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ON GENERALIZED φ-RECURRENT TRANS-SASAKIAN MANIFOLDS

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Abstract. The object of the present paper is to study generalized φ-recurrent trans-Sasakian manifolds.It is proved that a generalized φ-recurrent trans-Sasakian manifold is an Einstein manifold.Also we obtained a relation between the associated 1-forms A and B for a generalized φ-recurrent and generalized concircular φ-recurrent trans-Sasakian manifolds and finally proved that a three dimensional locally generalized φ-recurrent trans-Sasakian manifold is of constant curvature.

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1. Introduction

The notion of locally φ-symmetric Sasakian manifold was introduced by T. Takahashi [20] in 1977. φ-recurrent Sasakian manifold and generalized φ-recurrent Sasakian manifold were studied by the author [5] and [15] respectively.

Also J. A .Oubina in 1985 introduced a new class of almost contact metric structures which was a generalization of Sasakian [14], α -Sasakian [9], Kenmotsu [8], β-Kenmotsu [9] and cosymplectic [9] manifolds, which was called trans-Sasakian manifold [11]. After him many authors [3],[4], [5],[8],[10], [11], [12], [13], [14], [17], [18], [21], have studied various type of properties in trans-Sasakian manifold.

In the Gray-Hervella classification of almost Hermitian manifolds [7], there appears a class, W_4, of Hermitian manifolds which are closely related to locally conformal kaehler manifolds. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [14] if the product manifold $M \times R$ belongs to the class W_4. The class $C_6 \bigoplus C_5$ $([11],[12])$ coincides with the class of trans-Sasakian structures of type (α, β).In fact, in [12], local nature of the two subclasses, namely C_5 and C_6 structures, of trans-Sasakian structures are characterized completely.In [18], trans-Sasakian structures of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic [1],

β-Kenmotsu [9] and α -Sasakian [9] respectively.In [21], it is proved that trans-Sasakian structures are generalized quasi-Sasakian.Thus, trans-Sasakian structures also provide a large class of generalized quasi-Sasakian structures.

2. Preliminaries

Let M be an almost contact metric manifold [1] with an almost contact structure (φ,ξ,η,g), where φ is a $(1,1)$ tensor field , ξ is a vector field , η is a 1-form and g is a compatible Riemannian metric such that,

$$(2.1)(a) \quad \varphi^2 = -I + \eta \bigotimes \xi, (b) \;\; \eta(\xi) = 1, (c) \quad \varphi(\xi) = 0, (d) \quad \eta \; o \; \varphi = 0$$

$$(2.2) \qquad g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$$

$$(2.3)(a) \qquad g(X, \varphi Y) = -g(\varphi X, Y), \qquad (b) \;\; g(X,\xi) = \eta(X)$$

for all $X, Y \in TM$.

An almost contact structure (φ,ξ,η,g), on M is called trans-Sasakian structure [14] if $(M \times R, J, G)$ belongs to the class W_4 [7], where J is the almost complex structure on $M \times R$ defined by

$$J(X, fd/dt) = (\varphi X - f\xi, \eta(X)d/dt)$$

for all vector fields X on M and smooth functions f on $M \times R$, and G is the product metric on $M \times R$.This may be expressed by the condition [18]

$$(2.4) \qquad (\nabla_X \varphi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\varphi X, Y)\xi - \eta(Y)\varphi X)$$

for some smooth functions α and β on M.
From (2.4) it follows that

$$(2.5) \qquad \nabla_X \xi = -\alpha\varphi X + \beta(X - \eta(X)\xi)$$

$$(2.6) \qquad (\nabla_X \eta)Y = -\alpha g(\varphi X, Y) + \beta g(\varphi X, \varphi Y).$$

In [18], authors obtained some results which shall be useful for next sections. They are

$$(2.7) \;\; R(X,Y)\xi = (\alpha^2 - \beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\varphi X - \eta(X)\varphi Y) \\ + (Y\alpha)\varphi X - (X\alpha)\varphi Y + (Y\beta)\varphi^2 X - (X\beta)\varphi^2 Y$$

$$R(\xi, X)\xi = (\alpha^2 - \beta^2 - \xi\beta)(\eta(X)\xi - X) \tag{2.8}$$

$$2\alpha\beta + \xi\alpha = 0 \tag{2.9}$$

$$S(X, \xi) = (2n(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - (2n-1)X\beta - (\varphi X)\alpha \tag{2.10}$$

$$Q\xi = (2n(\alpha^2 - \beta^2) - \xi\beta)\xi - (2n-1)grad\ \beta + \varphi(grad\ \alpha) \tag{2.11}$$

where R is the curvature tensor, S is the Ricci-tensor and r is the scalar curvature.Also

$$g(QX, Y) = S(X, Y) \tag{2.12}$$

Q being the symmetric endomorphism of the tangent space at each point corresponding to the Ricci-tensor S.
When

$$\varphi(grad\ \alpha) = (2n-1)grad\ \beta, \tag{2.13}$$

then (2.10) and (2.11) reduces to

$$S(X, \xi) = 2n(\alpha^2 - \beta^2)\eta(X) \tag{2.14}$$

$$Q\xi = 2n(\alpha^2 - \beta^2)\xi. \tag{2.15}$$

Again a Sasakian manifold is said to be a φ-recurrent manifold if there exists a non zero 1-form A such that

$$\varphi^2((\nabla_W R)(X, Y)Z) = A(X)R(Y, Z)W \tag{2.16}$$

for all vector fields X,Y,Z,W orthogonal to ξ. A Riemannian manifold (M^{2n+1}, g) is called generalized recurrent [6], if its curvature tensor R satisfies the condition

$$(\nabla_X R)(Y, Z)W = A(X)R(Y, Z)W + B(X)[g(Z, W)Y - g(Y, W)Z] \tag{2.17}$$

where, A and B are two 1-forms, B is non zero and these are defined by

$$g(X, \rho_1) = A(X) \quad \text{and} \quad g(X, \rho_2) = B(X),\ \forall\ X \in TM \tag{2.18}$$

ρ_1 and ρ_2 being the vector fields associated to the 1-form A and B.

Definition 1. *Trans-Sasakian manifold (M^{2n+1}, g) is called generalized φ-recurrent if its curvature tensor R satisfies the condition*

$$\varphi^2((\nabla_W R)(X,Y)Z) = A(W)R(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y] \tag{2.19}$$

where A and B are two 1-forms, B is non zero and these are defined by

$$g(W,\rho_1) = A(W) \quad and \qquad g(W,\rho_2) = B(W),\ \forall\ W \in TM$$

ρ_1 and ρ_2 being the vector fields associated to the 1-form A and B.

The notion of generalized φ-recurrent Kenmotsu manifolds was introduced by A.Basari and C.Murathan[2] and also generalizing the notion of φ-recurrency, the authors D.A.Patil, D.G.Prakasha and C.S.Bagewadi[15] introduced the notion of generalized φ-recurrent Sasakian manifolds. Motivated by the above studies, we have studied of generalized φ-recurrent trans-Sasakian manifolds and obtained some interesting results.

3. On generalized φ-recurrent trans-Sasakian manifold

In this section we consider a generalized φ-recurrent trans-Sasakian manifold.Then by virtue of (2.1) and (2.19) we have

$$-(\nabla_W R)(X,Y)Z + \eta((\nabla_W R)(X,Y)Z)\xi = A(W)R(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y]. \tag{3.1}$$

From (3.1) it follows that

$$-g((\nabla_W R)(X,Y)Z,U) + \eta((\nabla_W R)(X,Y)Z)\eta(U) = A(W)g(R(X,Y)Z,U) + B(W)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \tag{3.2}$$

Let $\{e_i\}$, $i = 1,2,...,2n+1$, be an orthonormal basis of the tangent space at any point of the manifold.Then putting $X = U = e_i$ in (3.2) and taking summation over $i,\ 1 \le i \le 2n+1$, we get

$$-(\nabla_W S)(Y,Z) + \sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i,Y)Z)\eta(e_i) = A(W)S(Y,Z) + 2nB(W)g(Y,Z)]. \tag{3.3}$$

The second term of (3.3) by putting $Z = \xi$ takes the form $g((\nabla_W R)(e_i, Y)\xi, \xi)g(e_i, \xi)$. Consider

$$\begin{aligned} g((\nabla_W R)(e_i, Y)\xi, \xi) &= g(\nabla_W R(e_i, Y)\xi, \xi) - g(R(\nabla_W e_i, Y)\xi, \xi) \\ &\quad - g(R(e_i, \nabla_W Y)\xi, \xi) - g(R(e_i, Y)\nabla_W \xi, \xi) \end{aligned} \tag{3.4}$$

at $p \in M$.Since $\{e_i\}$ is an orthonormal basis, so $\nabla_X e_i = 0$ at p.
Using (2.7), (2.1)(a) and (2.3)(b), we have

$$\begin{aligned} g(R(e_i, \nabla_W Y)\xi, \xi) &= g((\alpha^2 - \beta^2)(\eta(\nabla_W Y)e_i - \eta(e_i)\nabla_W Y)) \\ &\quad + 2\alpha\beta(\eta(\nabla_W Y)\varphi e_i - \eta(e_i)\varphi(\nabla_W Y) \\ &\quad + (\nabla_W Y)\alpha(\varphi e_i) - (e_i\alpha)\varphi(\nabla_W Y) \\ &\quad + (\nabla_W Y)\beta\varphi^2 e_i - (e_i\beta)\varphi^(\nabla_W Y), \xi) = 0. \end{aligned} \tag{3.5}$$

Using (3.5) in (3.4) we obtain

$$g((\nabla_W R)(e_i, Y)\xi, \xi) = g(\nabla_W R(e_i, Y)\xi, \xi) - g(R(e_i, Y)\nabla_W \xi, \xi). \tag{3.6}$$

Since $(\nabla_W g) = 0$, we have $g(\nabla_W R)(e_i, Y)\xi, \xi) + g(R(e_i, Y)\xi, \nabla_W \xi) = 0$, which implies

$$g((\nabla_W R)(e_i, Y)\xi, \xi) = -g(R(e_i, Y)\xi, \nabla_W \xi) - g(R(e_i, Y)\nabla_W \xi, \xi). \tag{3.7}$$

Using (2.5) in (3.7) we get

$$\begin{aligned} g((\nabla_W R)(e_i, Y)\xi, \xi) &= -g(R(e_i, Y)\xi, -\alpha\varphi W + \beta(W - \eta(W)\xi)) \\ &\quad - g(R(e_i, Y) - \alpha\varphi W + \beta(W - \eta(W)\xi), \xi) \\ &= \alpha g(R(e_i, Y)\xi, \varphi W) - \beta g(R(e_i, Y)\xi, W) \\ &\quad + \alpha g(R(e_i, Y)\varphi W, \xi) - \beta g(R(e_i, Y)W, \xi) = 0. \end{aligned} \tag{3.8}$$

Replacing Z by ξ in (3.3) and using (2.3)(b), (2.13) and (2.14) we have

$$(\nabla_W S)(Y, \xi) = -[2n(\alpha^2 - \beta^2)A(W) + 2nB(W)]\eta(Y). \tag{3.9}$$

Now we know

$$(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi).$$

Using (2.5) and (2.14) in the above relation we get, after a brief calculation

$$(3.10)\quad (\nabla_W S)(Y,\xi) = 2n(\alpha^2-\beta^2)[-\alpha g(\varphi W,Y)+\beta g(\varphi Y,\varphi W)]+\alpha S(Y,\varphi W) - S(Y,\beta W)+2n\beta(\alpha^2-\beta^2)\eta(Y)\eta(W).$$

By virtue of (2.2), (3.10) reduces to

$$(3.11)\quad (\nabla_W S)(Y,\xi) = 2n(\alpha^2-\beta^2)[-\alpha g(Y,\varphi W)+\beta g(Y,W)] + \alpha S(Y,\varphi W)-\beta S(Y,W).$$

From (3.9) and (3.11) we have

$$(3.12)\quad 2n(\alpha^2-\beta^2)[-\alpha g(Y,\varphi W)+\beta g(Y,W)]+\alpha S(Y,\varphi W)-\beta S(Y,W) = -[2n(\alpha^2-\beta^2)A(W)+2nB(W)]\eta(Y).$$

Replacing $Y=\xi$ in (3.12) then using (2.1)(b), (2.3)(b), (2.13) and (2.14) we get

$$(3.13)\quad (\alpha^2-\beta^2)A(W)+B(W)=0.$$

Again replacing Y and W by φY and φW respectively in (3.12) and then using (2.1)(a), (2.3)(a), (2.12), (2.13) and (2.15) we obtain

$$(3.14)\quad S(Y,W)=2n(\alpha^2-\beta^2)g(Y,W)$$

and
$$S(\varphi Y,W)=2n(\alpha^2-\beta^2)g(\varphi Y,W).$$

Thus we can state

Theorem 1. *A generalized φ-recurrent trans-Sasakian manifold (M^{2n+1},g) satisfying $\varphi(grad\alpha)=(2n-1)grad\beta$, is an Einstein manifold and more over, the 1-forms A and B are related as $(\alpha^2-\beta^2)A+B=0$.*

Now from (2.19) and (2.16) we have

$$(3.15)\quad (\nabla_W R)(X,Y)Z=\eta((\nabla_W R)(X,Y)Z)\xi - aA(W)R(X,Y)Z+B(W)[g(Y,Z)X-g(X,Z)Y].$$

Using Bianchi's identity in (3.15) and use (3.13) we get

$$(3.16)\quad \begin{aligned} &A(W)R(X,Y)Z-(\alpha^2-\beta^2)A(W)[g(Y,Z)X-g(X,Z)Y]\\ &+A(X)R(Y,W)Z-(\alpha^2-\beta^2)A(X)[g(W,Z)Y-g(Y,Z)W]\\ &+A(Y)R(W,X)Z-(\alpha^2-\beta^2)A(Y)[g(X,Z)W-g(W,Z)X]=0.\end{aligned}$$

Putting $Y = Z = \{e_i\}$, where $\{e_i\}$ be an orthonormal basis of the tangent space at any point of the manifold, in (3.16) and taking summation over $i,\ 1 \leq i \leq 2n+1$, we get

$$S(W, \rho_1) = -6n(\alpha^2 - \beta^2)A(W) \tag{3.16}$$

From (3.16), we can state the following

Theorem 2. *In a generalized φ-recurrent trans-Sasakian manifold (M^{2n+1},g), n≥1, $6n(\alpha^2 - \beta^2)$ is the eigen value of the Ricci-tensor corresponding to the eigen vector ρ_1, where ρ_1 is the associated vector field of the 1-form A.*

4. On generalized concircular φ-recurrent trans-Sasakian manifold

Definition 2. *A trans-Sasakian manifold (M^{2n+1}, g) is called generalized concircular $\varphi-$recurrent if its concircular curvature tensor $\overline{C}$ (Yano, K; Kon, M, 1984)*

$$\overline{C}(X,Y)Z = R(X,Y)Z - \tfrac{r}{2n(2n+1)}[g(Y,Z)X - g(X,Z)Y]$$

satisfies the condition[21]

$$\varphi^2((\nabla_W \overline{C})(X,Y)Z) = A(W)\overline{C}(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y] \tag{4.1}$$

where A and B are two 1-forms, B is non zero and these are defined by

$$g(W, \rho_1) = A(W) \quad and \quad g(W, \rho_2) = B(W),\ \forall\ W \in TM$$

ρ_1 and ρ_2 being the vector fields associated to the 1-form A and B.

Let us consider a generalized concircular φ-recurrent trans-Sasakian manifold.Then by virtue of $(2.1)(a)$ and (4.1) we have

$$\begin{aligned} &-(\nabla_W \overline{C})(X,Y)Z + \eta((\nabla_W \overline{C})(X,Y)Z)\xi \\ &\quad = A(W)\overline{C}(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{4.2}$$

From (4.2) it follows that

$$\begin{aligned} &-g((\nabla_W \overline{C})(X,Y)Z,U) + \eta((\nabla_W \overline{C})(X,Y)Z)\eta(U) \\ &\quad = A(W)g(\overline{C}(X,Y)Z,U) + B(W)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \end{aligned} \tag{4.3}$$

Let $\{e_i\}$, $i = 1, 2, ..., 2n+1$, be an orthonormal basis of the tangent space at any point of the manifold.Then putting $Y = Z = \{e_i\}$ in (4.3) and taking summation over i, $1 \le i \le 2n+1$, we get

$$(4.4)\quad -(\nabla_W S)(X,U) + \tfrac{\nabla_W r}{2n+1} g(X,U) + (\nabla_W S)(X,\xi)\eta(U) - \tfrac{\nabla_W r}{2n+1}\eta(X)\eta(U) = A(W)[S(X,U) - \tfrac{r}{2n+1} g(X,U)] + 2nB(W)g(X,U)].$$

Replacing U by ξ in (4.4) and using (2.3)(b), (2.13) and (2.14) we have

$$(4.5)\qquad A(W)[2n(\alpha^2-\beta^2) - \tfrac{r}{2n+1}]\eta(X) + 2nB(W)\eta(X) = 0.$$

$$\eta(W) \neq 0, \qquad A(W)[2n(\alpha^2-\beta^2) - \tfrac{r}{2n+1}] + 2nB(W) = 0$$

$$(4.6)\qquad i.e \quad B(W) = A(W)[\tfrac{r}{2n+1} - (\alpha^2-\beta^2)].$$

So we get the following theorem

Theorem 3. *In a generalized concircular φ -recurrent trans-Sasakian manifold (M^{2n+1},g), the 1-forms A and B are related as (4.6).*

5. Three dimensional locally generalized φ-recurrent trans-Sasakian manifold

In a three dimensional Riemannian manifold (M^3, g), we get

$$(5.1)\qquad R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y + \tfrac{r}{2}[g(X,Z)Y - g(Y,Z)X]$$

where Q is the Ricci-operator that is $S(X,Y) = g(QX,Y)$ and r is the scalar curvature of the manifold.Now putting $Z = \xi$ in (5.1) and using (2.3)(b), (2.13) and (2.14) we get

$$(5.2)\qquad R(X,Y)\xi = \eta(Y)QX - \eta(X)QY + 2n(\alpha^2-\beta^2)\eta(Y)X - \eta(X)Y + \tfrac{r}{2}[\eta(X)Y - \eta(Y)X].$$

Using (2.7) and (5.2) we have

$$(5.3)\qquad (\alpha^2-\beta^2)(\eta(Y)X - \eta(X)Y) + 2\alpha\beta(\eta(Y)\varphi X - \eta(X)\varphi Y) + (Y\alpha)\varphi X - (X\alpha)\varphi Y + (Y\beta)\varphi^2 X - (X\beta)\varphi^2 Y$$

$$= \eta(Y)QX - \eta(X)QY + 2n(\alpha^2 - \beta^2)\eta(Y)X - \eta(X)Y + \tfrac{r}{2}[\eta(X)Y - \eta(Y)X].$$

Again, putting $X = \xi$ we obtain

$$QY = \{(\alpha^2 - \beta^2 - \xi\beta) - 2(\alpha^2 - \beta^2) + \tfrac{r}{2}\}Y + [4(\alpha^2 - \beta^2) - (\alpha^2 - \beta^2 - \xi\beta) - \tfrac{r}{2}]\eta(Y)\xi. \tag{5.4}$$

It follows from (5.4) that

$$S(Y, Z) = \{\tfrac{r}{2} - (\alpha^2 - \beta^2 + \xi\beta)\}g(Y, Z) + [3(\alpha^2 - \beta^2) + \xi\beta - \tfrac{r}{2}]\eta(Y)\eta(Z). \tag{5.5}$$

Thus from (5.1), (5.4), and (5.5) we have

$$\begin{aligned} R(X, Y)Z = {} & [(r + \tfrac{r}{2}) - 2(\alpha^2 - \beta^2 + \xi\beta)][g(Y, Z)X - g(X, Z)Y] \\ & + [3(\alpha^2 - \beta^2) + \xi\beta - \tfrac{r}{2}][g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi] \\ & + [3(\alpha^2 - \beta^2) + \xi\beta - \tfrac{r}{2}][\eta(Y)\eta(Z)X - \eta(Z)\eta(X)Y]. \end{aligned} \tag{5.6}$$

Taking the covariant differentiation to the both sides of the equation (5.6) we get

$$\begin{aligned} (\nabla_W R)(X, Y)Z = {} & \tfrac{3dr(W)}{2}[g(Y, Z)X - g(X, Z)Y] \\ & - \tfrac{3dr(W)}{2}[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \\ & + \eta(Y)\eta(Z)X - \eta(Z)\eta(X)Y] \\ & - [3(\alpha^2 - \beta^2) + \xi\beta - \tfrac{r}{2}][g(Y, Z)\eta(X) - g(X, Z)\eta(Y)](\nabla_W \xi) \\ & + [3(\alpha^2 - \beta^2) + \xi\beta - \tfrac{r}{2}][\eta(Y)X - \eta(X)Y](\nabla_W \eta)(Z) \\ & + [3(\alpha^2 - \beta^2) + \xi\beta - \tfrac{r}{2}][g(Y, Z)\xi - \eta(Z)Y](\nabla_W \eta)(X) \\ & - [3(\alpha^2 - \beta^2) + \xi\beta - \tfrac{r}{2}][g(X, Z)\xi - \eta(Z)X](\nabla_W \eta)(Y). \end{aligned} \tag{5.7}$$

We may assume that all the vector fields X, Y, Z, W are orthogonal to ξ

$$\begin{aligned} (\nabla_W R)(X, Y)Z = {} & \tfrac{3dr(W)}{2}[g(Y, Z)X - g(X, Z)Y] \\ & + [3(\alpha^2 - \beta^2) + \xi\beta - \tfrac{r}{2}][g(Y, Z)\nabla_W \eta)(X) - g(X, Z)\nabla_W \eta)(Y)]. \end{aligned} \tag{5.8}$$

Applying φ^2 to the both sides of (5.8)

$$\varphi^2((\nabla_W R)(X, Y)Z) = \tfrac{3dr(W)}{2}[g(Y, Z)X - g(X, Z)Y]. \tag{5.9}$$

Now,

$$A(W)R(X, Y)Z = [\tfrac{3dr(W)}{2} - B(W)][g(Y, Z)X - g(X, Z)Y]. \tag{5.10}$$

Putting $W = \{e_i\}$, where $\{e_i\}$, $i = 1, 2, 3$, be an orthonormal basis of the tangent space at any point of the manifold and taking summation over i, $1 \leq i \leq 3$, we get

$$R(X, Y)Z = \lambda[g(Y, Z)X - g(X, Z)Y]$$

where $\lambda = \frac{1}{A(e_i)}[\frac{3dr(e_i)}{2} - B(e_i)]$ is the scalar, since A is non zero one form. Therefore, (M^3, g) is of constant curvature λ.
Thus we the following theorem

Theorem 4. *A three dimensional locally generalized φ -recurrent trans-Sasakian manifold is a manifold of constant curvature.*

6. Example of generalized φ-recurrent trans-Sasakian manifold

Consider three dimensional manifold $M = \{(x, y, z) \in R^3 \setminus z \neq 0\}$, where (x, y, z) are the standard coordinates of R^3. The vector fields

$$e_1 = \frac{x}{z}\frac{\partial}{\partial x}, \qquad e_2 = \frac{y}{z}\frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial z} \tag{6.1}$$

are linearly independent at each point of M.Let g be the Riemannian metric defined by

$$\begin{aligned} &g(e_1, e_1) = 1,\ g(e_2, e_2) = 1,\ g(e_3, e_3) = 1, \\ &g(e_1, e_2) = 0,\ \ g(e_1, e_3) = 0,\ \ g(e_2, e_3) = 0. \end{aligned} \tag{6.2}$$

Let η be the $1-$form defined by $\eta = g(X, e_3)$ for any vector field $X \in \chi(M)$. Let φ be the $(1, 1)$ tensor field defined by

$$\varphi(e_1) = e_2, \quad \varphi(e_2) = -e_1, \quad \varphi(e_3) = 0. \tag{6.3}$$

Then using the linearity of φ and g we have

$$\eta(e_3) = 1, \quad \varphi^2 X = -X + \eta(Z)e_3, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any $X, Y \in \chi(M)$.Hence for $e_3 = \xi$, the structure defines an almost contact structure on M.Let ∇ be the Livi-Civita connection with respect to the metric g, then we obtain

(6.4) $$[e_1, e_2] = 0, \quad [e_2, e_3] = \tfrac{1}{z}e_2, \quad [e_1, e_3] = \tfrac{1}{z}e_1.$$

The Riemannian connection ∇ of the matric g is given by

(6.5) $$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Using (6.5), we get

$$2g(\nabla_{e_1} e_3, e_1) = 2g(\tfrac{1}{z}e_1, e_1) + 2g(e_2, e_1) = 2g(\tfrac{1}{z}e_1 + e_2, e_1),$$

since $g(e_1, e_2) = 0$.Hence

(6.6) $$\nabla_{e_1} e_3 = \tfrac{1}{z}e_1 + e_2$$

Again from (6.5) we obtain

$$2g(\nabla_{e_2} e_3, e_2) = 2g(\tfrac{1}{z}e_2, e_2) - 2g(e_1, e_2) = 2g(\tfrac{1}{z}e_2 - e_1, e_2),$$

since $g(e_1, e_2) = 0$.Hence we get

(6.7) $$\nabla_{e_2} e_3 = \tfrac{1}{z}e_2 - e_1$$

Again from (6.5) we obtain

(6.8) $$\nabla_{e_1} e_1 = -\tfrac{1}{z}e_1, \nabla_{e_1} e_2 = 0, \nabla_{e_2} e_1 = 0, \nabla_{e_2} e_2 = -\tfrac{1}{z}e_2, \\ \nabla_{e_3} e_1 = 0, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_3 = 0$$

The manifold M satisfies (2.5) with $\alpha = -1$ and $\beta = \frac{1}{z}$.Hence M is a trans Sasakian manifold.With the help of (6.6) ,(6.7) and (6.8) we get

(6.9) $$R(e_1, e_3)e_3 = -\tfrac{1}{z}e_2, \ R(e_3, e_1)e_3 = \tfrac{1}{z}e_2, \ R(e_1, e_2)e_3 = \tfrac{1}{z}(e_1 - e_2), \\ R(e_1, e_1)e_3 = 0, \ R(e_1, e_3)e_1 = 0, \ R(e_1, e_3)e_2 = 0$$

The vectors $\{e_1, e_2, e_3\}$ form a basis of the manifold M and so the vector can be written as $X = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$ where $\lambda_i \in \Re^3$, $i = 1, 2, 3$.Thus the covariant derivatives of the components of the curvature tensor are given by

$$(\nabla_X R)(e_1, e_3)e_3 = -(\tfrac{\lambda_1}{z} + \tfrac{\lambda_3}{z^2})e_1 + (\tfrac{2\lambda_3}{z^2} + \tfrac{\lambda_2}{z^2} - \tfrac{\lambda_1}{z^2} - \tfrac{\lambda_1}{z})e_2.$$

Applying φ^2 to both sides of the above equation, we obtain

$$\varphi^2((\nabla_X R)(e_1, e_3)e_3) = A(X)R(e_1, e_3)e_3 + B(X)[g(e_3, e_3)e_1 - g(e_1, e_3)e_3]$$

where $A(X) = \lambda_1 + \frac{\lambda_3}{z}$ and $B(X) = -(\frac{2\lambda_3}{z^2} + \frac{\lambda_2}{z^2} - \frac{\lambda_1}{z^2} - \frac{\lambda_1}{z})$.

This implies that there exist a generalized $\varphi-$recurrent trans-Sasakian manifold.

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