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A NOTE ON THE STABILITY OF AN EQUATION

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Abstract. If T is a map from a complete metric space to itself which satisfies a Lipschitz like condition, then it is shown that an equation of the form

$$(T - AI)(x) = 0,$$

for suitable real number A and I being the identity map, has the Hyers-Ulam stability.

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1. Introduction

In 2009, Li and Hua, [2], introduced the following notion of Hyers-Ulam stability for a polynomial equation. Let (X, d) be a complete metric space and $f : X \to X$. We say that the equation $f(x) = 0$ has the *Hyers-Ulam stability* if there exists a constant $K > 0$ such that for all $\varepsilon > 0$, if there is $y \in X$ with the property $d(f(y), 0) < \varepsilon$, then there exists $z \in X$ satisfying $f(z) = 0$ and $d(y, z) < K\varepsilon$.

The result of Li-Hua states that: If T is a contraction mapping from X to X, then the equation $(T - I)x = 0$ has the Hyers-Ulam stability, which is equivalent to saying that for every $\varepsilon > 0$, if

$$d(Tx - x, 0) \leq \varepsilon,$$

then there exists $z \in X$ satisfying $Tz - z = 0$ with $d(x, z) \leq K\varepsilon$ for some $K > 0$.

The main tool in Li-Hua's proof is the Banach contraction mapping theorem. Our objective here is to improve upon Li-Hua's result by using the notion of δ-Lipschitz condition (to be defined below) to induce a contraction mapping.

2. The results

Our main result reads:

Theorem 1. *Let (X, d) be a complete metric linear space and δ be a positive real number. If $T : X \to X$ satisfies the following δ-Lipschitz condition*

$$d(T(x), T(y)) = d(T(x-y), 0) \leq \delta\, d(x, y) \quad (x, y \in X),$$

then for all $A > \delta$, the equation

$$F_A(x) := (T - AI)x = 0$$

has the Hyers-Ulam stability, or equivalently, for $\varepsilon > 0$, if $d\left(F_A(y), 0\right) \leq \varepsilon$ $(y \in X)$, then there exists a (unique) $z \in X$ such that $F_A(z) = 0$ with $d(y, z) \leq K\varepsilon$ for some $K > 0$.

Proof. Defining $G(x) = \frac{1}{A}T(x)$, we see that for all $x, y \in X$,

$$d(G(x), G(y)) = d\left(\frac{1}{A}T(x), \frac{1}{A}T(y)\right) = \frac{1}{A}\, d(T(x), T(y)) = \frac{1}{A}\, d(T(x-y), 0) \leq \frac{\delta}{A}\, d\left(x, y\right),$$

showing that $G(x)$ is a contraction mapping. By the Banach contraction mapping theorem, [1, Section $5.1-2$], G has precisely one fixed point, in other words, there exists a (unique) $z \in X$ such that $G(z) = z$, i.e., $T(z) - Az = 0$. Thus, the equation $F_A(x) = 0$ has a solution $z \in X$.

Next, let $\varepsilon > 0$ and assume that there is $y \in X$ such that $d(F_A(y), 0) \leq \varepsilon$. Then

$$\begin{aligned} d(y, z) &= d\left(y - G(y) + G(y), z\right) = d(y - G(y), G(y) - G(z)) \\ &\leq d(y - G(y), 0) + d(G(y) - G(z), 0) = \frac{1}{A} d(F_A(y), 0) + d(G(y), G(z)) \\ &\leq \frac{1}{A}\varepsilon + \frac{\delta}{A} d(y, z), \end{aligned}$$

and so

$$d(y, z) \leq \frac{\varepsilon}{A - \delta},$$

with $A - \delta > 0$.

Specializing the metric space X to be a subset of $\mathbb{R}$, we obtain:

Corollary 2. *Let $\delta > 0$, $A > \delta$ and S be a complete subspace of $\mathbb{R}$. If $g : S \to S$ satisfies the δ-Lipschitz condition*

$$|g(x) - g(y)| \leq \delta |x - y| \qquad (x, y \in S),$$

then the equation

$$F_A(x) := g(x) - Ax = 0$$

has the Hyers-Ulam stability, or equivalently, for $\varepsilon > 0$*, if* $|F_A(y)| \le \varepsilon \quad (y \in S)$*, then there exists a (unique)* $z \in S$ *such that* $F_A(z) = 0$ *with* $|y - z| \le K\varepsilon$ *for some* $K > 0$*.*

Regarding Theorem 2.1 of [2], we have the following extension.

Corollary 3. *Let* $\ell \in \mathbb{N}$*, let* $n_1 > n_2 > \cdots > n_\ell \ge 2$ *be a sequence of positive integers, and let*

$$f(x) = A_1 x^{n_1} + A_2 x^{n_2} + \cdots + A_\ell x^{n_\ell} + Ax + b \in \mathbb{R}[x],$$

with $A_1(\ne 0), A_2, \ldots, A_\ell,\ A(\ne 0),\ b \in \mathbb{R}$*. If*

$$|A| \ge \sum_{t=1}^{\ell} |A_t| + |b| \tag{1}$$

and

$$(0 <) \quad \delta := \frac{1}{|A|} \sum_{t=1}^{\ell} n_t \, |A_t| < 1, \tag{2}$$

then the equation $f(x) = 0$ *has the Hyers-Ulam stability over* $[-1, 1]$*, or equivalently, for* $\varepsilon > 0$*, if*

$$|A_1 y^{n_1} + A_2 y^{n_2} + \cdots + A_\ell y^{n_\ell} + Ay + b| \le \varepsilon \quad (y \in [-1, 1]),$$

then there exists a (unique) $z \in [-1, 1]$ *such that*

$$A_1 z^{n_1} + A_2 z^{n_2} + \cdots + A_\ell z^{n_\ell} + Az + b = 0$$

with $|y - z| \le K\varepsilon$ *for some* $K > 0$*.*

Proof. Let

$$g(x) = \frac{-1}{A} \left(A_1 x^{n_1} + \cdots + A_\ell x^{n_\ell} + b \right) \quad (x \in [-1, 1]).$$

By (1), we see that $g([-1, 1]) \subseteq [-1, 1]$. Next, observe that for $x, y \in [-1, 1]$, we have

$$|g(x) - g(y)| = \frac{1}{|A|} \left| A_1 (x^{n_1} - y^{n_1}) + \cdots + A_\ell (x^{n_\ell} - y^{n_\ell}) \right| \le \frac{|x - y|}{|A|} \sum_{t=1}^{\ell} n_t \, |A_t| ,$$

and so by (2), $g(x)$ is δ-Lipschitz over $[-1,1]$. By Corollary 2, the function

$$g(x) - x = \frac{-1}{A} f(x)$$

and so also the function $f(x)$ has the Hyers-Ulam stability.

The case where $\delta < 1$ and $A = 1$ of Theorem 1 yields the following result which is Theorem 2.2 of [2].

Corollary 4. *Let (X, d) be a complete metric linear space. If T is a contraction mapping from X to X, then $(T - I)x = 0$ has the Hyers–Ulam stability. That is, for every $\varepsilon > 0$, if*

$$d(Tx - x, 0) < \varepsilon,$$

then there exists a unique $z \in X$ satisfying

$$Tz - z = 0$$

with

$$d(x, z) < K\varepsilon$$

for some $K > 0$.

References

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