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ON A CERTAIN CLASS OF HARMONIC UNIVALENT FUNCTIONS

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Abstract. The purpose of the present paper is to study a new class of univalent harmonic functions on unit disc satisfying the condition

$$\sum_{k=2}^{\infty} k^n \left\{(k-1)+\beta(k+1-2\alpha)\right\}(|a_k|+|b_k|) \leq 2\beta(1-\alpha)(1-|b_1|)$$

where $n \in N_0,\ 0 \leq \alpha < 1$ and $0 < \beta \leq 1$.

Sharp coefficient relation and distortion theorems are given for these functions. Results concerning the convolutions of functions satisfying the above inequalities with univalent, harmonic and convex functions in the unit disc and harmonic functions having positive real part are obtained.

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1. Introduction

Let U denote the open unit disc and S_H denote the class of all complex valued harmonic, orientation preserving, univalent functions f in U normalized, by $f(0) = f_z(0) - 1 = 0$ Each $f \in S_H$ can be expressed as $f = h + \overline{g}$, where h and g belong to the linear space $H(U)$ of all analytic functions in U.

Firstly, Clunie and Sheil-Small [3] studied S_H together with some geometric subclasses of S_H. They prove that although S_H is not compact, it is normal with respect to the topology of uniform convergence on compact subsets of U. Meanwhile the subclasses S_H^0 of S_H consisting of the functions having the property $f_{\overline{z}}(0) = 0$ is compact.

In this article we concentrate on a specific subclasses of univalent harmonic mappings.

For more basic results on the subject one may refer to the Duren [5], Ponnusamy and Rasila [8], [9].

2. The Class $S_H(n,\alpha,\beta)$

Let $U_r = \{z : |z| < r,\ 0 < r \leq 1\}$ and $U_1 = U$.

A harmonic, complex-valued, sense-preserving univalent mapping f defined on U can be written as

$$f(z) = h(z) + \overline{g(z)} \tag{1}$$

where

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

$$g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1 \tag{2}$$

are analytic in U.

Denote by $S_H(n;\alpha;\beta)$ the class of all functions of the form (1) that satisfy the condition

$$\sum_{k=2}^{\infty} k^n\{(k-1)+\beta(k+1-2\alpha)\}(|a_k|+|b_k|) \leq 2\beta(1-\alpha)(1-|b_1|), \tag{3}$$

where $n \in N_0,\ \beta \in (0,1],\ \alpha \in [0,1)$ and $0 \leq |b_1| < 1$.

The class $S_H(n,\alpha;\beta)$ with $b_1 = 0$ will be denoted by $S_H^0(n,\alpha;\beta)$.

We note that by specializing the parameter in $S_H(n,\alpha;\beta)$ we obtain the following known subclasses of S_H studied earlier by various authors.

1. $S_H(0,\alpha;1) = HS(\alpha)$ and $S_H(1,\alpha;1) = HK(\alpha)$ were studied by Ozturk and Yalcin [7] and see also [6].
2. $S_H(0,0;1) = HS$ and $S_H(1,0;1) = HK$ were studied by Avci and Zlotkiewicz [2].

If h, g, H, G are of the form (1) and if

$$f(z) = h(z) + \overline{g(z)} \text{ and } F(z) = H(z) + \overline{G(z)},$$

then the convolution of f and F is defined to be the function:

$$(f * F)(z) = z + \sum_{k=2}^{\infty} a_k A_k z^k + \overline{\sum_{k=1}^{\infty} b_k B_k z^k},$$

while the Integral convolution is defined by:

$$(f \diamond F)(z) = z + \sum_{k=2}^{\infty} \frac{a_k A_k z^k}{k} + \sum_{k=1}^{\infty} \frac{b_k B_k z^k}{k}.$$

The δ-neighborhood of f is the set

$$N_\delta(f) = \left\{ F : \sum_{k=2}^{\infty} k(|a_k - A_k| + |b_k - B_k|) + |b_1 - B_1| \le \delta \right\}$$

(see [11], [5]).

In this case, let us define the generalized δ-neighborhood of f to be the set;

$$N(f) = \left\{ F : \sum_{k=2}^{\infty} (k-\alpha)(|a_k - A_k| + |b_k - B_k|) + (1-\alpha)|b_1 - B_1| \le (1-\alpha)\delta \right\}.$$

3. Main Results

First, we show that the class $S_H(n, \alpha, \beta)$ is univalent and sense-preserving in U.

Theorem 1. *The class $S_H(n, \alpha, \beta)$ consist of univalent sense preserving harmonic mappings.*

Proof. If $z_1 \neq z_2$, then

$$\begin{aligned}
\left| \frac{f(z_1) - f(z_2)}{h(z_1) - h(z_2)} \right| &\ge 1 - \left| \frac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)} \right| \\
&= 1 - \left| \frac{\sum_{k=1}^{\infty} b_k (z_1^k - z_2^k)}{(z_1 - z_2) + \sum_{k=2}^{\infty} a_k (z_1^k - z_2^k)} \right| \\
&> 1 - \frac{\sum_{k=1}^{\infty} k|b_k|}{1 - \sum_{k=2}^{\infty} k|a_k|} \\
&\ge 1 - \frac{\sum_{k=1}^{\infty} \frac{k^n[(k-1) + \beta(k+1-2\alpha)]}{2\beta(1-\alpha)} |b_k|}{1 - \sum_{k=2}^{\infty} \frac{k^n[(k-1) + \beta(k+1-2\alpha)]}{2\beta(1-\alpha)} |a_k|} \\
&\ge 0,
\end{aligned}$$

which proves univalence.

Note that f is sense preserving in U. This is because

$$|h'(z)| \geq 1 - \sum_{k=2}^{\infty} k|a_k|\ |z|^{k-1} > 1 - \sum_{k=2}^{\infty} k|a_k|$$
$$\geq 1 - \sum_{k=2}^{\infty} \frac{k^n[(k-1)+\beta(k+1-2\alpha)]}{2\beta(1-\alpha)}|a_k| \geq \sum_{k=1}^{\infty} \frac{k^n[(k-1)+\beta(k+1-2\alpha)]}{2\beta(1-\alpha)}|b_k|$$
$$\geq \sum_{k=1}^{\infty} k|b_k| > \sum_{k=1}^{\infty} k|b_k|\ |z|^{k-1} \geq |g'(z)|.$$

The following theorem gives the distortion bounds for functions in $S_H(n,\alpha,\beta)$ which yields a covering result for this class.

Theorem 2. *If* $f \in S_H(n,\alpha,\beta)$, *then*

$$|f(z)| \leq (1+|b_1|)|z| + \frac{2\beta(1-\alpha)}{2^n[1+3\beta-2\alpha\beta]}(1-|b_1|)|z|^2$$

and

$$|f(z)| \geq (1-|b_1|)\left(|z| - \frac{2\beta(1-\alpha)}{2^n(1+3\beta-2\alpha\beta)}|z|^2\right)$$

Proof. We only prove te right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f \in S_H(n,\alpha,\beta)$

$$|f(z)| \leq (1+|b_1|)|z| + \sum_{k=2}^{\infty}(|a_k|+|b_k|)|z|^k$$

$$|f(z)| \leq |z|(1+|b_1|) + |z|^2 \sum_{k=2}^{\infty}(|a_k|+|b_k|)$$
$$\leq |z|(1+|b_1|) + |z|^2 \frac{2\beta(1-\alpha)}{2^n(1+3\beta-2\alpha\beta)} \sum_{k=2}^{\infty} \frac{k^n[(k-1)+\beta(k+1-2\alpha)]}{2\beta(1-\alpha)}(|a_k|+|b_k|)$$
$$\leq |z|(1+|b_1|) + |z|^2 \frac{2\beta(1-\alpha)}{2^n(1+3\beta-2\alpha\beta)}(1-|b_1|)$$

The results are sharp for the functions

$$f_\theta(z) = z + |b_1|e^{i\theta}\overline{z} + \frac{2\beta(1-\alpha)}{2^n(1+3\beta-2\alpha\beta)}(1-|b_1|)z^2$$

and
$$f_\theta(z) = z + |b_1|e^{i\theta}\overline{z} + \frac{2\beta(1-\alpha)}{2^n(1+3\beta-2\alpha\beta)}(1-|b_1|)\overline{z}^2.$$

The following covering result follows from the left inequality in Theorem 3.3.

Corollary 3. *Let f of the form (2.2) be so that $f \in S_H(n,\alpha,\beta)$ then*

$$\left\{ w : |w| < \frac{2^n(1+3\beta-2\alpha\beta) - 2\beta(1-\alpha)}{2^n(1+3\beta-2\alpha\beta)} - \frac{2\beta(1-\alpha) - 2^n(1+3\beta-2\alpha\beta)}{2^n(1+3\beta-2\alpha\beta)}|b_1| \right\} \subset f(U)$$

Next we determine the extreme points of closed convex hulls of $S_H^0(n\alpha,\beta)$

Theorem 4. *The extreme points of $S_H^0(n,\alpha,\beta)$ are only the functions of the form $z + a_k z^k$ or $z + \overline{b_l z^l}$ with*

$$|a_k| = \frac{2\beta(1-\alpha)}{k^n[(k-1)+\beta(k+1-2\alpha)]},\ |b_l| = \frac{2\beta(1-\alpha)}{l^n[(l-1)+\beta(l+1-2\alpha)]}$$

$$0 \le \alpha < 1,\ 0 < \beta \le 1$$

Proof. The proof of above theorem is similar to the corresponding theorem of [4]. Therefore we omit the details involved.

Remark 1. *For $n = 0,\ \beta = 1,\ n = 1,\ \beta = 1$ the above results in [7].*

Let K_H^0 denote the class of harmonic univalent functions of the form (1) *with $b_1 = 0$ that map U onto convex domains. It is Known [3, theorem 5.10] that the sharp inequalities $|A_k| \le \frac{k+1}{2},\ |B_k| \le \frac{k-1}{2}$.*

Theorem 5. *Suppose that $F(z) = z + \sum_{k=2}^{\infty} A_k z^k + \overline{B_k z^k}$ belongs to K_H^0. Then $f \in S_H^0(n,\alpha;\beta)$ then $f * F \in S_H^0(n-1,\alpha;\beta)$ and $f \diamond F \in HS^0(n,\alpha;\beta)\quad n \in N$.*

Proof. Since $f \in S_H^0(n,\alpha;\beta)$ then

$$\sum_{k=2}^{\infty} k^n \left\{(k-1)+\beta(k+1-2\alpha)\right\}(|a_k|+|b_k|) \le 2\beta(1-\alpha)(1-|b_1|). \tag{4}$$

Now using (3.1),

$$\begin{aligned}
&\sum_{k=2}^{\infty} k^{n-1}[(k-1)+\beta(k+1-2\alpha)](|a_k A_k|+|b_k B_k|)\\
=&\sum_{k=2}^{\infty} k^{n}[(k-1)+\beta(k+1-2\alpha)](|a_k|\left|\frac{A_k}{k}\right|+|b_k|\left|\frac{B_k}{k}\right|)\\
\leq&\sum_{k=2}^{\infty} k^{n}[(k-1)+\beta(k+1-2\alpha)](|a_k|+|b_k|)\\
\leq& 2\beta(1-\alpha)\\
\Rightarrow&\ f * F \in S_H^0(n-1,\alpha,\beta).
\end{aligned}$$

Similarly, It can be easily seen that $f \diamond F \in S_H^0(n,\alpha,\beta)$ if $f \in S_H^0(n,\alpha,\beta)$. Let P_H^0 denote the class of functions F complex and harmoinic in U, $f = h + \overline{g}$ such that $\mathrm{Re} f(z) > 0,\ z \in U$ and

$$H(z) = 1 + \sum_{k=1}^{\infty} A_k z^k,\ G(z) = \sum_{k=2}^{\infty} B_k z^k.$$

It is Known [4, Theorem 3] that the sharp inequalities $|A_k| \leq k+1,\ |B_k| \leq k-1$ are true.

Theorem 6. *Suppose that*

$$F(z) = 1 + \sum_{k=1}^{\infty} (A_k z^k + \overline{B_k z^k})$$

belongs to P_H^0. *Then* $f \in S_H^0(n,\alpha,\beta)$ *and for* $3/2 \leq |A_1| \leq 2,\ 1/A_1\ f * F \in S_H^0(n-1,\alpha,\beta)$ *and* $1/A_1\ f \diamond F \in S_H^0(n,\alpha,\beta)$

Proof. The proof of this theorem is much akin that of Theorem 3.5, so we omit the details.

Theorem 7. *Let* $f(z) = z + \overline{b_1 z} + \sum_{k=2}^{\infty}(a_k z^k + \overline{b_k z^k})$ *is a member of* $S_H(n,\alpha,\beta)$. *If* $\delta \leq (1-|b_1|)\left\{1 - \frac{\beta}{2^{n-1}}\right\}$ *then* $N_\delta(f) \subset S_H(\alpha)$, *for* $n > 1$.

Proof. Let $f \in S_H^0(n,\alpha,\beta)$ and $F(z) = z+\overline{B_1 z}+\sum_{k=2}^{\infty}(A_k z^k+\overline{B_k z^k})$ belong to $N_\delta(f)$. We have

$$(1-\alpha)|B_1| + \sum_{k=2}^{\infty}(k-\alpha)(|A_k|+|B_k|)$$
$$\leq (1-\alpha)|B_1-b_1| + (1-\alpha)|b_1| + \sum_{k=2}^{\infty}(k-\alpha)(|A_k-a_k|+|B_k-b_k|) + \sum_{k=2}^{\infty}(k-\alpha)(|a_k|+|b_k|)$$
$$\leq (1-\alpha)\delta + (1-\alpha)|b_1| + \frac{1}{2^n}\sum_{k=2}^{\infty}k^n[(k-1)+\beta(k+1-2\alpha)](|a_k|+|b_k|)$$
$$\leq (1-\alpha)\delta + (1-\alpha)|b_1| + 1/2^n 2\beta(1-\alpha)(1-|b_1|)$$
$$= (1-\alpha)\delta + (1-\alpha)\left[|b_1| + \frac{\beta(1-|b_1|)}{2^{n-1}}\right]$$
$$\leq 1-\alpha.$$

Hence for $\delta \leq_(1-|b_1|)\left\{1-\frac{\beta}{2^{n-1}}\right\}$.

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