A NOTE ON SUBCLASSES OF UNIVALENT FUNCTIONS DEFINED BY A GENERALIZED SĂLĂGEAN OPERATOR

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Abstract.The object of this paper is to derive some inclusion relations regarding a new class denoted by $S_n^m(\lambda,\alpha)$ using the generalized Sălăgean operator.

Key words and phrases: univalent, Sălăgean operator, differential subordination.

2000 Mathematics Subject Classification: 30C45.

1.Introduction

Let $\mathcal{A}_n$ denote the class of functions of the form

$$f(z)=z+\sum_{k=n+1}^{\infty}a_kz^k, n\in N^*=\{1,2,\ldots\} \tag{1}$$

analytic and univalent in the unit disc of the complex plane

$$U=\{z\in C:\ |z|<1\} \tag{2}$$

with $\mathcal{A}_1=\mathcal{A}$.

F.M. Al-Oboudi in [1] defined, for a function in $\mathcal{A}_n$, the following differential operator:

$$D^0f(z)=f(z) \tag{3}$$

$$D_\lambda^1f(z)=D_\lambda f(z)=(1-\lambda)f(z)+\lambda zf'(z) \tag{4}$$

$$D_\lambda^mf(z)=D_\lambda(D_\lambda^{m-1}f(z)),\lambda>0. \tag{5}$$

When $\lambda=1$, we get the Sălăgean operator [6].

If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if there is a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$ for $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(u) \subset g(U)$.

We shall use the following lemmas to prove our results.

LEMMA 1.1.(Miller and Mocanu [3]).*Let h be a convex function with $h(0) = a$ and let $\gamma \in C^*$ be a complex number with Re $\gamma > 0$.If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z)$$

then

$$p(z) \prec q(z) \prec h(z)$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{(\gamma/n)-1} dt.$$

The function q is convex and is the best (a, n)-dominant.

LEMMA 1.2.(Miller and Mocanu [4]).*Let q be a convex function in U and let*

$$h(z) = q(z) + n\alpha zq'(z)$$

where $\alpha > 0$ and n is a positive integer. If

$$p(z) = q(0) + p_n z^n + \ldots \in \mathcal{H}[q(0), n]$$

and

$$p(z) + \alpha zp'(z) \prec h(z)$$

then

$$p(z) \prec q(z)$$

and this result is sharp.

2.Main results

Definition 2.1. *Let $f \in \mathcal{A}_n$, $n \in N^*$. We say that the function f is in the class $S_n^m(\lambda, \alpha)$, $\lambda > 0$, $\alpha \in [0, 1)$, $m \in N$, if f satisfies the condition*

$$Re[D_\lambda^m f(z)]' > \alpha, z \in U. \tag{6}$$

Theorem 2.1.*If $\alpha \in [0, 1)$, $m \in N$ and $n \in N^*$ then*

$$S_n^{m+1}(\lambda, \alpha) \subset S_n^m(\lambda, \delta) \tag{7}$$

where

$$\delta = \delta(\lambda, \alpha, n) = 2\alpha - 1 + 2(1-\alpha)\frac{1}{n\lambda}\beta\left(\frac{1}{\lambda n}\right) \tag{8}$$

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{t+1} dt \tag{9}$$

is the Beta function.

Proof. Let $f \in S_n^{m+1}(\lambda, \alpha)$. By using the properties of the operator D_λ^m, we get

$$D_\lambda^{m+1} f(z) = (1-\lambda) D_\lambda^m f(z) + \lambda z (D_\lambda^m f(z))' \tag{10}$$

If we denote by

$$p(z) = (D_\lambda^m f(z))' \tag{11}$$

where $p(z) = 1 + p_n z^n + \ldots$, $p(z) \in \mathcal{H}[1, n]$ then after a short computation we get

$$(D_\lambda^{m+1} f(z))' = p(z) + \lambda z p'(z), z \in U. \tag{12}$$

Since $f \in S_n^{m+1}(\lambda, \alpha)$, from Definition 2.1 one obtains

$$Re(D_\lambda^{m+1} f(z))' > \alpha, z \in U.$$

Using (12) we get

$$Re(p(z) + \lambda z p'(z)) > \alpha$$

which is equivalent to

$$p(z) + \lambda z p'(z) \prec \frac{1 + (2\alpha - 1)z}{1 + z} \equiv h(z). \tag{13}$$

Making use of Lemma 1.1 we have

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{1}{n\lambda z^{1/\lambda n}} \int_0^z \frac{1 + (2\alpha - 1)t}{1 + t} t^{(1/\lambda n) - 1} dt.$$

The function q is convex and is the best $(1, n)$-dominant.

Since

$$(D_\lambda^m f(z))' \prec 2\alpha - 1 + 2(1 - \alpha)\frac{1}{n\lambda} \cdot \frac{1}{z^{1/\lambda n}} \int_0^z \frac{t^{(1/\lambda n) - 1}}{t + 1} dt$$

it results that

$$Re(D_\lambda^m f(z))' > q(1) = \delta \tag{14}$$

where

$$\delta = \delta(\lambda, \alpha, n) = 2\alpha - 1 + 2(1 - \alpha)\frac{1}{n\lambda}\beta\left(\frac{1}{\lambda n}\right) \tag{15}$$

$$\beta\left(\frac{1}{\lambda n}\right) = \int_0^1 \frac{t^{(1/\lambda n) - 1}}{t + 1} dt. \tag{16}$$

From (14) we deduce that $f \in S_n^m(\lambda, \alpha, \delta)$ and the proof of the theorem is complete.

Making use of Lemma 1.2 we now prove the next theorems.

Theorem 2.2.*Let $q(z)$ be a convex function, $q(0) = 1$ and let h be a function such that*

$$h(z) = q(z) + n\lambda z q'(z), \lambda > 0. \tag{17}$$

If $f \in \mathcal{A}_n$ and verifies the differential subordination

$$(D_\lambda^{m+1} f(z))' \prec h(z) \tag{18}$$

then

$$(D_\lambda^m f(z))' \prec q(z) \tag{19}$$

and the result is sharp.

Proof. From (12) and (18) one obtains

$$p(z) + \lambda z p'(z) \prec q(z) + n\lambda z q'(z) \equiv h(z)$$

then, by using Lemma 1.2 we get

$$p(z) \prec q(z)$$

or

$$(D_\lambda^m f(z))' \prec q(z), z \in U$$

and this result is sharp.

THEOREM 2.3. *Let q be a convex function with $q(0) = 1$ and let h be a function of the form*

$$h(z) = q(z) + nzq'(z), \lambda > 0, \ z \in U. \tag{20}$$

If $f \in \mathcal{A}_n$ verifies the differential subordination

$$(D_\lambda^m f(z))' \prec h(z), z \in U \tag{21}$$

then

$$\frac{D_\lambda^m f(z)}{z} \prec q(z) \tag{22}$$

and this result is sharp.

Proof. If we let

$$p(z) = \frac{D_\lambda^m f(z)}{z}, z \in U$$

then we obtain

$$(D_\lambda^m f(z))' = p(z) + zp'(z), z \in U.$$

The subordination (21) becomes

$$p(z) + zp'(z) \prec q(z) + nzq'(z)$$

and from Lemma 1.2 we have (22). The result is sharp.

Remark.

a) For $n = 1$ these results were obtained in [2].

b) For $n = 1$, $\lambda = 1$ the results were obtained in [5].

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