha_1+1)f(z))'}{\lambda H_p^{l,m}(\alpha_1)f(z) + \delta H_p^{l,m}(\alpha_1+1)f(z)} - p\right]$$

By Setting $\theta(w) := 1$ and $\phi(w) := \gamma/w$, and also by letting $Q(z) = zq'(z)\phi(q(z)) = \gamma\frac{zq'(z)}{q(z)}$ and $h(z) = \theta(q(z)) + Q(z) = 1 + \gamma\frac{zq'(z)}{q(z)}$, we observe that $Q(z)$ is univalent starlike in $\mathcal{U}$ and $\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$.

Thus assertions of Theorem 2 follows by an applications of Lemma 1. This completes the proof of Theorem 2.

Taking $\lambda = 0, \delta = 1, \gamma = 1$ and $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le A < B \le 1)$, we get the following:

Corollary 3. *If $f \in \mathcal{A}_p$*

$$1 + \eta\left[\frac{z(H_p^{l,m}(\alpha_1+1)f(z))'}{H_p^{l,m}(\alpha_1+1)f(z)} - p\right] \prec 1 + \frac{(A-B)z}{(1+Az)(1+Bz)}$$

then

$$\left[\frac{H_p^{l,m}(\alpha_1+1)f(z)}{z^p}\right]^\eta \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Setting $\lambda = 0, \delta = 1, \gamma = 1, p = 1, l = m+1, \alpha_1 = 0, \alpha_j = 1 (j = 2, 3, \cdots, m+1), \beta_j = 1 (j = 1, 2, \cdots, m)$ and $q(z) = (1+Bz)^{\eta(A-B)/B}$, which is univalent if and only if $|(\eta(A-B)/B)-1| \le 1$ or $|(\eta(A-B)/B+1| \le 1$ (see[17]), Theorem 2 reduced to the following:

Corollary 4. *If $f \in \mathcal{A}$*

$$1 + \eta\left[\frac{zf'(z)}{f(z)} - 1\right] \prec 1 + \frac{\eta(A-B)z}{1+Bz}$$

then

$$\left[\frac{f(z)}{z}\right]^\eta \prec (1+Bz)^{\eta(A-B)/B}$$

and $(1+Bz)^{\eta(A-B)/B}$ is the best dominant.

Setting $\lambda = 0, \delta = 1, \eta = 1, \gamma = \frac{1}{b}$ $(b \in \mathbb{C} \setminus \{0\}), p = 1, l = m+1, \alpha_1 = 0, \alpha_j = 1 (j = 2, 3, \cdots, m+1), \beta_j = 1 (j = 1, 2, \cdots, m)$ and $q(z) = \frac{1}{(1-z)^{2b}}$ in Theorem 2, we get the following:

Corollary 5.[25] *If $f \in \mathcal{A}$*

$$1 + \frac{1}{b}\left[\frac{zf'(z)}{f(z)} - 1\right] \prec \frac{1+z}{1-z}$$

then

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^{2b}}$$

and $\frac{1}{(1-z)^{2b}}$ is the best dominant.

Setting $\lambda = 0, \delta = 1, \eta = 1, \gamma = \frac{1}{b}$ $(b \in \mathbb{C} \setminus \{0\}), p = 1, l = m+1, \alpha_1 = \beta_1, \alpha_j = 1 (j = 1, 2, \cdots, m+1), \beta_j = 1 (j = 1, 2, \cdots, m)$ and $q(z) = \frac{1}{(1-z)^{2b}}$ in Theorem 2, we get the following:

Corollary 6.[25] *If $f \in \mathcal{A}$*

$$1 + \frac{1}{b}\frac{zf''(z)}{f'(z)} \prec \frac{1+z}{1-z}$$

then

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^{2b}}$$

and $\frac{1}{(1-z)^{2b}}$ is the best dominant.

Theorem 3. *Let the function $f \in \mathcal{A}_p$ satisfying the subordination condition:*

$$\Omega(z) \prec \mu q(z) + \zeta + \gamma z q'(z), \tag{12}$$

where

$$\Omega(z) := \left[\frac{\lambda H_p^{l,m}(\alpha_1)f(z) + \delta H_p^{l,m}(\alpha_1+1)f(z)}{(\lambda+\delta)z^P}\right]^{\eta} \times \left\{\mu + \gamma\eta\left(\frac{\lambda z(H_p^{l,m}(\alpha_1)f(z))' + \delta z(H_p^{l,m}(\alpha_1+1)f(z))'}{\lambda H_p^{l,m}(\alpha_1)f(z) + \delta H_p^{l,m}(\alpha_1+1)f(z)} - p\right)\right\} + \zeta, \tag{13}$$

with $\lambda+\delta\neq 0, \gamma, \eta \neq 0$ *for* $\lambda, \delta, \eta, \gamma, \zeta, \mu \in \mathbb{C}$ *and* q *be univalent in* u *satisfies*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\Re\left(\frac{\mu}{\gamma}\right)\right\}, \tag{14}$$

then

$$\left[\frac{\lambda H_p^{l,m}(\alpha_1)f(z)+\delta H_p^{l,m}(\alpha_1+1)f(z)}{(\lambda+\delta)z^P}\right]^\eta \prec q(z) \tag{15}$$

and q *is the best dominant.*

Proof. Consider

$$h(z) := \left[\frac{\lambda H_p^{l,m}(\alpha_1)f(z)+\delta H_p^{l,m}(\alpha_1+1)f(z)}{(\lambda+\delta)z^P}\right]^\eta$$

Logarithmic differentiation yields

$$\frac{zh'(z)}{h(z)} = \eta\left(\frac{\lambda z(H_p^{l,m}(\alpha_1)f(z))'+\delta z(H_p^{l,m}(\alpha_1+1)f(z))'}{\lambda H_p^{l,m}(\alpha_1)f(z)+\delta H_p^{l,m}(\alpha_1+1)f(z)} - p\right)$$

Therefore

$$zh'(z) = \eta h(z)\left(\frac{\lambda z(H_p^{l,m}(\alpha_1)f(z))'+\delta z(H_p^{l,m}(\alpha_1+1)f(z))'}{\lambda H_p^{l,m}(\alpha_1)f(z)+\delta H_p^{l,m}(\alpha_1+1)f(z)} - p\right)$$

Setting $\theta(w) = \mu w + \zeta,\ \phi(w) = \gamma$ and

$$Q(z) = zq'(z)\phi(q(z)) = \gamma z q'(z)$$
$$p(z) = \theta(q(z)) + Q(z) = \mu q(z) + \zeta + \gamma z q'(z).$$

Therefore from (14), it is observed that Q is starlike in $\mathcal{U}$ and also

$$\Re\left(\frac{zp'(z)}{Q(z)}\right) = \Re\left\{\frac{\mu}{\gamma}+1+\frac{zq''(z)}{q'(z)}\right\} > 0.$$

Thus assertion of Theorem 3 followed by application of Lemma 1. This completes the proof of Theorem 3.

Setting $\lambda = 0, \delta = 1, \gamma = 1$ and $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq A < B \leq 1)$ in Theorem 3, we get the following:

Corollary 7. *If $f \in \mathcal{A}_p$ and $\Re(\mu) > 0$. Suppose that*

$$\Re\left(\frac{1-Bz}{1+Bz}\right) > \max\{0, -\Re(\mu)\}.$$

If

$$\left[\frac{H_p^{l,m}(\alpha_1+1)f(z)}{z^p}\right]^\eta \left\{\mu + \eta\left(\frac{z(H_p^{l,m}(\alpha_1+1)f(z))'}{H_p^{l,m}(\alpha_1+1)f(z)} - p\right)\right\} + \zeta \prec \mu\frac{1+Az}{1+Bz} + \zeta + \frac{(A-B)z}{(1+Bz)^2},$$

then

$$\left[\frac{H_p^{l,m}(\alpha_1+1)f(z)}{z^p}\right]^\eta \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Again setting $\lambda = 0, \delta = 1, \gamma = 1, p = 1, l = m+1, \alpha_1 = 0, \alpha_j = 1 (j = 2, 3, \cdots, m+1), \beta_j = 1 (j = 1, 2, \cdots, m)$ and $q(z) = \frac{1+z}{1-z}$ in Theorem 3, we get the following:

Corollary 8. *Let $f \in \mathcal{A}$ and*

$$\left[\frac{f(z)}{z}\right]^\eta \left\{\mu + \eta\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} + \zeta \prec \mu\frac{1+z}{1-z} + \zeta + \frac{2z}{(1-z)^2},$$

then

$$\left[\frac{f(z)}{z}\right]^\eta \prec \frac{1+z}{1-z}$$

and $\frac{1+z}{1-z}$ is the best dominant.

4. Superordination Results Involving Dziok-Srivastava Operator

We have the following superordination results:

Theorem 4. *Let the function $f \in \mathcal{A}_p$, suppose that $\frac{H_p^{l,m}(\alpha_1)f(z)}{z^p} \in \mathcal{H}[q(0), 1] \cup Q$ and*

$$(1-\tau)\frac{H_p^{l,m}(\alpha_1)f(z)}{z^p} + \tau\frac{H_p^{l,m}(\alpha_1+1)f(z)}{z^p} \qquad (\tau > 0) \tag{16}$$

is univalent in $\mathcal{U}$, where $H_p^{l,m}(\alpha_1)$ is defined by (4). If q be convex univalent in $\mathcal{U}$ and

$$q(z)+\frac{\tau z q'(z)}{\alpha_1} \prec (1-\tau)\frac{H_p^{l,m}(\alpha_1)f(z)}{z^p}+\tau\frac{H_p^{l,m}(\alpha_1+1)f(z)}{z^p} \tag{17}$$

then

$$q(z) \prec \frac{H_p^{l,m}(\alpha_1)f(z)}{z^p} \quad (z\in\mathcal{U}), \tag{18}$$

and q is the best subordinant.

Proof. Write

$$p(z)=\frac{H_p^{l,m}(\alpha_1)f(z)}{z^p} \qquad (z\in\mathcal{U}).$$

Differentiation followed by applications of (5) gives

$$p(z)+\frac{zp'(z)}{\alpha_1}=\frac{H_p^{l,m}(\alpha_1+1)f(z)}{z^p}$$

Now the hypothesis (16) of Theorem 4 became

$$(1-\tau)\frac{H_p^{l,m}(\alpha_1)f(z)}{z^p}+\tau\frac{H_p^{l,m}(\alpha_1+1)f(z)}{z^p}=p(z)+\frac{\tau z p'(z)}{\alpha_1}$$

Therefore by application of Lemma 5 to the resulting equation, we get

$$q(z)+\frac{\tau z q'(z)}{\alpha_1} \prec p(z)+\frac{\tau z p'(z)}{\alpha_1}$$

implies

$$q(z) \prec p(z)=\frac{H_p^{l,m}(\alpha_1)f(z)}{z^p},$$

and q is the best subordinant. The proof of Theorem 4 is completed.

By taking $q(z)=\frac{1+Az}{1+Bz}; \quad -1\le B<A\le 1$, in Theorem 4, we get the following:

Corollary 9. *Let the function $f\in\mathcal{A}_p$, suppose that $\frac{H_p^{l,m}(\alpha_1)f(z)}{z^p}\in\mathcal{H}[q(0),1]\cup Q$ and*

$$(1-\tau)\frac{H_p^{l,m}(\alpha_1)f(z)}{z^p}+\tau\frac{H_p^{l,m}(\alpha_1+1)f(z)}{z^p} \qquad (\tau>0)$$

is univalent in $\mathcal{U}$*, where* $H_p^{l,m}(\alpha_1)$ *is defined by (4). If* q *be convex univalent in* $\mathcal{U}$ *and*

$$\frac{1+Az}{1+Bz}+\frac{\tau(A-B)z}{\alpha_1(1+Bz)^2}\prec(1-\tau)\frac{H_p^{l,m}(\alpha_1)f(z)}{z^p}+\tau\frac{H_p^{l,m}(\alpha_1+1)f(z)}{z^p}$$

then

$$\frac{1+Az}{1+Bz}\prec\frac{H_p^{l,m}(\alpha_1)f(z)}{z^p}\quad(z\in\mathcal{U}),$$

and $\frac{1+Az}{1+Bz}$ *is the best subordinant.*

The proof of Theorem 5 and Theorem 6 are similar to previous theorems, therefore we state the theorems without proof.

Theorem 5. *Let the function* q *be convex univalent in* $\mathcal{U}$ *and* $\lambda,\delta,\gamma,\eta\in\mathbb{C}$ *with* $\lambda+\delta\neq 0,\gamma,\eta\neq 0$*. Let* $f\in\mathcal{A}_p$ *and suppose that*

$$\left[\frac{\lambda H_p^{l,m}(\alpha_1)f(z)+\delta H_p^{l,m}(\alpha_1+1)f(z)}{(\lambda+\delta)z^p}\right]^\eta\in\mathcal{H}[q(0),1]\cap Q \text{ and} \tag{19}$$

$$1+\gamma\eta\left[\frac{\lambda z(H_p^{l,m}(\alpha_1)f(z))'+\delta z(H_p^{l,m}(\alpha_1+1)f(z))'}{\lambda H_p^{l,m}(\alpha_1)f(z)+\delta H_p^{l,m}(\alpha_1+1)f(z)}-p\right] \tag{20}$$

is univalent in $\mathcal{U}$*. If*

$$1+\gamma\frac{zq'(z)}{q(z)}\prec 1+\gamma\eta\left[\frac{\lambda z(H_p^{l,m}(\alpha_1)f(z))'+\delta z(H_p^{l,m}(\alpha_1+1)f(z))'}{\lambda H_p^{l,m}(\alpha_1)f(z)+\delta H_p^{l,m}(\alpha_1+1)f(z)}-p\right] \tag{21}$$

then

$$q(z)\prec\left[\frac{\lambda H_p^{l,m}(\alpha_1)f(z)+\delta H_p^{l,m}(\alpha_1+1)f(z)}{(\lambda+\delta)z^p}\right]^\eta \tag{22}$$

and q *is the best subordinant.*

Theorem 6. *Let the function* q *be convex univalent in the open unit disc* $\mathcal{U}$ *and* $\lambda,\delta,\gamma,\eta,\zeta,\mu\in\mathbb{C}$ *with* $\lambda+\delta,\gamma,\eta\neq 0$*. Let* $f\in\mathcal{A}_p$ *and suppose that*

$$\frac{H_p^{l,m}(\alpha_1)f(z)}{z^p}\in\mathcal{H}[q(0),1]\cap Q \text{ and } \Re\left(\frac{\mu q'(z)}{\gamma}\right)>0. \tag{23}$$

If

$$\mu q(z) + \zeta + \gamma z q'(z) \prec \Omega \tag{24}$$

then

$$q(z) \prec \left[\frac{\lambda H_p^{l,m}(\alpha_1)f(z) + \delta H_p^{l,m}(\alpha_1+1)f(z)}{(\lambda+\delta)z^p}\right]^\eta \tag{25}$$

and q is the best subordinant.

5.Sandwich-Type Theorems

By combining subordination and superordination results of Theorems 1 with 4, 2 with 5, and 3 with 6, discussed in Section3 and Section4 respectively, we get the following Sandwich-type results:

Theorem 7. *Let the function q_1 be convex univalent, q_2 be univalent in $\mathcal{U}$ and $\tau(>0) \in \mathbb{C}$. Suppose q_1 satisfies $\Re(\tau) > 0$ and q_2 satisfies (7). If $H_p^{l,m}(\alpha_1)f(z)/z^p \in \mathcal{H}[q(0),1] \cap Q$,*

$$(1-\tau)\frac{H_p^{l,m}(\alpha_1)f(z)}{z^p} + \tau\frac{H_p^{l,m}(\alpha_1+1)f(z)}{z^p}$$

is univalent in $\mathcal{U}$ and

$$q_1(z) + \frac{\tau z q_1'(z)}{\alpha_1} \prec (1-\tau)\frac{H_p^{l,m}(\alpha_1)f(z)}{z^p} + \tau\frac{H_p^{l,m}(\alpha_1+1)f(z)}{z^p} \prec q_2(z) + \frac{\tau z q_2'(z)}{\alpha_1}$$

implies

$$q_1(z) \prec \frac{H_p^{l,m}(\alpha_1)f(z)}{z^p} \prec q_2(z)$$

where q_1 and q_2 are respectively the best subordinant and the dominant.

Theorem 8. *Let the function q_1 be convex univalent, q_2 satisfying (10) be univalent in $\mathcal{U}$ and $0 < \tau \in \mathbb{C}$. Suppose that $\left[\frac{\lambda H_p^{l,m}(\alpha_1)f(z)+\delta H_p^{l,m}(\alpha_1+1)f(z)}{(\lambda+\delta)z^p}\right]^\eta \in \mathcal{H}[q(0),1] \cap Q$,*

$$1 + \gamma\eta\left[\frac{\lambda z(H_p^{l,m}(\alpha_1)f(z))' + \delta z(H_p^{l,m}(\alpha_1+1)f(z))'}{\lambda H_p^{l,m}(\alpha_1)f(z) + \delta H_p^{l,m}(\alpha_1+1)f(z)} - p\right]$$

is univalent in $\mathcal{U}$. *If*

$$1+\gamma\frac{zq_1'(z)}{q_1(z)} \prec 1+\gamma\eta\left[\frac{\lambda z(H_p^{l,m}(\alpha_1)f(z))'+\delta z(H_p^{l,m}(\alpha_1+1)f(z))'}{\lambda H_p^{l,m}(\alpha_1)f(z)+\delta H_p^{l,m}(\alpha_1+1)f(z)}-p\right] \prec 1+\gamma\frac{zq_2'(z)}{q_2(z)}$$

then

$$q_1(z) \prec \left[\frac{\lambda H_p^{l,m}(\alpha_1)f(z)+\delta H_p^{l,m}(\alpha_1+1)f(z)}{(\lambda+\delta)z^p}\right]^{\eta} \prec q_2(z)$$

where q_1 *and* q_2 *are respectively the best subordinant and the dominant.*

Theorem 9. *Let the function* q_1 *satisfying (23) be convex univalent,* q_2 *satisfying (14) be univalent in* $\mathcal{U}$ *and* $\lambda,\delta,\mu,\gamma,\eta,\zeta\in\mathbb{C};\lambda+\delta\neq 0,\gamma,\eta\neq 0$. *Suppose that* $\left[\frac{\lambda H_p^{l,m}(\alpha_1)f(z)+\delta H_p^{l,m}(\alpha_1+1)f(z)}{(\lambda+\delta)z^p}\right]^{\eta}\in\mathcal{H}[q(0),1]\cap Q,\Omega$ *is univalent in* $\mathcal{U}$. *If*

$$\mu q_1(z)+\zeta+\gamma q_1'(z) \prec \Omega \prec \mu q_2(z)+\zeta+\gamma q_2'(z)$$

then

$$q_1(z) \prec \left[\frac{\lambda H_p^{l,m}(\alpha_1)f(z)+\delta H_p^{l,m}(\alpha_1+1)f(z)}{(\lambda+\delta)z^p}\right]^{\eta} \prec q_2(z)$$

where q_1 *and* q_2 *are respectively the best subordinant and the dominant.*

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