ON SOME STARLIKE FUNCTIONS WITH NEGATIVE COEFFICIENTS

Mugur Acu, Irina Dorca and Shigeyoshi Owa

Abstract. In this paper we study a class of starlike functions with negative coefficients defined by using a modified Sălăgean operator.

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1. Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc U,

$$A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$$

and $S = \{f \in A : f \text{ is univalent in } U\}$.

In [5] the subfamily T of S consisting of functions f of the form

$$f(z) = z - \sum_{j=2}^{\infty} a_j z^j,\ a_j \geq 0, j = 2, 3, ...,\ z \in U \tag{1}$$

was introduced.

The purpose of this paper is to define a subclass of starlike functions with negative coefficients and to give some properties of its by using a modified Sălăgean operator.

2. Preliminary results

Let D^n be the Sălăgean differential operator (see [4]) $D^n : A \to A$, $n \in \mathbb{N}$, defined as:

$$D^0 f(z) = f(z)$$
$$D^1 f(z) = Df(z) = zf'(z)$$
$$D^n f(z) = D(D^{n-1} f(z))$$

Remark 0.1 *If* $f \in T, f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, ...$, $z \in U$ *then* $D^n f(z) = z - \sum_{j=2}^{\infty} j^n a_j z^j$.

Definition 0.1 *[1] Let* $\beta, \lambda \in \mathbb{N}$, $\beta \geq 0$, $\lambda \geq 0$ *and* $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$. *We denote by* D_λ^β *the linear operator defined by*

$$D_\lambda^\beta : A \to A\,,$$

$$D_\lambda^\beta f(z) = z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^\beta \, a_j z^j\,.$$

Theorem 0.1 *[4] If* $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, ...$, $z \in U$ *then the next assertions are equivalent:*

(i) $\sum_{j=2}^{\infty} j a_j \leq 1$

(ii) $f \in T$

(iii) $f \in T^*$, *where* $T^* = T \bigcap S^*$ *and* S^* *is the well-known class of starlike functions.*

Definition 0.2 *[4] Let* $\alpha \in [0, 1)$ *and* $n \in \mathbb{N}$, *then*

$$S_n(\alpha) = \left\{ f \in A : Re\frac{D^{n+1} f(z)}{D^n f(z)} > \alpha, z \in U \right\}$$

is the set of n-starlike functions of order α.

Definition 0.3 *[3] Let $\alpha \in [0,1), \beta \in (0,1]$ and let $n \in \mathbb{N}$; we define the class $T_n(\alpha,\beta)$ of n-starlike functions of order α and type β with negative coefficients by*

$$T_n(\alpha,\beta) = \{f \in A : |J_n(f,\alpha;z)| < \beta, z \in U\},$$

where

$$J_n(f,\alpha;z) = \frac{\dfrac{D^{n+1}f(z)}{D^n f(z)} - 1}{\dfrac{D^{n+1}f(z)}{D^n f(z)} + 1 - 2\alpha}, \ z \in U$$

Remark 0.2 *The class $T_n(\alpha,1)$ is the class of n-starlike functions of order α with negative coefficients i.e. $T_n(\alpha,1) = T \bigcap S_n(\alpha)$.*

Theorem 0.2 *[3] Let $\alpha \in [0,1), \beta \in (0,1]$ and $n \in \mathbb{N}$. The function f of the form (1) is in $T_n(\alpha,\beta)$ if and only if*

$$\sum_{j=2}^{\infty} j^n[j-1+\beta(j+1-2\alpha)]a_j \leq 2\beta(1-\alpha)$$

Definition 0.4 *[3] Let $I_c : A \to A$ be the integral operator defined by $f = I_c(F)$, where $c \in (-1,\infty)$, $F \in A$ and*

$$f(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}F(t)dt. \tag{2}$$

We note if $F \in A$ is a function of the form (1), then

$$f(z) = I_cF(z) = z - \sum_{j=2}^{\infty}\frac{c+1}{c+j}a_j z^j. \tag{3}$$

Remark 0.3 *In [3] is showed that if $F \in T_n(\alpha,\beta)$ then $f = I_c(F) \in T_n(\alpha,\beta)$.*

Remark 0.4 *From Remark 0.2 and Theorem 0.2, for $f(z)$ of the form (1), we have $f \in T_n(\alpha,1) = T_n(\alpha)$ iff*

$$\sum_{j=2}^{\infty} j^n(j-\alpha)a_j \leq 1-\alpha, \ \text{where } \alpha \in [0,1)$$

We denote by $f * g$ the modified Hadamard product of two functions $f(z)$, $g(z) \in T$, $f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j$, $(a_j \geq 0,\ j = 2, 3, ...)$ and $g(z) = z - \sum\limits_{j=2}^{\infty} b_j z^j$, $(b_j \geq 0,$ j=2,3,...), is defined by

$$(f * g)(z) = z - \sum_{j=2}^{\infty} a_j b_j z^j .$$

An analytic function f is set to be subordinate to an analytic function g if $f(z) = g(w(z))$, $z \in U$, for some analytic function w with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$. We denote this subordination by $f \prec g$.

Theorem 0.3 *[2] If f and g are analytic in U with $f \prec g$, then for $\mu > 0$ and $z = re^{i\theta}$ $(0 < r < 1)$, we have*

$$\int_0^{2\pi} |f(z)|^{\mu} d\theta \leq \int_0^{2\pi} |g(z)|^{\mu} d\theta .$$

Following, we define a certain subclass of starlike functions with negative coefficients by using a modified Sălăgean operator and give several properties of it.

3. Main results

Definition 0.5 *Let $f \in T$, $f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, ...$, $z \in U$. We say that f is in the class $TL_{\beta}(\alpha)$ if:*

$$Re \frac{D_{\lambda}^{\beta+1} f(z)}{D_{\lambda}^{\beta} f(z)} > \alpha, \quad \alpha \in [0, 1), \quad \lambda \geq 0, \quad \beta \geq 0, \quad z \in U. \tag{4}$$

Remark 0.5 *We can see from Definition 0.5 that the class $TL_{\beta}(\alpha)$ contains all the functions $f \in T$, $f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j = 2, 3, \ldots$, $z \in U$, which satisfy the condition (4).*

Theorem 0.4 *Let $\alpha \in [0,1)$, $\lambda \geq 0$ and $\beta \geq 0$. The function $f \in T$ of the form (1) is in the class $TL_\beta(\alpha)$ iff*

$$\sum_{j=2}^{\infty}[(1+(j-1)\lambda)^{\beta}(1+(j-1)\lambda-\alpha)]a_j < 1-\alpha. \qquad (5)$$

Proof. Let $f \in TL_\beta(\alpha)$, with $\alpha \in [0,1)$, $\lambda \geq 0$ and $\beta \geq 0$. We have

$$Re\frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)} > \alpha.$$

If we take $z \in [0,1)$, $\beta \geq 0$, $\lambda \geq 0$, we have (see Definition 0.1):

$$\frac{1-\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta+1}a_j z^{j-1}}{1-\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}a_j z^{j-1}} > \alpha. \qquad (6)$$

From (5) we obtain:

$$1-\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta+1}a_j z^{j-1} > \alpha-\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}\alpha a_j z^{j-1},$$

$$\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}(1+(j-1)\lambda-\alpha)a_j z^{j-1} < 1-\alpha.$$

Letting $z \to 1^-$ along the real axis we have:

$$\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta}(1+(j-1)\lambda-\alpha)a_j < 1-\alpha.$$

Conversely, let take $f \in T$ for which the relation (4) hold.

The condition $Re\frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)} > \alpha$ is equivalent with

$$\alpha - Re\left(\frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)}-1\right) < 1\,. \qquad (7)$$

We have

$$\alpha - Re\left(\frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)} - 1\right) \leq \alpha + \left|\frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)} - 1\right|$$

$$= \alpha + \left|\frac{D_\lambda^{\beta+1}f(z) - D_\lambda^{\beta}f(z)}{D_\lambda^{\beta}f(z)}\right| = \alpha + \left|\frac{\sum_{j=2}^{\infty}(1+(j-1)\lambda)^\beta a_j[(j-1)\lambda]z^{j-1}}{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta a_j z^{j-1}}\right|$$

$$\leq \alpha + \frac{\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta a_j|1-j|\lambda|z|^{j-1}}{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta a_j|z|^{j-1}} = \alpha + \frac{\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta a_j(j-1)\lambda|z|^{j-1}}{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta a_j|z|^{j-1}}$$

$$< \alpha + \frac{\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta a_j(j-1)\lambda}{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta a_j} = \frac{\alpha + \sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta a_j[(j-1)\lambda - \alpha]}{1-\sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta a_j} < 1.$$

Thus

$$\alpha + \sum_{j=2}^{\infty}[1+(j-1)\lambda]^\beta a_j[(j-1)\lambda + 1 - \alpha] < 1,$$

which is the condition (4).

Remark 0.6 *Using the condition (4) it is easy to prove that* $TL_{\beta+1}(\alpha) \subseteq TL_\beta(\alpha)$*, where* $\beta \geq 0$*,* $\alpha \in [0,1)$ *and* $\lambda \geq 0$*.*

Theorem 0.5 *If* $f(z) = z - \sum_{j=2}^{\infty} a_j z^j \in TL_\beta(\alpha), (a_j \geq 0,\ j = 2,3,...)$*,* $g(z) = z - \sum_{j=2}^{\infty} b_j z^j \in TL_\beta(\alpha)$*,* $(b_j \geq 0,\ j = 2,3,...)$*,* $\alpha \in [0,1)$*,* $\lambda \geq 0$*,* $\beta \geq 0$*, then* $f(z) * g(z) \in TL_\beta(\alpha)$*.*

Proof. We have

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[(j-1)\lambda+1-\alpha]a_j < 1-\alpha$$

and

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[(j-1)\lambda+1-\alpha]b_j < 1-\alpha.$$

We know that $f(z)*g(z) = z-\sum\limits_{j=2}^{\infty}a_jb_jz^j$. From $g(z)\in T$, by using Theorem 0.1, we have $\sum\limits_{j=2}^{\infty}jb_j\leq 1$. We notice that $b_j<1,\quad j=2,3,\dots$.

Thus,

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[(j-1)\lambda+1-\alpha]a_jb_j < \sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[(j-1)\lambda+1-\alpha]a_j < 1-\alpha.$$

This means that $f(z)*g(z)\in TL_{\beta}(\alpha),\quad \beta\geq 0,\quad \alpha\in[0,1)$ and $\lambda\geq 0$.

Theorem 0.6 *If* $F(z) = z-\sum\limits_{j=2}^{\infty}a_jz^j \in TL_{\beta}(\alpha)$, *then* $f(z) = I_cF(z) \in TL_{\beta}(\alpha)$, *where* I_c *is the integral operator defined by (2).*

Proof. We have $f(z) = z-\sum\limits_{j=2}^{\infty}b_jz^j$, where $b_j = \dfrac{c+1}{c+j}a_j,\quad c\in(-1,\infty)$, j=2,3,... .

Thus $\quad b_j<a_j,\quad$ j=2,3 ... and using the condition (5) for $F(z)$ we obtain

$$\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[(j-1)\lambda+1-\alpha]b_j < \sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[(j-1)\lambda+1-\alpha]a_j < 1-\alpha.$$

This completes our proof.

We consider the integral operator $I_{c+\delta}: A\to A$, $0<u\leq 1$, $1\leq\delta<\infty$, $0<c<\infty$, defined by

$$f(z) = I_{c+\delta}(F(z)) = (c+\delta)\int_0^1 u^{c+\delta-2}F(uz)du. \tag{8}$$

Remark 0.7 *For* $F(z) = z + \sum_{j=2}^{\infty} a_j z^j$. *From (7) we obtain*

$$f(z) = z + \sum_{j=2}^{\infty} \frac{c+\delta}{c+j+\delta-1} a_j z^j.$$

Also, we notice that $0 < \frac{c+\delta}{c+j+\delta-1} < 1$, where $0 < c < \infty$, $j \geq 2$, $1 \leq \delta < \infty$.

Remark 0.8 *It is easy to prove that for* $F(z) \in T$ *and* $f(z) = I_{c+\delta}(F(z))$, *we have* $f(z) \in T$, *where* $I_{c+\delta}$ *is the integral operator defined by (7).*

Theorem 0.7 *Let* $F(z)$ *be in the class* $TL_\beta(\alpha)$, $F(z) = z - \sum_{j=2}^{\infty} a_j z^j$, $a_j \geq 0$, $j \geq 2$. *Then* $f(z) = I_{c+\delta}(F(z)) \in TL_\beta(\alpha)$, *where* $I_{c+\delta}$ *is the integral operator defined by (7) .*

Proof. From $F(z) \in TL_\beta(\alpha)$ we have (see Theorem 0.4)

$$\sum_{j=2}^{\infty} [(1+(j-1)\lambda)^\beta (1+(j-1)\lambda-\alpha)] a_j < 1-\alpha$$

where $\lambda \geq 0$, $\beta \geq 0$, $0 < c < \infty$ and $1 \leq \delta < \infty$. Let $f(z) = z - \sum_{j=2}^{\infty} b_j z^j$, where (see Remark 0.7)

$$b_j = \frac{c+\delta}{c+\delta+j-1} a_j \geq 0 \ \ \textit{and} \ \ 0 < \frac{c+\delta}{c+\delta+j-1} < 1.$$

From Remark 0.8 we obtain $f(z) \in T$. We have

$$[(1+(j-1)\lambda)^\beta (1+(j-1)\lambda-\alpha)] b_j < [(1+(j-1)\lambda)^\beta (1+(j-1)\lambda-\alpha)] a_j.$$

Thus,

$$\sum_{j=2}^{\infty} [(1+(j-1)\lambda)^\beta (1+(j-1)\lambda-\alpha)] b_j \leq \sum_{j=2}^{\infty} [(1+(j-1)\lambda)^\beta (1+(j-1)\lambda-\alpha)] a_j < 1-\alpha.$$

This completes our proof.

Theorem 0.8 *Let* $f_1(z) = z$ *and*

$$f_j(z) = z - \frac{1-\alpha}{(1+(j-1)\lambda)^\beta(1-\alpha+(j-1)\lambda)} z^j,\ j = 2, 3, ...$$

Then $f \in TL_\beta(\alpha)$ *iff it can be expressed in the form* $f(z) = \sum\limits_{j=1}^{\infty} \lambda_j f_j(z)$*, where* $\lambda_j \geq 0$ *and* $\sum\limits_{j=1}^{\infty} \lambda_j = 1$.

Proof. Let $f(z) = \sum\limits_{j=1}^{\infty} \lambda_j f_j(z)$, $\lambda_j \geq 0$, j=1,2, ... , with $\sum\limits_{j=1}^{\infty} \lambda_j = 1$. We obtain

$$f(z) = \sum_{j=1}^{\infty} \lambda_j f_j(z) = \sum_{j=1}^{\infty} \lambda_j \left(z - \frac{1-\alpha}{[1+(j-1)\lambda]^\beta[1-\alpha+(j-1)\lambda]} z^j \right)$$

$$= \sum_{j=1}^{\infty} \lambda_j z - \sum_{j=1}^{\infty} \lambda_j \frac{1-\alpha}{[1+(j-1)\lambda]^\beta[1-\alpha+(j-1)\lambda]} z^j$$

$$= z - \sum_{j=2}^{\infty} \lambda_j \frac{1-\alpha}{[1+(j-1)\lambda]^\beta[1-\alpha+(j-1)\lambda]} z^j.$$

We have

$$\sum_{j=2}^{\infty} [1+(j-1)\lambda]^\beta[1-\alpha+(j-1)\lambda]\lambda_j \frac{1-\alpha}{[1+(j-1)\lambda]^\beta[1-\alpha+(j-1)\lambda]}$$

$$= (1-\alpha)\sum_{j=2}^{\infty} \lambda_j = (1-\alpha)(\sum_{j=1}^{\infty} \lambda_j - \lambda_1) < 1-\alpha$$

which is the condition (5) for $f(z) = \sum\limits_{j=1}^{\infty} \lambda_j f_j(z)$. Thus $f(z) \in TL_\beta(\alpha)$.

Conversely, we suppose that $f(z) \in TL_\beta(\alpha), f(z) = z - \sum\limits_{j=2}^{\infty} a_j z^j, a_j \geq 0$ and we take $\lambda_j = \dfrac{[1+(j-1)\lambda]^\beta[1-\alpha+(j-1)\lambda]}{1-\alpha} a_j \geq 0$, j=2,3, ... , with $\lambda_1 = 1 - \sum\limits_{j=2}^{\infty} \lambda_j$.

Using the condition (5), we obtain

$$\sum_{j=2}^{\infty}\lambda_j=\frac{1}{1-\alpha}\sum_{j=2}^{\infty}[1+(j-1)\lambda]^{\beta}[1-\alpha+(j-1)\lambda]a_j<\frac{1}{1-\alpha}(1-\alpha)=1.$$

Then $f(z)=\sum\limits_{j=1}^{\infty}\lambda_j f_j$, where $\lambda_j\geq 0$, j=1,2, ... and $\sum\limits_{j=1}^{\infty}\lambda_j=1$. This completes our proof.

Corolary 0.1 *The extreme points of $TL_{\beta}(\alpha)$ are $f_1(z)=z$ and*

$$f_j(z)=z-\frac{1-\alpha}{(1+(j-1)\lambda)^{\beta}(1-\alpha+(j-1)\lambda)}z^j,\ j=2,3,...\ .$$

Theorem 0.9 *Let $f(z)\in TL_{\beta}(\alpha)$, $\beta\geq 0$, $\lambda\geq 0$, $\alpha\in[0,1)$, $\mu>0$ and $f_j(z)=z-\dfrac{1-\alpha}{[(1+(j-1)\lambda)^{\beta}(1+(j-1)\lambda-\alpha)]}z^j$, j=2,3... . Then for $z=re^{i\theta}(0<r<1)$, we have*

$$\int_0^{2\pi}|f(z)|^{\mu}d\theta\leq\int_0^{2\pi}|f_j(re^{i\theta})|^{\mu}d\theta.$$

Proof We have to show that

$$\int_0^{2\pi}\left|1-\sum_{j=2}^{\infty}a_jz^{j-1}\right|^{\mu}d\theta\leq\int_0^{2\pi}\left|1-\frac{1-\alpha}{[(1+(j-1)\lambda)^{\beta}(1+(j-1)\lambda-\alpha)]}z^{j-1}\right|^{\mu}d\theta.$$

From Theorem 0.3 we deduce that it is sufficiently to prove that

$$1-\sum_{j=2}^{\infty}a_jz^{j-1}\prec 1-\frac{1-\alpha}{[(1+(j-1)\lambda)^{\beta}(1+(j-1)\lambda-\alpha)]}z^{j-1}.$$

Considering

$$1-\sum_{j=2}^{\infty}a_jz^{j-1}=1-\frac{1-\alpha}{[(1+(j-1)\lambda)^{\beta}(1+(j-1)\lambda-\alpha)]}w(z)^{j-1}$$

we find that

$$\{w(z)\}^{j-1} = \frac{[(1+(j-1)\lambda)^{\beta}(1+(j-1)\lambda-\alpha)]}{1-\alpha}\sum_{j=2}^{\infty} a_j z^{j-1}$$

which readily yields $w(0) = 0$.

By using the condition (5), we can write

$$1-\alpha > [(1+\lambda)^{\beta}(1+\lambda-\alpha)]a_2 + [(1+2\lambda)^{\beta}(1+2\lambda-\alpha)]a_3 + ...$$

$$+[(1+(j-1)\lambda)^{\beta}(1+(j-1)\lambda-\alpha)]a_j + [(1+j\lambda)^{\beta}(1+j\lambda-\alpha)]a_{j+1} + ...$$

$$\geq \sum_{i=2}^{\infty}[(1+(j-1)\lambda)^{\beta}(1+(j-1)\lambda-\alpha)]a_i$$

$$= [(1+(j-1)\lambda)^{\beta}(1+(j-1)\lambda-\alpha)]\sum_{i=2}^{\infty} a_i\,.$$

Thus

$$\sum_{j=2}^{\infty} a_j < \frac{1-\alpha}{[(1+(j-1)\lambda)^{\beta}(1+(j-1)\lambda-\alpha)]}$$

and

$$|\{w(z)\}|^{j-1} = \left|\frac{[(1+(j-1)\lambda)^{\beta}(1+(j-1)\lambda-\alpha)]}{1-\alpha}\sum_{j=2}^{\infty} a_j z^{j-1}\right|$$

$$\leq \frac{[(1+(j-1)\lambda)^{\beta}(1+(j-1)\lambda-\alpha)]}{1-\alpha}\sum_{j=2}^{\infty} a_j\,|z|^{j-1} < |z| < 1.$$

This completes our theorem's proof.

Remark 0.9 *We notice that, in the particular case, obtained for $\lambda = 1$ and $\beta \in \mathbb{N}$, we find similarly results for the class $T_n(\alpha)$ of the n-starlike functions of order α with negative coefficients (inclusive the necessary and sufficiently condition presented in Remark 0.4).*

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Mugur Acu
Department of Mathematics
University "Lucian Blaga" of Sibiu
Str. Dr. I. Raţiu, No 5–7, Sibiu, România.
email:*acu_mugur@yahoo.com*

Irina Dorca
Department of Mathematics
University of Piteşti
Str. Târgu din Vale No 1, Piteşti, România.
email:*irina.dorca@gmail.com*

Shigeyoshi Owa
Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502, Japan.
email:*owa@math.kindai.ac.jp*