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DYNAMIC BIFURCATION DIAGRAMS FOR SOME MODELS IN ECONOMICS AND BIOLOGY

by
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Abstract. A method to determine the static bifurcation diagram (sbd) and dynamical bifurcation diagram (dbd) is proposed and the results for a few concrete problems are shown.
Keywords: bifurcation theory, bifurcation diagrams

1. Historical premises

For a long time, we were working in hydrodynamic stability theory (hsth) [1], just the period during which this field was quickly moving from an analytical towards a geometric and topological treatment. In addition, more and more dynamical and bifurcational arguments were invoked in hsth to provide interdisciplinary studies on concrete fluid flows. Firstly, this change was imposed by the impossibility to obtain general results on the involved spectral problems. In exchange, the geometric view favored a better formalization of the intuition and experimental observations in the form of much simpler mathematical models. Indeed, in the years '60 of the last century, the idea of deterministic chaos was formulated [2], producing a genuine revolution in the scientific thinking and eventually leading to the new paradigm of our times. The key argument in the new trend was the thermal convection, a central topic of hsth. Soon, a macroscopic pattern formation as a result of a microscopic self-organization as a critical parameter was crossed, became the explanation sheared with by the greatest part of phenomena. The physical (in the largest sense) background was the matter of synergetics [3], [4], while its mathematical counterpart proved to be nonlinear dynamics and bifurcation theory.

Whence the decision of the most specialists in hsth to supplement their studies with applications of the two theories to concrete finite-dimensional models. We chose models from biology and economics, less investigated than those from mechanics.

Their simple form proved to be associated with very rich dynamics and bifurcations, as the title of a famous paper [5] warned the scientists long ago in a similar contexts.

In spite of their very low dimensions (mostly two), a lot of open problems remained due to the lack of theoretical results and constructive algorithms.

2. Ten concrete Cauchy problems

We investigated them together with my former PhDs. The results were published as monographs belonging to the Series of Applied and Industrial Mathematics of University of Pitesti (Romania) [6] or issued elsewhere [7], [8], as articles in Romanian and foreign journals, as contributions in the Proceedings of annual conferences (CAIMs) organized by the Romanian Society of Applied and Industrial Mathematics (ROMAI) [9]. The packages of programs used belonged to our former PhDs [13], [15], [16], or available on the Web [10], [11].

The first model studied was the FitzHugh Nagumo (F-N) model, namely the Cauchy problem for the system of ordinary differential equations

$$\begin{cases} \dot{x}_1 = c\left(x_1 + x_2 - x_1^3/3\right), \\ \dot{x}_2 = -\left(x_1 - a + bx_2\right)/c, \end{cases} \tag{1}$$

involving two unknown functions $x_1, x_2 : \mathbf{R} \to \mathbf{R}$ and three real parameters *a, b, c.*

In [7] there were determined (mainly theoretically and also numerically) the sbd and its asymptotic behavior as one parameter tends to infinity, the type of the equilibria, by the linearization only, the proof of the existence of saddle node equilibria by center manifold theory, curves of Hopf (H), Bogdanov-Takens (BT), Bautin (Ba), homoclinic, breaking saddle-connection bifurcation values, and we investigated numerically the canard phenomenon and explained it by using the curve of inflection points of the phase trajectories. Two types of bifurcations seemed to be new [18]. One parameter was kept fixed in a certain range. Transient regimes were investigated too. In [6], vol.8, the case of all three parameters was considered and all types of bifurcations were found in a systematic way. The stratum in the parameter space corresponding to the presence of three coexisting limit cycles was determined theoretically, whereas [7] yielded only a numerical estimation. By means of two diffeomorphisms a result from [7] was immediately regained, and in a very elegant way. It was determined the organizing center of the dbd. Among the open problems we quote: the evaluation of the size of the limit cycles, the systematic study of the canard phenomenon, the nonsatifactory investigation of the Ba bifurcation, the study of the dbd a little bit beyond the double-zero bifurcation values.

The F-N model with a forcing term, involving five parameters reads

$$\begin{cases} \dot{x}_1 = c\left(x_1 + x_2 - x_1^3/3 - A\cos\omega t\right), \\ \dot{x}_2 = -\left(x_1 - a + bx_2\right)/c \end{cases} \tag{2}$$

It was analyzed in [6] , vol. 8, [12] , mostly by asymptotic and numerical methods. The Poincaré map was used to derive Feigenbaum-like diagrams for the associated discrete system. An interesting graph containing the regions in the (ω, A) plane, corresponding to solutions the period of which are multiple of $2\pi/\omega$ were determined numerically. Two main types of saddle-node and period doubling bifurcations were detected also numerically.

The Rayleigh (R) model, containing two real parameters,

$$\begin{cases} \dot{x}_1 = x_2, \\ \varepsilon\dot{x}_2 = -ax_1 + x_2 - x_2^3/3, \end{cases} \tag{3}$$

and the R model with forcing involving four parameters

$$\begin{cases} \dot{x}_1 = x_2, \\ \varepsilon\dot{x}_2 = -ax_1 + x_2 - x_2^3/3 + g\sin\omega t, \end{cases} \tag{4}$$

were investigated in [6], vol. 8 and [13], [14]. The existence and uniqueness of the limit cycle was proved in two ways, one of which based on very fine bounds determined by the curve of inflection points of phase trajectories. The variation of the limit cycle with the parameters ε and a was deduced numerically. The codimension-one bifurcations of 1-periodic solution was investigated by asymptotic methods. These methods were also used to approximate the limit cycle when the forcing term was absent. By means of the Poincaré map the deterministic chaos was investigated and the alternating periodic and chaotic regimes were found. An interesting succession of attractors was revealed. The metamorphoses of the basins of attraction with fractal boundaries was shown.

For the two R models a lot of work remains to be done.

In [15], [17], for the Goodwin model, involving four parameters,

$$\begin{cases} \dot{x}_1 = x_1(a - x_2), \\ \dot{x}_2 = x_2(cx_1 - d), \end{cases} \tag{5}$$

an incipient investigation of the dbd was performed. Here the systematic treatment based on the normal forms and unfoldings theory is not done.

The Gray-Scott model

$$\begin{cases} \dot{x}_1 = -(a+b)x_1 + x_2{}^2 - x_1 x_2{}^2 + b, \\ \dot{x}_2 = -bx_1 - (a+d)x_2 + x_2{}^2 - x_1 x_2{}^2 + b + ac, \end{cases} \tag{6}$$

was studied in [8]. From the very beginning the presence of the four real parameters imposed a very cumbersome study of the equilibria (their multiplicity, their type upon the linearization). The topological type of equilibria was determined only for a few strata in the parameter space, corresponding to cusp, saddle-node, B-T, H,and Ba bifurcations. For the saddle-node bifurcation an original variant of the Liapunov-Schmidt was used. The perturbed bifurcation theory was applied to find the topological distinct sections of the sbd. However, this is the deepest analysis depending on four parameters for this model especially for the static case.

In spite of the fact that the study in [8] is the most complete available, a lot of open problems exist: the systematic application of the perturbed bifurcation theory for the investigation of the higher dimensional manifolds (e.g. sbd, the manifolds of values of codimension-one or –two bifurcations), the systematic derivation of the normal forms at the singularity and the unfoldings around them, *y compris* the degenerated ones, the systematic investigation of the sbd and dbd for the case when at least one parameter is null.

A model of evolution of the capital and working force of a firm

$$\begin{cases} \dot{x}_1 = cx_1^2 x_2 + bx_1, \\ \dot{x}_2 = x_1 + \alpha_2 x_2 - 1 \end{cases} \tag{7}$$

was investigated in [19] and then in many other papers, e.g. [20]. Before its processing, this model contained eight economical parameters. The sbd was deduced for the general case when all three parameters were nonnull as well as for the cases when at least one parameter was null. This type of investigation from [19] is the most complete in our group. The local dbd was determined around all nondegenerated and some of the degenerated bifurcations. The normal form theory, the centre manifold theory and the unfolding theory were systematically and extensively applied. The degeneration of some bifurcations was given only up to certain orders. Sometimes the connection between the old and new parameters was given. The perturbed bifurcation was not applied.

The advertizing model

$$\begin{cases} \dot{x}_1 = -\gamma\left(x_1 + \varphi x_2 + 2x_1 x_2 + x_2^2 + x_1 x_2^2\right), \\ \dot{x}_2 = x_1 + x_2 + 2x_1 x_2 + x_2^2 + x_1 x_2^2, \end{cases} \tag{8}$$

was studied in [19]. The sbd was derived while the investigation of the dbd was incomplete.

The multiplicator-accelerator model

$$\begin{cases} \dot{x}_1 = (a-1)x_1 - abx_2, \\ \dot{x}_2 = x_1 - bx_2, \end{cases} \tag{9}$$

was treated in [21], [22]. Even in the linear case, the comparison of its dbd with the standard dbd revealed the presence of an extra stratum in the parameter space. This phenomenon is due to the nonlinear dependence of one of the equations on the parameters.

In [22] the demand-supply model

$$\begin{cases} \dot{x}_1 = ax_1 - x_2 + b, \\ \dot{x}_2 = x_1 - cx_2^2 \end{cases} \tag{10}$$

was investigated. The sbd and dbd were determined by means of the three mentioned theories used in [19] too. It was found that the presence of the three parameters made the study almost as elaborate as in [19], even if the model (10) contained only quadratic nonlinearities. As open problems we quote the lack of the relationship between the initial and final parameters. The systematic study of the case of at least one null parameter was also absent.

3. Main steps of the method

After nine years of collaboration with our former PhD students on the basis of the mentioned theories from [23], [24], [25], the following main steps were revealed in determining the sbd and dbd.

1. The reduction of the number of parameters.

2. The separate and systematic investigation of the cases where at least one parameter vanishes.
3. The determination of the equilibria and, correspondingly, of the sbd.
4. The use of the perturbed bifurcation theory to determine all topological nonequivalent sections in sbd.
5. The determination of the topologic type of the equilibria.
6. The determination, in the parameter space, for each nonhyperbolic equilibrium point of the manifolds S, H and Q corresponding to multiple equilibria, to H bifurcation values and to double-zero bifurcation values.
7. The application of the normal form theory to derive the normal form of the vector field at a given singularity.
8. The determination of the associated miniversal unfolding for each degenerated or nondegenerated bifurcation.
9. The supplementation of the parametric portrait with the new manifolds corresponding to global bifurcations.
10. The construction of the local dbd for each nonhyperbolic equilibrium followed by a cumulative dbd.
11. The interpretation of various types of transient dynamices.

Our practice showed us that, at least hitherto, this is the less time consuming way to perform the most complete study on dynamics and bifurcation of Cauchy problems for ordinary differential equations.

References:

[1] Georgescu, A., Hydrodynamic stability theory, Kluwer, Dordrecht, 1985.
[2] Lorenz, E. N., Deterministic nonperiodic flows, Atmos. Sci., **20** (1963), 130-141.
[3] Springer Series in Synergetics, Berlin.
[4] Georgescu, A., Synergetics-a new synthesis of science, Ed. Tehnica, Bucharest, 1987. (Romanian)
[5] May, R., Simple mathematical models with very complicated dynamics, Nature, **261** (1976), 459-467.
[6] Series of Applied and Industrial Mathematics of University of Pitesti. (Romanian)

1. Georgescu, A., M. Moroianu, I. Oprea, Bifurcation theory. Principles and applications, 1999.

4. Ion, A. V., Global attractors and inertial manifolds for two problems in fluid mechanics, 2000.

5. Suciu N., On the relationship between the microscopic and macroscopic modelling of thermodynamic processes, 2001.
6. Ungureanu L., L. Ungureanu, Elements of economical dynamics, 2000.
8. Sterpu M., Dynamics and bifurcations for two generalized Van der Pol models, 2001.
[7] Rocsoreanu, C., A. Georgescu, N. Giurgiteanu, FitzHugh-Nagumo model. Bifurcation and dynamics, Kluwer, Dordrecht, 2000.
[8] Curtu, R., Dynamics and bifurcations for the Gray-Scott model with uncatalized conversion, Ed. Transilvania, Brasov, 2002 . (Romanian)
[9] Conference on Applied and Industrial Mathematics (CAIM), ROMAI, 1993-2003.
[10] Ermentrout, B., XPPAUT, http://www.math.pitt.edu/bard/xpp/xpp.html.
[11] Doedel, E., AUTO, ftp://ftp.cs.concordia.ca/pub/doedel/auto
[12] Georgescu A., M. Sterpu, Dynamics and bifurcation in the periodically forced FitzHugh-Nagumo system, Int. J. Chaos Theory and Applications, **5**, 2(2000), 63-80.
[13] Bazavan, P., Three dimensional computational study of the attractors of the Van der Pol type equations, Ph. D. Thesis, Institute of Mathematics of Romanian Academy, 2001. (Romanian)
[14] Sterpu, M., A. Georgescu, Codimension three bifurcations for the FitzHugh Nagumo dynamical scheme, Mathematical Reports, **3(53)**, 3(2001), 287-292.
[15] Giurgiteanu, N., Contributions to the study of the systems of differential equations by numerical methods and applications to biology, PhD Thesis, Institute of Mathematics of Romanian Academy, 1997. (Romanian).
[16] Giurgiteanu, N., Economic and biologic dynamics, Ed Europa, Craiova, 1997. (Romanian).
[17] Georgescu, A., N. Giurgiteanu, C. Rocsoreanu, Bifurcation in the Goodwin model from economics II, Bul. St. Seria Mat.-Inf., Univ. Pitesti, **3**(1999), 105-112.
[18] Rocsoreanu, C., A. Georgescu, N. Giurgiteanu, Connection between saddles for the FitzHugh-Nagumo system, Int. J. Bif. Chaos, **11,**2(2001), 533-540.
[19] Ungureanu, L., Structural stability and bifurcation in some mathematical models of economic dynamics, PhD Thesis, University of Pitesti, 2002. (Romanian).
[20] Ungureanu, L., A. Georgescu, M. Popescu, Static bifurcation diagrams for a microeconomic model, Bul. Acad. Rep. Moldova, **3(40)**(2002), 21-26.

[21] Georgescu, A., C. Georgescu, The multiplicator-accelerator continuous model. Linear case, Bul. St. Seria Mat.-Inf., Univ. Pitesti, **1**(1997), 95-104.
[22] Georgescu, C., Contributions to the study of some ordinary differential equations from economics, PhD Thesis, University of Pitesti, 2004. (Romanian).
[23] Arrowsmith, D. K., C. M. Place, An introduction to the dynamical systems, Cambridge University Press, 1990.
[24] Y. Kuznetsov, Elements of Applied Bifurcation Theory, Applied Mathematical Sciences, 112, Springer, New York, 1995.
[25] Chow, S. N., C. Li, D. Wang, Normal forms and bifurcation of planar vector fields, Cambridge University Press, 1994.

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