Acta Universitatis Apulensis
ISSN: 1582-5329
No. 37/2014
pp. 111-123

EXISTENCE OF SOLUTIONS FOR NONLOCAL PROBLEMS IN ORLICZ-SOBOLEV SPACES VIA GENUS THEORY

N. T. Chung

Abstract. Using the genus theory, introduced by Krasnoselskii, we study the existence of weak solutions for a class of nonlocal problems in Orlicz-Sobolev spaces. Our results are natural extensions from the previous ones in [2, 14]. To our knowledge, this is the first contribution to the study of nonlocal problems in this class of spaces.

2000 *Mathematics Subject Classification*: 35J60, 35J70, 58E05, 76A02.

Keywords: Nonlocal problems, Orlicz-Sobolev spaces, Genus theory, Existence, Multiplicity.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$. Assume that $a : (0, \infty) \to \mathbb{R}$ is a function such that the mapping, defined by

$$\varphi(t) := \begin{cases} a(|t|)t & \text{for } t \neq 0, \\ 0, & \text{for } t = 0, \end{cases}$$

is an odd, increasing homeomorphisms from $\mathbb{R}$ onto $\mathbb{R}$. For the function φ above, let us define

$$\Phi(t) = \int_0^t \varphi(s)ds \quad \text{for all } t \in \mathbb{R},$$

on which will be imposed some suitable conditions later.

In this article, we are concerned with a class of nonlocal problems in Orlicz-Sobolev spaces of the form

$$\begin{cases} -M\Big(\int_\Omega \Phi(|\nabla u|)dx\Big) \operatorname{div}\Big(a(|\nabla u|)\nabla u\Big) &= f(x,u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

Firstly, it should be noticed that if $\varphi(t) = p|t|^{p-2}t$ for all $t \in \mathbb{R}$, $p > 1$ then problem (1) becomes the well-known p-Kirchhoff-type equation

$$\begin{cases} -M\Big(\int_\Omega |\nabla u|^p dx\Big)\Delta_p u &= f(x,u) \quad \text{in } \Omega, \\ \qquad\qquad\qquad\qquad u &= 0 \qquad\quad\ \text{on } \partial\Omega, \end{cases} \tag{2}$$

which has been intensively studied in recent years, see the papers [3, 6, 14, 19, 20, 24, 25]. In the case when $p(.)$ is a function, problem (2) has also been studied by many authors, see for examples [2, 8, 9, 13, 15, 16]. Since the first equation in (2) contains an integral over Ω, it is no longer a pointwise identity; therefore it is often called a nonlocal problem. This problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density, see [7]. Moreover, problem (2) is related to the stationary version of the Kirchhoff equation which is presented by Kirchhoff in 1883, see [18] for details.

We point out the fact that if $M(t) \equiv 1$ and the function $\varphi(t)$ is defined above, problem (1) becomes a nonlinear and non-homogeneous problem, namely,

$$\begin{cases} -\operatorname{div}\Big(a(|\nabla u|)\nabla u\Big) &= f(x,u) \quad \text{in } \Omega, \\ \qquad\qquad\qquad u &= 0 \qquad\quad\ \text{on } \partial\Omega, \end{cases} \tag{3}$$

which has been studied by some authors in Orlicz-Sobolev spaces, we refer to [4, 11, 12, 17, 21, 22].

In this article, motivated by the works mentioned above, we shall study the existence of solutions for nonlocal problems of type (1). It is clear that this is a natural extension from the earlier studies on nonlocal problems in classical Sobolev spaces and on nonlinear non-homogeneous problems in Orlicz-Sobolev spaces. To our knowledge, this is the first contribution to the study of nonlocal problems in this class of spaces. More precisely, using the ideas firstly introduced in the paper [14] and developed in [2] we want to illustrate how to handle problem (1) in Orlicz-Sobolev spaces by using the genus theory.

In order to study problem (1), let us introduce the functional spaces where it will be discussed. We will give just a brief review of some basic concepts and facts of the theory of Orlicz and Orlicz-Sobolev spaces, useful for what follows, for more details we refer the readers to the books by Adams [1], M.M. Rao et al. [23], the papers by Clément et al. [11, 12], M. Mihăilescu et al. [21, 22] and F. Cammaroto et al. [4].

For $\varphi : \mathbb{R} \to \mathbb{R}$ and Φ introduced at the start of the paper, we can see that Φ is a Young function, that is, $\Phi(0) = 0$, Φ is convex, and $\lim_{t\to\infty} \Phi(t) = +\infty$. Furthermore, since $\Phi(t) = 0$ if and only if $t = 0$, $\lim_{t\to 0} \frac{\Phi(t)}{t} = 0$, and $\lim_{t\to\infty} \frac{\Phi(t)}{t} = +\infty$, the function Φ is then called an N-function. The function Φ^* defined by the formula

$$\Phi^*(t) = \int_0^t \varphi^{-1}(s)ds \text{ for all } t \in \mathbb{R}$$

is called the complementary function of Φ and it satisfies the condition

$$\Phi^*(t) = \sup\{st - \Phi(s) : \ s \geq 0\} \quad \text{for all } t \geq 0.$$

We observe that the function Φ^* is also an N-function in the sense above and the following Young inequality holds

$$st \leq \Phi(s) + \Phi^*(t) \quad \text{for all } s, t \geq 0.$$

The Orlicz class defined by the N-function Φ is the set

$$K_\Phi(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \ \int_\Omega \Phi(|u(x)|)dx < \infty \right\}$$

and the Orlicz space $L_\Phi(\Omega)$ is then defined as the linear hull of the set $K_\Phi(\Omega)$. The space $L_\Phi(\Omega)$ is a Banach space under the following Luxemburg norm

$$\|u\|_\Phi := \inf \left\{ k > 0 : \ \int_\Omega \Phi\Big(\frac{u(x)}{k}\Big)dx \leq 1 \right\}$$

or the equivalent Orlicz norm

$$\|u\|_{L_\Phi} := \sup \left\{ \Big| \int_\Omega u(x)v(x)dx \Big| : \ v \in K_{\Phi^*}(\Omega), \ \int_\Omega \Phi^*(|v(x)|)dx \leq 1 \right\}.$$

For Orlicz spaces, the Hölder inequality reads as follows (see [23]):

$$\int_\Omega uvdx \leq 2\|u\|_{L_\Phi(\Omega)}\|u\|_{L^*_\Phi(\Omega)} \quad \text{for all } u \in L_\Phi(\Omega) \text{ and } v \in L_{\Phi^*}(\Omega).$$

The Orlicz-Sobolev space $W^1L_\Phi(\Omega)$ built upon $L_\Phi(\Omega)$ is the space defined by

$$W^1L_\Phi(\Omega) := \left\{ u \in L_\Phi(\Omega) : \ \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega), \ i = 1, 2, ..., N \right\}.$$

and it is a Banach space with respect to the norm

$$\|u\|_{1,\Phi} := \|u\|_\Phi + \||\nabla u|\|_\Phi.$$

We now introduce the Orlicz-Sobolev space $W_0^1L_\Phi(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^1L_\Phi(\Omega)$. It turns out that the space $W_0^1L_\Phi(\Omega)$ can be renormed by using as an equivalent norm

$$\|u\| := \||\nabla u|\|_\Phi.$$

For an easier manipulation of the spaces defined above, we define the numbers

$$\varphi_0 := \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} \quad \text{and} \quad \varphi^0 := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)}.$$

Throughout this paper, we assume that

$$1 < \varphi_0 \le \frac{t\varphi(t)}{\Phi(t)} \le \varphi^0 < \infty, \quad \forall t \ge 0, \tag{4}$$

which assures that Φ satisfies the Δ_2-condition, i.e.,

$$\Phi(2t) \le K\Phi(t), \quad \forall t \ge 0, \tag{5}$$

where K is a positive constant, see [22, Proposition 2.3].

In this paper, we also need the following condition

$$\text{the function } t \mapsto \Phi(\sqrt{t}) \text{ is convex for all } t \in [0, \infty). \tag{6}$$

We notice that Orlicz-Sobolev spaces, unlike the Sobolev spaces they generalize, are in general neither separable nor reflexive. A key tool to guarantee these properties is represented by the Δ_2-condition (5). Actually, condition (5) assures that both $L_\Phi(\omega)$ and $W_0^1L_\Phi(\Omega)$ are separable, see [1]. Conditions (5) and (6) assure that $L_\Phi(\Omega)$ is a uniformly convex space and thus, a reflexive Banach space (see [22]); consequently, the Orlicz-Sobolev space $W_0^1L_\Phi(\Omega)$ is also a reflexive Banach space. The following important lemma will be used throughout this paper.

Proposition 1 (see [4, 21, 22])**.** *Let $u \in W_0^1L_\Phi(\Omega)$. Then we have*

(i) $\|u\|^{\varphi^0} \le \int_\Omega \Phi(|\nabla u(x)|)dx \le \|u\|^{\varphi_0}$ *if* $\|u\| < 1$.

(ii) $\|u\|^{\varphi_0} \le \int_\Omega \Phi(|\nabla u(x)|)dx \le \|u\|^{\varphi^0}$ *if* $\|u\| > 1$.

We also find that with the help of condition (4), the Orlicz-Sobolev space $W_0^1L_\Phi(\Omega)$ is continuously embedded in the classical Sobolev space $W_0^{1,\varphi_0}(\Omega)$, as a result, $W_0^1L_\Phi(\Omega)$ is continuously and compactly embedded in the classical Lebesgue space $L^q(\Omega)$ for all $1 \le q < \varphi_0^* := \frac{N\varphi_0}{N-\varphi_0}$. On the theories of Lebesgue spaces with variable exponent used in this paper, we refer the readers to [9, 15]. Before stating and proving the main result of this paper in the next section, the rest of this section is devoted to present some examples of functions $\varphi : \mathbb{R} \to \mathbb{R}$ which are odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ and satisfy conditions (5) and (6), the readers can find them in the papers [4, 21].

Example 1.

(1) *Let* $\varphi(t) = p|t|^{p-2}t$, $t \in \mathbb{R}$, $p > 1$. *A simple computation shows that* $\varphi_0 = \varphi^0 = p$. *In this case, the corresponding Orlicz space* $L_\Phi(\Omega)$ *is the classical Lebesgue space* $L^p(\Omega)$ *while the Orlicz-Sobolev space* $W_0^1L_\Phi(\Omega)$ *is the classical Sobolev space* $W_0^{1,p}(\Omega)$. *Therefore, we obtain the p-Kirchhoff-type problems as in [3, 6, 14, 19, 20, 24, 25] and the references cited there.*

(2) *Let* $\varphi(t) = log(1+|t|^s)|t|^{p-2}t$, $t \in \mathbb{R}$, $p, s > 1$. *Then we can deduce that* $\varphi_0 = p$ *and* $\varphi^0 = p + s$.

(3) *Let* $\varphi(t) = \frac{|t|^{p-2}t}{log(1+|t|)}$ *if* $t \neq 0$, $\varphi(0) = 0$ *with* $p > 2$. *Then we can deduce that* $\varphi_0 = p - 1$ *and* $\varphi^0 = p$.

2. Main result

In this section, we will use Krasnoselskii's genus theory to get the existence of solutions for problem (1). For simplicity, we denote $X = W_0^1L_\Phi(\Omega)$, $C_+(\overline{\Omega}) := \{p: \ p \in C(\overline{\Omega}),\ p(x) > 1 \text{ for all } x \in \overline{\Omega}\}$, $p^+ = \sup_{x\in\Omega} p(x)$, $p^- = \inf_{x\in\Omega} p(x)$. In the following, when there is no misunderstanding, we always use C_i to denote positive constants. Firstly, we recall some basic notations of Krasnoselskii's genus, we refer the readers to the book [5] for details.

Let Y be a real Banach space. Let

$$\mathcal{R} = \{E \subset Y\backslash\{0\} : \ E \text{ is compact and } E = -E\}.$$

Definition 1. *Let* $E \in \mathcal{R}$ *and* $Y = \mathbb{R}^N$. *The genus* $\gamma(E)$ *of* E *is defined by*

$$\gamma(E) = \min\left\{k \geq 1;\ \textit{there exists an odd continuous mapping } \phi : E \to \mathbb{R}^k\backslash\{0\}\right\}.$$

If such a mapping ϕ *does not exist for any* $k > 0$, *we set* $\gamma(E) = \infty$.

Note that if E is a subset, which consists of finitely many pairs of points, then $\gamma(E) = 1$. Moreover, from the definition, $\gamma(\emptyset) = 0$. A typical example of a set of genus k is a set, which is homeomorphic to a $(k-1)$ dimensional sphere via an odd map.

Lemma 1. *Let* $Y = \mathbb{R}^N$ *and* $\partial\Omega$ *be the boundary of an open, symmetric and bounded subset* $\Omega \subset \mathbb{R}^N$ *with* $0 \in \Omega$. *Then we have* $\gamma(\partial\Omega) = N$.

From Lemma 1, we conclude the following remark.

Remark 1. *Let us denote by S the unit sphere in Y. Then we have*

(i) $\gamma(S^{N-1}) = N$;

(ii) *If Y is of infinite dimension and separable then $\gamma(S) = \infty$.*

Definition 2. *A function $u \in X = W_0^1 L_\Phi(\Omega)$ is said to be a weak solution of problem (1) if it holds that*

$$M\Big(\int_\Omega \Phi(|\nabla u|)dx\Big)\int_\Omega a(|\nabla u|)\nabla u \cdot \nabla v dx - \int_\Omega f(x,u)v dx = 0$$

for all $v \in X$.

Our first result is given by the following theorem.

Theorem 2. *Assume that*

(M_0) *$M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that*

$$m_1 t^{\alpha_1 - 1} \leq M(t) \leq m_2 t^{\alpha_2 - 1}$$

for all $t \in \mathbb{R}^+$, where $m_2 \geq m_1 > 0$ and $\alpha_2 \geq \alpha_1 > 1$;

(F_0) *$f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous funtion such that*

$$C_1|t|^{q(x)-1} \leq f(x,t) \leq C_2|t|^{r(x)-1}$$

for all $t \in \mathbb{R}$ and all $x \in \overline{\Omega}$, where C_1, C_2 are two positive constants and the functions $q, r \in C_+(\overline{\Omega})$ satisfy $1 < q^- \leq q^+ < r^- \leq r^+ < \varphi_0^* = \frac{N\varphi_0}{N-\varphi_0}$;

(E_1) *$\varphi_0 < r^-$ and $r^+ < \varphi_0 \alpha_1$.*

Then problem (1) has a non-trivial weak solution in X. In addition, if the following condition holds

(F_1) *$f(x,t) = -f(x,-t)$ for all $t \in \mathbb{R}$ and all $x \in \Omega$,*

then problem (1) has infinitely many weak solutions.

Let us define the energy functional $\mathcal{J} : X := W_0^1 L_\Phi(\Omega) \to \mathbb{R}$ by the formula

$$\begin{aligned}\mathcal{J}(u) &= \widehat{M}\Big(\int_\Omega \Phi(|\nabla u|)dx\Big) - \int_\Omega F(x,u)dx \\ &= \mathcal{M}(u) - \mathcal{F}(u), \quad u \in X,\end{aligned} \tag{7}$$

where

$$\begin{aligned}\mathcal{M}(u) &= \widehat{M}\Big(\int_\Omega \Phi(|\nabla u|)dx\Big), \quad \widehat{M}(t) := \int_0^t M(s)ds,\\ \mathcal{F}(u) &= \int_\Omega F(x,u)dx, \quad F(x,t) = \int_0^t f(x,s)ds.\end{aligned} \tag{8}$$

By Proposition 1 and the continuous embeddings obtained from the hypotheses (M_0), (F_0), some standard arguments assure that the functional $\mathcal{J}$ is well-defined on X and $\mathcal{J} \in C^1(X)$ with the derivative given by

$$\mathcal{J}'(u)(v) = M\Big(\int_\Omega \Phi(|\nabla u|)dx\Big)\int_\Omega a(|\nabla u|)\nabla u\cdot\nabla v dx - \int_\Omega f(x,u)v dx$$

for all $u, v \in X$. Thus, weak solutions of problem (1) are exact the critical points of the functional $\mathcal{J}$.

In order to prove Theorem 2, we shall use the following result, which was introduced by Clark, see [10].

Proposition 2. *Let $J \in C^1(Y,\mathbb{R})$ be a functional satisfying the (PS) condition. Furthermore, let us suppose that*

(i) J is bounded from below and even;

(ii) There is a compact set $K \in \mathcal{R}$ such that $\gamma(K) = k$ and $\sup_{x\in K} J(x) < J(0)$.

Then J possesses at least k pairs of distinct critical points, and their corresponding critical values are less than $J(0)$.

Lemma 3. *Suppose that (M_0), (F_0) and (E_1) are satisfied. Then we have that the following assertions hold:*

(i) The functional $\mathcal{J}$ given by formula (7) is coercive and bounded from below.

(ii) The functional $\mathcal{J}$ is weakly lower semi-continuous.

Proof. (i) By the condition (M_0) and Proposition 1, for any $u \in X$ with $\|u\| > 1$ we have

$$\begin{aligned}\mathcal{J}(u) &= \widehat{M}\Big(\int_\Omega \Phi(|\nabla u|)dx\Big) - \int_\Omega F(x,u)dx\\ &\geq \frac{m_0}{\alpha_1}\Big(\int_\Omega \Phi(|\nabla u|)dx\Big)^{\alpha_1} - \frac{C_2}{r^-}\int_\Omega |u|^{r(x)}dx\\ &\geq \frac{m_0}{\alpha_1}\|u\|^{\alpha_1\varphi_0} - \frac{C_2}{r^-}c^{r^+}\|u\|^{r^+}.\end{aligned} \tag{9}$$

Since $r^+ < \alpha_1\varphi_0$, relation (9) shows that the functional $\mathcal{J}$ is coercive and bounded from below.

(ii) Let $\{u_m\} \subset X$ be a sequence that converges weakly to u in X. Then, from the proof of [22, Lemma 4.3] we deduce that the functional

$$u \mapsto \int_\Omega \Phi(|\nabla u|)dx$$

is weakly lower semi-continuous, i.e.,

$$\int_\Omega \Phi(|\nabla u|)dx \leq \liminf_{m\to\infty} \int_\Omega \Phi(|\nabla u_m|)dx. \tag{10}$$

Combining (10) with the continuity and monotonicity of the function $\psi : \mathbb{R}^+ \to \mathbb{R}$, $t \mapsto \psi(t) = \widehat{M}(t)$, we get

$$\begin{aligned} \liminf_{m\to\infty} \mathcal{M}(u_m) &= \liminf_{m\to\infty} \widehat{M}\Big(\int_\Omega \Phi(|\nabla u_m|)dx\Big) \\ &\geq \widehat{M}\Big(\liminf_{m\to\infty} \int_\Omega \Phi(|\nabla u_m|)dx\Big) \\ &\geq \widehat{M}\Big(\int_\Omega \Phi(|\nabla u|)dx\Big) \\ &= \mathcal{M}(u). \end{aligned} \tag{11}$$

On the other hand, by (E_1), the space X is compactly embedded in the space $L^{r(x)}(\Omega)$. For this reason, using (F_0) and the Hölder inequality (see [9, 15]) we have

$$\begin{aligned} &|\mathcal{F}(u_m) - \mathcal{F}(u)| \\ &\leq \int_\Omega |F(x,u_m) - F(x,u)|dx \\ &= \int_\Omega |f(x, u + \theta_m(u_m - u))||u_m - u|dx \\ &\leq C_2 \int_\Omega |u + \theta_m(u_m - u)|^{r(x)-1}|u_m - u|dx \\ &\leq C_2 \||u + \theta_m(u_m - u)|^{r(x)-1}\|_{L^{\frac{r(x)}{r(x)-1}}(\Omega)} \|u_m - u\|_{L^{r(x)}(\Omega)}, \quad 0 < \theta_m < 1, \end{aligned}$$

which tends to 0 as $m \to \infty$. Hence,

$$\lim_{m\to\infty} \mathcal{F}(u_m) = \mathcal{F}(u). \tag{12}$$

From (11), (12) and the definition of $\mathcal{J}$, the lemma is proved.

Lemma 4. *Suppose that* (M_0), (F_0) *and* (E_1) *are satisfied. Then the functional* $\mathcal{J}$ *satisfies the* (PS) *condition.*

Proof. Let $\{u_m\} \subset X$ be a sequence such that

$$\mathcal{J}(u_m) \to \overline{c} > 0, \quad \mathcal{J}'(u_m) \to 0 \text{ in } X^*, \tag{13}$$

where X^* is the dual space of X.

Since the functional $\mathcal{J}$ is coercive, it follows from (13) that the sequence $\{u_m\}$ is bounded in X. On the other hand, by conditions (5) and (6), the Banach space X is reflexive. Thus, there exists $u \in X$ such that passing to a subsequence, still denoted by $\{u_m\}$, it converges weakly to u in X. Therefore, $\{u_m\}$ converges strongly to u in $L^{r(x)}(\Omega)$. Using the Hölder inequality we deduce that

$$\begin{aligned}
&\Big|\mathcal{F}'(u_m)(u_m - u)\Big| \\
&= \Big|\int_\Omega f(x, u_m)(u_m - u)dx\Big| \\
&\le C_2 \int_\Omega |u_m|^{r(x)-1}|u_m - u|dx \\
&\le c_2 \||u_m|^{r(x)-1}\|_{L^{\frac{r(x)}{r(x)-1}}(\Omega)} \|u_m - u\|_{L^{r(x)}(\Omega)},
\end{aligned} \tag{14}$$

which tends to 0 as $m \to \infty$.

On the other hand, by (13), we have

$$\lim_{m\to\infty} \mathcal{J}'(u_m)(u_m - u) = 0. \tag{15}$$

From (14) and (15) and the definition of the functional $\mathcal{J}'$, we get

$$\lim_{m\to\infty} \mathcal{M}'(u_m)(u_m - u) = 0. \tag{16}$$

Using Proposition 1, since $\{u_m\}$ is bounded in X, passing to a subsequence, if necessary, we may assume that

$$\int_\Omega \Phi(|\nabla u_m|)dx \to t_1 \ge 0 \text{ as } m \to \infty.$$

If $t_1 = 0$ then $\{u_m\}$ converges strongly to $u = 0$ in X and the proof is finished. If $t_1 > 0$ then since the function M is continuous, we get

$$M\Big(\int_\Omega \Phi(|\nabla u_m|)dx\Big) \to M(t_1) \text{ as } m \to \infty.$$

Thus, by (M_0), for sufficiently large m, we have

$$M\Big(\int_\Omega \Phi(|\nabla u_m|)dx\Big) \geq C_4 > 0. \tag{17}$$

From (16), (17), it follows that

$$\lim_{m\to\infty}\int_\Omega a(|\nabla u_m|)\nabla u_m \cdot (\nabla u_m - \nabla u)dx = 0.$$

Thus, using [21, Lemma 5], $\{u_m\}$ converges strongly to u in X and the functional $\mathcal{J}$ satisfies the Palais-Smale condition.

Proof of Theorem 2. Firstly, if the conditions (M_0), (F_0) and (E_1) are satisfied then it follows from Lemma 3 that problem (1) admits a weak solution as a global minimizer of the functional $\mathcal{J}$.

We now consider the case when the additional condition (F_1) is satisifed. It is clear that $\mathcal{J}$ is even. Set (see [5])

$$\begin{aligned} \mathcal{R}_k &= \{E \subset \mathcal{R} : \ \gamma(E) \geq k\}, \\ c_k &= \inf_{E\in\mathcal{R}_k}\sup_{u\in E}\mathcal{J}(u), \quad k = 1, 2, ..., \end{aligned}$$

then we have

$$-\infty < c_1 \leq c_2 \leq ... \leq c_k \leq c_{k+1} \leq ...$$

Now, we will show that $c_k < 0$ for every $k \in \mathbb{N}$. From (5) and (6), X is a reflexive and separable Banach space. For any $k \in \mathbb{N}$, we can choose a k-dimensional linear subspace X_k of X such that $X_k \subset C_0^\infty(\Omega)$. As the norms on X_k are equivalent, there exists $r_k \in (0, 1)$ such that $u \in X_k$ with $\|u_k\| \leq r_k$ implies that $\|u\|_{L^\infty(\Omega)} \leq \delta$.

Set $S_{r_k}^{(k)} = \{u \in X_k : \ \|u\| = r_k\}$. By the compactness of $S_{r_k}^{(k)}$ and the condition (F_0), there exists a constant $\eta_k > 0$ such that

$$\int_\Omega F(x, u)dx \geq \frac{C_1}{q^+}\int_\Omega |u|^{q(x)}dx \geq \eta_k \text{ for all } u \in S_{r_k}^{(k)}. \tag{18}$$

From (18), using again (M_0) and (F_0), for $u \in S_{r_k}^{(k)}$ and $t \in (0, 1)$, we have

$$\begin{aligned} \mathcal{J}(tu) &= \widehat{M}\Big(\int_\Omega \Phi(|\nabla tu|)dx\Big) - \int_\Omega F(x, tu)dx \\ &\leq \frac{m_2}{\alpha_2}\Big(\int_\Omega \Phi(|\nabla tu|)dx\Big)^{\alpha_2} - \frac{C_1}{q^+}\int_\Omega |tu|^{q(x)}dx \\ &\leq \frac{m_2}{\alpha_2}\|tu\|^{\varphi_0\alpha_2} - \frac{C_1}{q^+}\int_\Omega |tu|^{q(x)}dx \\ &\leq \frac{m_2}{\alpha_2}t^{\varphi_0\alpha_2}r_k^{\varphi_0\alpha_2} - t^{q^+}\eta_k. \end{aligned} \tag{19}$$

Because $q^+ < r^- \leq r^+ < \varphi_0\alpha_1 \leq \varphi_0\alpha_2$, we can find $t_k \in (0,1)$ and $\epsilon_k > 0$ such that $\mathcal{J}(t_k u) \leq -\epsilon_k$ for all $u \in S_{r_k}^{(k)}$, that is, $\mathcal{J}(u) \leq -\epsilon_k < 0$ for all $u \in S_{r_k}^{(k)}$.

It is clear that $\gamma(S_{t_k r_k}^{(k)}) = k$, so $c_k \leq -\epsilon_k < 0$. Finally, by Lemmas 3 and 4 and above results, we can apply Proposition 2 in order to deduce that the functional $\mathcal{J}$ admits at least k pairs of distinct critical points, and since k is arbitrary, we obtain infinitely many critical points of $\mathcal{J}$. The proof is completed.

Theorem 5. *Suppose that the conditions* (M_0), (F_0) *and*

(E_2) $r^+ < \varphi_0$

are satisfied. Then problem (1) has a weak solution. In addition, if the condition (F_1) *is satisfied then problem (1) has a sequence of weak solutions* $\{\pm u_k : k = 1, 2, ...\}$ *such that* $\mathcal{J}(\pm u_k) < 0$.

Proof. Since $r^+ < \varphi_0 < \varphi_0\alpha_1$, using (M_0), (F_0), and the similar argument as in the proof of Lemma 3, we can show the coerciveness of $\mathcal{J}$ and that $\mathcal{J}$ is weak lower semi-continuous, so $\mathcal{J}$ attains it minimum on X, that is, problem (1) has a weak solution. Moreover, by help of coerciveness, we know that $\mathcal{J}$ satisfies (PS) condition on X (see Lemma 4), and from (F_1), $\mathcal{J}$ is even.

In the rest of the proof, since we develope the same arguments which we used in the proof of the Theorem 2, we omit the details. Therefore, if we follow the similar steps as we did in (18) and (19), and consider the fact that $q^+ < r^- < \varphi_0 < \alpha_2\varphi_0$, we can find $t_k \in (0,1)$ and $\epsilon_k > 0$ such that

$$J(u) \leq -\epsilon_k < 0 \text{ for all } u \in S_{t_k r_k}^{(k)}.$$

Obviously, $\gamma(S_{t_k r_k}^{(k)}) = k$, so $c_k \leq -\epsilon_k < 0$. By Krasnoselskii's genus, each c_k is a critical value of J, hence there is a sequence of weak solutions $\{\pm u_k : k = 1, 2, ...\}$ such that $J(\pm u_k) < 0$.

Acknowledgements. The author would like to thank the referees for their helpful comments and suggestions which improved the presentation of the original manuscript. This paper was done when the author was working at the Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore, as a Research Fellow.

References

[1] R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] M. Avci, B. Cekic and R.A. Mashiyev, *Existence and multiplicity of the solutions of the $p(x)$-Kirchhoff type equation via genus theory*, Math. Methods Appl. Sci. 34(14) (2011), 1751-1759.

[3] A. Bensedik and M. Bouchekif, *On an elliptic equation of Kirchhoff-type with a potential asymptotically linear at infinity*, Math. Comput. Modelling. 49 (2009), 1089-1096.

[4] F. Cammaroto and L. Vilasi, *Multiple solutions for a non-homogeneous Dirichlet problem in Orlicz-Sobolev spaces*, Appl. Math. Comput. 218 (2012), 11518-11527.

[5] K.C. Chang, *Critical point theory and applications*, Shanghai Scientific and Technology press: Shanghai, 1986.

[6] C.Y. Chen, Y.C. Kuo and T.F. Wu, *The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions*, J. Differential Equations. 250 (2011), 1876-1908.

[7] M. Chipot and B. Lovat, *Some remarks on nonlocal elliptic and parabolic problems*, Nonlinear Anal. 30 (7) (1997), 4619-4627.

[8] N.T. Chung, *Multiple solutions for a $p(x)$-Kirchhoff-type equation with sign-changing nonlinearities*, Complex Variables and Elliptic Equations. 58 (12) (2013), 1637-1646.

[9] N.T. Chung, *Multiplicity results for a class of $p(x)$-Kirchhoff type equations with combined nonlinearities*, E. J. Qualitative Theory of Diff. Equ. Vol. 2012, No. 42 (2012), 1-13.

[10] D.C. Clarke, *A variant of the Lusternik-Schnirelman theory*, Indiana Univ. Math. J. 22 (1972), 65-74.

[11] Ph. Clément, M. Garciá-Huidobro, R. Manásevich, and K. Schmitt, *Mountain pass type solutions for quasilinear elliptic equations*, Calc.Var. 11 (2000), 33-62.

[12] Ph. Clément, B. dePagter, G. Sweers, and F. deThin, *Existence of solutions to a semilinear elliptic system through Orlicz-Sobolev spaces*, Mediterr. J. Math. 1 (2004), 241-267.

[13] F. Colasuonno and P. Pucci, *Multiplicity of solutions for $p(x)$-polyharmonic elliptic Kirchhoff equations*, Nonlinear Anal. 74 (2011), 5962-5974.

[14] F.J.S.A. Corrêa and G.M. Figueiredo, *On a p-Kirchhoff equation via Krasnoselskii's genus*, Appl. Math. Letters. 22 (2009), 819-822.

[15] G. Dai and R. Hao, *Existence of solutions for a $p(x)$-Kirchhoff-type equation*, J. Math. Anal. Appl. 359 (2009), 275-284.

[16] X.L. Fan, *On nonlocal $p(x)$-Laplacian Dirichlet problems*, Nonlinear Anal. 72 (2010), 3314-3323.

[17] F. Fang and Z. Tan, *Existence and Multiplicity of solutions for a class of quasilinear elliptic equations: An Orlicz-Sobolev setting*, J. Math. Anal. Appl. 389 (2012), 420-428.

[18] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, Germany, 1883.

[19] D. Liu, *On a p-Kirchhoff equation via fountain theorem and dual fountain theorem*, Nonlinear Anal. 72 (2010), 302-308.

[20] T.F. Ma, *Remarks on an elliptic equation of Kirchhoff type*, Nonlinear Anal. 63 (2005),1967-1977.

[21] M. Mihăilescu and D. Repovš, *Multiple solutions for a nonlinear and nonhomogeneous problem in Orlicz-Sobolev spaces*, Appl. Math. Comput. 217 (2011), 6624-6632.

[22] M. Mihăilescu and V. Rădulescu, *Neumann problems associated to nonhomogeneous differential operators in Orlicz-Sobolev spaces*, Ann. Inst. Fourier. 6 (2008), 2087-2111.

[23] M.M. Rao and Z.D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker Inc., New York, 1991.

[24] B. Ricceri, *On an elliptic Kirchhoff-type problem depending on two parameters*, Journal of Global Optimization. 46(4) 2010, 543-549.

[25] J.J. Sun and C.L. Tang, *Existence and multiplicity of solutions for Kirchhoff type equations*, Nonlinear Anal. 74 (2011), 1212-1222.

Nguyen Thanh Chung
Department of Mathematics,
Quang Binh University,
312 Ly Thuong Kiet, Dong Hoi, Quang Binh, Vietnam
email: *ntchung82@yahoo.com*