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SOME PROPERTIES OF SUBCLASSES OF P - VALENT FUNCTIONS DEFINED BY DIFFERENTIAL SUBORDINATION

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Abstract. In this paper, we introduce and study some properties of subclasses of p- valent functions which are defined by differential subordination. Coefficient inequalities, some properties of neighborhoods, distortion and covering theorems, radius of starlikeness and convexity for these subclasses are given.

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1. Introduction

Let $\mathcal{T}_p(j)$ be the class of analytic functions f of the form

$$f(z)=z^p-\sum_{k=j+p}^{\infty}a_kz^k, \quad (a_k\geq 0,\ j,p\in\mathbb{N}=\{1,2,...\}), \tag{1}$$

defined in the open unit disc $\mathcal{U}=\{z\in\mathbb{C}\,:\,|z|<\ 1\}$.
Let Ω be the class of functions ω analytic in $\mathcal{U}$ such that $\omega(0)=0,\ |\omega(z)|<1$.
For any two functions f and g in $\mathcal{T}_p(j)$, f is said to be subordinate to g denoted $f\prec g$, if there exists an analytic functions $\omega\in\Omega$ such that $f(z)=g(\omega(z))$ [3].

Definition 1. *Let $n\in\mathbb{N}$ and $\lambda\in\mathbb{R},\ \lambda\geq 0.$ We define the operator $D_\lambda^{n,p}:\mathcal{T}_p(j)\to\mathcal{T}_p(j)$ is defined as $D_\lambda^{0,p}f(z)=f(z)$,*
$D_\lambda^{1,p}f(z)=(1-\lambda)f(z)+\frac{\lambda}{p}zf'(z)=D_\lambda f(z)$ *and* $D_\lambda^{n,p}f(z)=D_\lambda\left(D_\lambda^{n-1,p}f(z)\right)$.

Further, if $f(z)=z^p-\sum_{k=j+p}^{\infty}a_kz^k$, then we have,

$$D_\lambda^{n,p}f(z)=z^p-\sum_{k=j+p}^{\infty}\left[1+\left(\frac{k}{p}-1\right)\lambda\right]^n a_kz^k. \tag{2}$$

Remark 1. *It is easy to observe that for* $p = 1$, $\jmath = 1$ *we get the Al - Oboudi operator [1] and for* $p = 1$, $\jmath = 1$, $\lambda = 1$, *the Sălăgean's differential operator [7].*

For any function $f \in \mathcal{T}(j)$ and $\delta \geq 0$, the (j, δ) - neighborhood of f is defined as,

$$\mathcal{N}_{j,\delta}(f) = \left\{ g(z) = z^p - \sum_{k=j+p}^{\infty} b_k z^k \in \mathcal{T}_p(j) : \quad \sum_{k=j+p}^{\infty} k|a_k - b_k| \leq \delta \right\}. \tag{3}$$

In particular, for the function $e(z) = z^p$, we see that,

$$\mathcal{N}_{j,\delta}(e) = \left\{ g(z) = z^p - \sum_{k=j+p}^{\infty} b_k z^k \in \mathcal{T}_p(j) : \quad \sum_{k=j+p}^{\infty} k|b_k| \leq \delta \right\}. \tag{4}$$

The concept of neighborhoods was first introduced by Goodman [4] and then generalized by Ruscheweyh [5] .

Definition 2. *A function* $f \in \mathcal{T}_p(j)$ *is said to be in the class* $\mathcal{T}_j(n, m, p, A, B, \lambda)$ *if*

$$\frac{D_\lambda^{n+m,p} f(z)}{D_\lambda^{n,p} f(z)} \prec \frac{1 + Az}{1 + Bz}, \quad z \in \mathcal{U}, \tag{5}$$

where, $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, $\lambda \geq 1$ *and* $-1 \leq B < A \leq 1$.

We observe that $\mathcal{T}_j(n, m, 1, 1 - 2\alpha, -1, 1) \equiv \mathcal{T}_j(n, m, \alpha)$ [2],
$\mathcal{T}_j(0, 1, 1, 1 - 2\alpha, -1, 1) \equiv \mathcal{S}_j^\star(\alpha)$ [6], the class of starlike functions of order α and
$\mathcal{T}_j(1, 1, 1, 1 - 2\alpha, -1, 1) \equiv \mathcal{C}_j(\alpha)$ [6], the class of convex functions of order α.

2. Neighborhoods for the class $\mathcal{T}_j(n, m, p, A, B, \lambda)$

Theorem 1. *A function* $f \in \mathcal{T}_p(j)$ *belongs to the class* $\mathcal{T}_j(n, m, p, A, B, \lambda)$ *if and only if*

$$\sum_{k=j+p}^{\infty} [1 + (\frac{k}{p} - 1)\lambda]^n \left\{ (1 - B)[1 + (\frac{k}{p} - 1)\lambda]^m - (1 - A) \right\} a_k \leq A - B \tag{6}$$

for $n \in \mathbb{N}_0$, $m \in \mathbb{N}$, $\lambda \geq 1$ *and* $-1 \leq B < A \leq 1$.

Proof. Let $f \in \mathcal{T}_j(n, m, p, A, B, \lambda)$. Then,

$$\frac{D_\lambda^{n+m,p} f(z)}{D_\lambda^{n,p} f(z)} \prec \frac{1+Az}{1+Bz}, \quad z \in \mathcal{U}. \tag{7}$$

Therefore, there exists an analytic function ω such that

$$\omega(z) = \frac{D_\lambda^{n,p} f(z) - D_\lambda^{n+m,p} f(z)}{BD_\lambda^{n+m,p} f(z) - AD_\lambda^{n,p} f(z)} \tag{8}$$

Hence,

$$|\omega(z)| = \left| \frac{D_\lambda^{n,p} f(z) - D_\lambda^{n+m,p} f(z)}{BD_\lambda^{n+m,p} f(z) - AD_\lambda^{n,p} f(z)} \right|$$

$$= \left| \frac{\sum_{k=j+p}^{\infty} [1+(\frac{k}{p}-1)\lambda]^n \left\{ [1+(\frac{k}{p}-1)\lambda]^m - 1 \right\} a_k z^k}{(A-B)z^p + \sum_{k=j+p}^{\infty} [1+(\frac{k}{p}-1)\lambda]^n \left\{ B[1+(\frac{k}{p}-1)\lambda]^m - A \right\} a_k z^k} \right| < 1.$$

Thus,

$$\Re \left\{ \frac{\sum_{k=j+p}^{\infty} [1+(\frac{k}{p}-1)\lambda]^n \left\{ [1+(\frac{k}{p}-1)\lambda]^m - 1 \right\} a_k z^k}{(A-B)z^p + \sum_{k=j+p}^{\infty} [1+(\frac{k}{p}-1)\lambda]^n \left\{ B[1+(\frac{k}{p}-1)\lambda]^m - A \right\} a_k z^k} \right\} < 1. \tag{9}$$

Taking $|z| = r$, for sufficiently small r with $0 < r < 1$, the denominator of (9) is positive and so it is positive for all r with $0 < r < 1$, since $\omega(z)$ is analytic for $|z| < 1$. Then, the inequality (9) yields

$$\sum_{k=j+p}^{\infty} [1+(\frac{k}{p}-1)\lambda]^n \left\{ [1+(\frac{k}{p}-1)\lambda]^m - 1 \right\} a_k r^k$$

$$< (A-B)r^p + B \sum_{k=j+p}^{\infty} [1+(\frac{k}{p}-1)\lambda]^{n+m} a_k r^k - A \sum_{k=j+p}^{\infty} [1+(\frac{k}{p}-1)\lambda]^n a_k r^k.$$

Equivalently,

$$\sum_{k=j+p}^{\infty} [1+(\frac{k}{p}-1)\lambda]^n \left\{ (1-B)[1+(\frac{k}{p}-1)\lambda]^m - (1-A) \right\} a_k r^k \leq (A-B)r^p$$

and (6) follows upon letting $r \to 1$.
Conversely, for $|z| = r,\ 0 < r < 1$, we have $r^k < r^p$. That is,

$$\sum_{k=j+p}^{\infty} [1 + (\frac{k}{p} - 1)\lambda]^n \left\{(1-B)[1 + (\frac{k}{p} - 1)\lambda]^m - (1-A)\right\} a_k r^k$$

$$\leq \sum_{k=j+p}^{\infty} [1 + (\frac{k}{p} - 1)\lambda]^n \left\{(1-B)[1 + (\frac{k}{p} - 1)\lambda]^m - (1-A)\right\} a_k r^p \leq (A-B)r^p.$$

From (6), we have

$$\left|\sum_{k=j+p}^{\infty} [1 + (\frac{k}{p} - 1)\lambda]^n \left\{[1 + (\frac{k}{p} - 1)\lambda]^m - 1\right\} a_k z^k\right|$$

$$\leq \sum_{k=j+p}^{\infty} [1 + (\frac{k}{p} - 1)\lambda]^n \left\{[1 + (\frac{k}{p} - 1)\lambda]^m - 1\right\} a_k r^k$$

$$< (A-B)r^p + \sum_{k=j+p}^{\infty} \left\{B[1 + (\frac{k}{p} - 1)\lambda]^m - A\right\} [1 + (\frac{k}{p} - 1)\lambda]^n a_k r^k$$

$$< \left|(A-B)z^p + \sum_{k=j+p}^{\infty} \left\{B[1 + (\frac{k}{p} - 1)\lambda]^m - A\right\} [1 + (\frac{k}{p} - 1)\lambda]^n a_k z^k\right|.$$

This proves that

$$\frac{D_\lambda^{n+m,p} f(z)}{D_\lambda^{n,p} f(z)} \prec \frac{1+Az}{1+Bz}, \quad z \in \mathcal{U}$$

and hence $f \in \mathcal{T}_j(n, m, p, A, B, \lambda)$.

Theorem 2. *If*

$$\delta = \frac{(A-B)}{(1 + \frac{j}{p}\lambda)^{n-1}\left[(1-B)(1 + \frac{j}{p}\lambda)^m - (1-A)\right]}, \tag{10}$$

then $\mathcal{T}_j(n, m, p, A, B, \lambda) \subset N_{j,\delta}(e)$.

Proof. It follows from (6), that if $f \in \mathcal{T}_j(n, m, p, A, B, \lambda)$, then

$$(1 + \frac{j}{p}\lambda)^{n-1}\left[(1-B)(1 + \frac{j}{p}\lambda)^m - (1-A)\right] \sum_{k=j+p}^{\infty} k a_k \leq (A-B), \tag{11}$$

which implies,

$$\sum_{k=j+p}^{\infty} ka_k \leq \frac{(A-B)}{(1+\frac{j}{p}\lambda)^{n-1}\left[(1-B)(1+\frac{j}{p}\lambda)^m-(1-A)\right]} = \delta. \tag{12}$$

Using (4), we get the result.

3. Neighborhoods for the classes $\mathcal{R}_j(n,p,A,B,\lambda)$ and $\mathcal{P}_j(n,p,A,B,\lambda)$

Definition 3. *A function $f \in \mathcal{T}_p(j)$ is said to be in the class $\mathcal{R}_j(n,p,A,B,\lambda)$ if it satisfies*

$$(D_\lambda^{n,p} f(z))' \prec \frac{1+Az}{1+Bz}, \quad (z \in \mathcal{U}), \tag{13}$$

where $-1 \leq B < A \leq 1$, $\lambda \geq 1$ and $n \in \mathcal{N}_0$.

Definition 4. *A function $f \in \mathcal{T}_p(j)$ is said to be in the class $\mathcal{P}_j(n,p,A,B,\lambda)$ if it satisfies*

$$\frac{D_\lambda^{n,p} f(z)}{z} \prec \frac{1+Az}{1+Bz}, \quad (z \in \mathcal{U}), \tag{14}$$

where $-1 \leq B < A \leq 1$, $\lambda \geq 1$ and $n \in \mathcal{N}_0$.

Lemma 3. *A function $f \in \mathcal{T}_p(j)$ belongs to the class $\mathcal{R}_j(n,p,A,B,\lambda)$ if and only if*

$$\sum_{k=j+p}^{\infty} (1-B)[1+(\frac{k}{p}-1)\lambda]^{n+1} a_k \leq A-B. \tag{15}$$

Lemma 4. *A function $f \in \mathcal{T}_p(j)$ belongs to the class $\mathcal{P}_j(n,p,A,B,\lambda)$ if and only if*

$$\sum_{k=j+p}^{\infty} (1-B)[1+(\frac{k}{p}-1)\lambda]^{n} a_k \leq A-B. \tag{16}$$

Theorem 5. *$\mathcal{R}_j(n,p,A,B,\lambda) \subset \mathcal{N}_{j,\delta}(e)$, where*

$$\delta = \frac{(A-B)}{[1+\frac{j}{p}\lambda]^n(1-B)}. \tag{17}$$

Proof. If $f \in \mathcal{R}_j(n,p,A,B,\lambda)$, we have,

$$[1+\frac{j}{p}\lambda]^n \sum_{k=j+p}^{\infty} (1-B) k a_k \leq A-B, \tag{18}$$

which implies,

$$\sum_{k=j+p}^{\infty} ka_k \leq \frac{(A-B)}{[1+\frac{j}{p}\lambda]^n(1-B)} = \delta.$$

Theorem 6. *$\mathcal{P}_j(n,p,A,B,\lambda) \subset \mathcal{N}_{j,\delta}(e)$, where*

$$\delta = \frac{(A-B)}{[1+\frac{j}{p}\lambda]^{n-1}(1-B)}. \tag{19}$$

Proof. If $f \in \mathcal{P}_j(n,p,A,B,\lambda)$, we have,

$$[1+\frac{j}{p}\lambda]^{n-1} \sum_{k=j+p}^{\infty} (1-B)ka_k \leq A-B, \tag{20}$$

which implies,

$$\sum_{k=j+p}^{\infty} ka_k \leq \frac{(A-B)}{[1+\frac{j}{p}\lambda]^{n-1}(1-B)} = \delta.$$

Thus, in view of the condition (4), we get the required result of Theorem 6.

4. Neighborhood of the class $\mathcal{K}_j^{\lambda}(n,m,p,A,B,C,D)$

Definition 5. *A function $f \in \mathcal{T}_p(j)$ is said to be in the class $\mathcal{K}_j^{\lambda}(n,m,p,A,B,C,D)$ if it satisfies*

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{A-B}{1-B}, \quad z \in \mathcal{U}, \tag{21}$$

for $-1 \leq B < A \leq 1,\ -1 \leq D < C \leq 1,\ \lambda \geq 1$ and $g \in \mathcal{T}_j(n,m,p,C,D,\lambda)$.

Theorem 7. *For $g \in \mathcal{T}_j(n,m,p,C,D,\lambda)$ we have $\mathcal{N}_{j,\delta}(g) \subset \mathcal{K}_j^{\lambda}(n,m,p,A,B,C,D)$ and*

$$\frac{1-A}{1-B} = 1 - \frac{[1+\frac{j}{p}\lambda]^{n-1}\left[(1-D)[1+\frac{j}{p}\lambda]^m - (1-C)\right]\delta}{[1+\frac{j}{p}\lambda]^n\left[(1-D)[1+\frac{j}{p}\lambda]^m - (1-C)\right] - (C-D)} \tag{22}$$

where

$$\delta \leq (1-D)(1+\frac{j}{p}\lambda) - (C-D)[1+\frac{j}{p}\lambda]^{1-n}\left\{(1-D)[1+\frac{j}{p}\lambda]^m - (1-C)\right\}^{-1}.$$

Proof. Let $f \in \mathcal{N}_{j,\delta}(g)$ for $g \in \mathcal{T}_j(n,m,p,C,D,\lambda)$. Then,

$$\sum_{k=j+p}^{\infty} k\,|a_k - b_k| \le \delta, \quad \text{and} \quad \sum_{k=j+p}^{\infty} b_k \le \frac{C-D}{[1+\frac{j}{p}\lambda]^n\left[(1-D)[1+\frac{j}{p}\lambda]^m-(1-C)\right]}. \tag{23}$$

Consider,

$$\begin{aligned}
\left|\frac{f(z)}{g(z)}-1\right| &\le \frac{\sum_{k=j+p}^{\infty}|a_k-b_k|}{1-\sum_{k=j+p}^{\infty}b_k} \\
&\le \frac{[1+\frac{j}{p}\lambda]^{n-1}\left\{(1-D)[1+\frac{j}{p}\lambda]^m-(1-C)\right\}\delta}{[1+\frac{j}{p}\lambda]^n\left\{(1-D)[1+\frac{j}{p}\lambda]^m-(1-C)\right\}-(C-D)} \\
&= \frac{A-B}{1-B}.
\end{aligned}$$

This implies that $f \in \mathcal{K}_j^{\lambda}(n,m,p,A,B,C,D)$.

5. Distortion and covering Theorems

Theorem 8. *If $f \in \mathcal{T}_j(n,m,p,A,B,\lambda)$, then*

$$r^p - \frac{A-B}{(1+\frac{j}{p}\lambda)^n\left\{(1-B)(1+\frac{j}{p}\lambda)^m-(1-A)\right\}}r^{j+p} \le |f(z)| \le$$

$$r^p + \frac{A-B}{(1+\frac{j}{p}\lambda)^n\left\{(1-B)(1+\frac{j}{p}\lambda)^m-(1-A)\right\}}r^{j+p} \quad (0<|z|=r<1),$$

with equality for

$$f(z) = z^p - \frac{A-B}{(1+\frac{j}{p}\lambda)^n\left\{(1-B)(1+\frac{j}{p}\lambda)^m-(1-A)\right\}}r^{j+p} \quad (z=\pm r) \tag{24}$$

Proof. In view of Theorem 1, we have

$$(1+\frac{j}{p}\lambda)^n\left\{(1-B)(1+\frac{j}{p}\lambda)^m-(1-A)\right\}\sum_{k=j+p}^{\infty}a_k$$

$$\leq \sum_{k=j+p}^{\infty}[1+(\frac{k}{p}-1)\lambda]^n\left\{(1-B)[1+(\frac{k}{p}-1)\lambda]^m-(1-A)\right\}a_k \leq A-B.$$

Hence

$$|f(z)| \leq r^p + \sum_{k=j+p}^{\infty} a_k r^k \leq r^p + r^{j+p}\sum_{k=j+p}^{\infty} a_k$$
$$\leq r^p + \frac{A-B}{(1+\frac{j}{p}\lambda)^n\left\{(1-B)(1+\frac{j}{p}\lambda)^m-(1-A)\right\}}r^{j+p}$$

and

$$|f(z)| \geq r^p - \sum_{k=j+p}^{\infty} a_k r^k \geq r^p - r^{j+p}\sum_{k=j+p}^{\infty} a_k$$
$$\geq r^p - \frac{A-B}{(1+\frac{j}{p}\lambda)^n\left\{(1-B)(1+\frac{j}{p}\lambda)^m-(1-A)\right\}}r^{j+p}.$$

This completes the proof.

Theorem 9. *Any function $f \in \mathcal{T}_j(n,m,p,A,B,\lambda)$ maps the disk $|z|<1$ on to a domain that contains the disk*

$$|w| < 1 - \frac{A-B}{(1+\frac{j}{p}\lambda)^n\left\{(1-B)(1+\frac{j}{p}\lambda)^m-(1-A)\right\}}.$$

Proof. The proof follows upon letting $r \to 1$ in Theorem 8.

Theorem 10. *If $f \in \mathcal{T}_j(n,m,p,A,B,\lambda)$,then*

$$1 \quad - \frac{(A-B)}{(1+\frac{j}{p}\lambda)^{n-1}\left\{(1-B)(1+\frac{j}{p}\lambda)^m-(1-A)\right\}}r^j \leq \left|f'(z)\right| \leq$$

$$1 \quad + \frac{A-B}{(1+\frac{j}{p}\lambda)^{n-1}\left\{(1-B)(1+\frac{j}{p}\lambda)^m-(1-A)\right\}}r^j \quad (0<|z|=r<1),$$

with equality for

$$f(z) = z^p - \frac{A-B}{(1+\frac{j}{p}\lambda)^{n-1}\left\{(1-B)(1+\frac{j}{p}\lambda)^m-(1-A)\right\}}z^{j+p} \quad (z=\pm r) \qquad (25)$$

Proof. We have

$$|f'(z)| \leq 1 + \sum_{k=j+p}^{\infty} k\, a_k\, |z|^{k-1} \leq 1 + r^j \sum_{k=j+p}^{\infty} k\, a_k.$$

In view of Theorem 1,

$$\sum_{k=j+p}^{\infty} k\, a_k \leq \frac{A-B}{(1+\frac{j}{p}\lambda)^{n-1}\left\{(1-B)(1+\frac{j}{p}\lambda)^m - (1-A)\right\}}$$

Thus

$$|f'(z)| \leq 1 + \frac{A-B}{(1+\frac{j}{p}\lambda)^{n-1}\left\{(1-B)(1+\frac{j}{p}\lambda)^m - (1-A)\right\}} r^j.$$

On the other hand,

$$\begin{aligned} |f'(z)| &\geq 1 - \sum_{k=j+p}^{\infty} k\, a_k\, |z|^{k-p} \geq 1 - r^j \sum_{k=j+p}^{\infty} k\, a_k \\ &\geq 1 - \frac{A-B}{(1+\frac{j}{p}\lambda)^{n-1}\left\{(1-B)(1+\frac{j}{p}\lambda)^m - (1-A)\right\}} r^j. \end{aligned}$$

This completes the proof.

6. Radii of starlikeness and convexity

In this section, we find the radius of starlikeness of order ρ and the radius of convexity of order ρ for functins in the class $\mathcal{T}_j(n, m, p, A, B, \lambda)$.

Theorem 11. *If $f \in \mathcal{T}_j(n, m, p, A, B, \lambda)$, then f is starlike of order $\rho, (0 \leq \rho < p)$ in $|z| < r_1(n, m, p, A, B, \lambda, \rho)$ where*
$r_1(n, m, p, A, B, \lambda, \rho) =$

$$\inf_k \left[\frac{[1+(\frac{k}{p}-1)\lambda]^n\left\{(1-B)[1+(\frac{k}{p}-1)\lambda]^m - (1-A)\right\}(p-\rho)}{(k-\rho)(A-B)}\right]^{\frac{1}{k-p}}$$

Proof. It is sufficient to show that $\left|z\dfrac{f'(z)}{f(z)} - p\right| \leq p - \rho$ $(0 \leq \rho < p)$ for $|z| < r_1(n, m, p, A, B, \lambda, \rho)$.

We have

$$\left| z\frac{f'(z)}{f(z)} - p \right| \leq \frac{\sum_{k=j+p}^{\infty} (k-p)a_k|z|^{k-p}}{1 - \sum_{k=j+p}^{\infty} a_k|z|^{k-p}}$$

Thus $\left| z\frac{f'(z)}{f(z)} - p \right| \leq p - \rho$ if

$$\sum_{k=j+p}^{\infty} \frac{(k-\rho)a_k|z|^{k-p}}{(p-\rho)} \leq 1. \tag{26}$$

Hence, by Theorem (1), (26) will be true if
$\frac{(k-\rho)|z|^{k-p}}{(p-\rho)} \leq$
$\frac{[1+(\frac{k}{p}-1)\lambda]^n \left\{(1-B)[1+(\frac{k}{p}-1)\lambda]^m - (1-A)\right\}}{(A-B)}$
or if

$$|z| \leq \left[\frac{[1+(\frac{k}{p}-1)\lambda]^n \left\{(1-B)[1+(\frac{k}{p}-1)\lambda]^m - (1-A)\right\}(p-\rho)}{(k-\rho)(A-B)} \right]^{\frac{1}{k-p}} \tag{27}$$

$(k \geq j+p)$. This completes the proof.

Theorem 12. *If* $f \in \mathcal{T}_j(n,m,p,A,B,\lambda)$, *then* f *is convex of order* $\rho, (0 \leq \rho < p)$ *in* $|z| < r_2(n,m,p,A,B,\lambda,\rho)$ *where*

$$r_2(n,m,p,A,B,\lambda,\rho) =$$

$$\inf_k \left[\frac{[1+(\frac{k}{p}-1)\lambda]^n \left\{(1-B)[1+(\frac{k}{p}-1)\lambda]^m - (1-A)\right\}(p-\rho)}{k\,(k-\rho)(A-B)} \right]^{\frac{1}{k-p}}$$

Proof. It is sufficient to show that $\left| 1 + z\frac{f''(z)}{f'(z)} - p \right| \leq p - \rho$ $(0 \leq \rho < p)$ for $|z| < r_2(n,m,p,A,B,\lambda,\rho)$.

We have

$$\left|1+z\frac{f''(z)}{f'(z)}-p\right| \leq \frac{\displaystyle\sum_{k=j+p}^{\infty} k\,(k-p)a_k|z|^{k-p}}{1-\displaystyle\sum_{k=j+p}^{\infty} k\,a_k|z|^{k-p}}$$

Thus $\left|1+z\frac{f''(z)}{f'(z)}-p\right| \leq p-\rho$ if

$$\sum_{k=j+p}^{\infty} \frac{k\,(k-\rho)a_k|z|^{k-p}}{(p-\rho)} \leq 1. \tag{28}$$

Hence, by Theorem (1), (28) will be true if

$$\frac{k\,(k-\rho)|z|^{k-p}}{(p-\rho)} \leq$$

$$\frac{[1+(\frac{k}{p}-1)\lambda]^n \left\{(1-B)[1+(\frac{k}{p}-1)\lambda]^m-(1-A)\right\}}{(A-B)}$$

or if

$$|z| \leq \left[\frac{[1+(\frac{k}{p}-1)\lambda]^n\left\{(1-B)[1+(\frac{k}{p}-1)\lambda]^m-(1-A)\right\}(p-\rho)}{k\,(k-\rho)(A-B)}\right]^{\frac{1}{k-p}} \tag{29}$$

$(k \geq j+p)$. This completes the proof.

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