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ON THE FEKETE-SZEGÖ INEQUALITY FOR CERTAIN CLASS OF ANALYTIC FUNCTIONS

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Abstract. In the present investigation, we introduce $\mathcal{S}_g^\alpha(\varphi)$, the class of functions $f \in \mathcal{A}$ satisfying

$$1+\frac{z(f*g)'(z))}{(f*g)(z)}+\frac{z(f*g)''(z)}{(f*g)'(z)}-\frac{(1-\alpha)z^2(f*g)''(z)+z(f*g)'(z)}{(1-\alpha)z(f*g)'(z)+\alpha(f*g)(z)} \prec \varphi(z) \ \ (\alpha \geq 0)$$

where g is a fixed normalized analytic function defined in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Recently many authors have discussed Fekete-Szegö inequality for several classes defined in terms of subordination by taking $\varphi(\mathbb{D})$ symmetric with respect to the real axis and starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$. This paper is dedicated to find the sharp bounds of the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for functions in the class $\mathcal{S}_g^\alpha(\varphi)$, where φ is an analytic function with positive real part in the unit disk $\mathbb{D}$ with $\varphi(0) = 1$ and $\varphi'(0) > 0$. Further the Fekete-Szegö inequality for some special classes are derived using our main results.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Further the subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. For any two analytic functions f and g, we say that f is *subordinate* to g or g is *superordinate* to f, denoted by $f \prec g$, if there exists a Schwarz function w with $|w(z)| \leq |z|$ such that $f(z) = g(w(z))$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$.

Let φ be an analytic function with positive real part in the unit disc $\mathbb{D}$ with $\varphi(0) = 1$ and $\varphi'(0) > 0$, which maps the unit disc $\mathbb{D}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let $\mathcal{P}(\varphi)$ be the class of analytic functions p in $\mathbb{D}$ with $p(0) = 1$ and $p(\mathbb{D}) \subset \varphi(\mathbb{D})$ or equivalently $p \prec \varphi$. Denote by $\mathcal{P} := \mathcal{P}((1+z)/(1-z))$, the class of normalized analytic functions with positive real part in the unit disk $\mathbb{D}$. Let $\mathcal{S}^*(\varphi)$ be the class of functions $f \in \mathcal{S}$ such that $zf'(z)/f(z) \in \mathcal{P}(\varphi)$ and $\mathcal{K}(\varphi)$ be the class of functions $f \in \mathcal{S}$ such that $1 + zf''(z)/f'(z) \in \mathcal{P}(\varphi)$. These classes were introduced and studied by Ma and Minda [9]. The classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ reduces to several well-known classes for a suitable choice of φ. For example consider $\mathcal{S}^*((1+Az)/(1+Bz)) =: \mathcal{S}^*[A,B]$ $(-1 \leq B < A \leq 1)$, the class of Janowski [5] starlike functions. The classes $\mathcal{S}^*((1+(1-2\beta)z)/(1-z)) =: \mathcal{S}^*(\beta)$ and $\mathcal{K}((1+(1-2\beta)z)/(1-z)) =: \mathcal{K}(\beta)$ $(0 \leq \beta < 1)$ are the classes of starlike and convex functions of order β respectively, for $\beta = 0$, they reduce to the well-known classes of starlike and convex functions respectively.

In geometric function theory, finding bound for the coefficient a_n is an important problem, as it reveals the geometric properties of the corresponding function. For example, the bound for the second coefficient a_2 of functions in the class $\mathcal{S}$ gives the growth and distortion bounds as well as covering theorems. In 1933, Fekete and Szegö [4] obtained the sharp bound for $|a_3 - \mu a_2^2|$ as a function of the real parameter μ and proved that

$$|a_2^2 - \mu a_3| \leq 1 + 2\exp\left(-\frac{2\mu}{1-\mu}\right) \quad (0 \leq \mu \leq 1),$$

for functions in the class $\mathcal{S}$. Later the problem of finding sharp bound for the non-linear functional $|a_3 - \mu a_2^2|$ of any compact family of functions $f \in \mathcal{S}$ is identified as Fekete-Szegö problem. In the recent years several authors have investigated the Fekete-Szegö inequality for various subclasses of analytic functions. For ready reference one can see [1, 3, 7, 8, 11–14, 16–18].

For $f \in \mathcal{A}$ given by (1) and g given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{2}$$

the Hadamard product(or convolution) of f and g, denoted by $f * g$, is defined as

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z).$$

In this article φ is assumed to be an analytic function with positive real part in the unit disk $\mathbb{D}$, and has the Taylor's series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots ,$$

with $B_1 > 0$ and B_2 is any real number.

Definition 1. *A function $f \in \mathcal{A}$ of the form (1) is said to be in the class $\mathcal{S}_g^\alpha(\varphi)$ if it satisfies*

$$1 + \frac{z(f*g)'(z))}{(f*g)(z)} + \frac{z(f*g)''(z)}{(f*g)'(z)} - \frac{(1-\alpha)z^2(f*g)''(z) + z(f*g)'(z)}{(1-\alpha)z(f*g)'(z) + \alpha(f*g)(z)} \prec \varphi(z), \alpha \geq 0). \tag{3}$$

Note that the above class $\mathcal{S}_g^\alpha(\varphi)$, in fact generalizes several known classes a few are enlisted below:

Remark 1. *For $g(z) = z/(1-z)$, we have $\mathcal{S}_g^0(\varphi) =: \mathcal{S}^*(\varphi)$ and $\mathcal{S}_g^1(\varphi) =: \mathcal{K}(\varphi)$.*

Remark 2. *If we take $g(z) = z + \sum_{n=2}^\infty n^m z^n$, then $(f*g)(z)$ reduces to the Sălăgean [15] differential operator $\mathcal{D}^m$ defined by*

$$\mathcal{D}^m f(z) = z + \sum_{n=2}^{\infty} n^m a_n z^n, \quad m \in \{0, 1, 2, 3, \ldots\}.$$

Further, if we set $\varphi(z) = (1+z)/(1-z)$ and $g = z + \sum_{n=2}^\infty n^m z^n$ in the above Definition 1, then the class $\mathcal{S}_g^\alpha(\varphi)$ reduces to the class $\mathcal{HS}_m^*(\alpha)$, introduced by Răducanu [14], who investigated the relationship property between the classes $\mathcal{HS}_m^*(\alpha)$ and $\mathcal{S}^*$ and obtained the Fekete-Szegö inequality for the class $\mathcal{HS}_m^*(\alpha)$.

In the present investigation, we derive the Fekete-Szegö inequality for the class $\mathcal{S}_g^\alpha(\varphi)$ and deduce the same for some special classes too. The following lemmas are required in order to prove our main results. Lemma 1 of Ali *et al.* [2], is a reformulation of the corresponding result for functions with positive real part due to Ma and Minda [9].

Let Ω be the class of analytic functions w, normalized by the condition $w(0) = 0$ and satisfying $|w(z)| < 1$.

Lemma 1. *[2] If $w(z) := w_1 z + w_2 z^2 + \cdots \in \Omega$ $(z \in \mathbb{D})$, then*

$$|w_2 - t w_1^2| \leq \begin{cases} -t & (t \leq -1), \\ 1 & (-1 \leq t \leq 1), \\ t & (t \geq 1). \end{cases} \tag{4}$$

For $t < -1$ or $t > 1$, equality holds if and only if $w(z) = z$ or one of its rotations. For $-1 < t < 1$, equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if $w(z) = z(\lambda + z)/(1 + \lambda z)$ $(0 \le \lambda \le 1)$ or one of its rotations, while for $t = 1$, equality holds if and only if $w(z) = -z(\lambda+z)/(1+\lambda z)$ $(0 \le \lambda \le 1)$ or one of its rotations. Also the sharp upper bound in the inequality (4) can be improved as follows when $-1 < t < 1$:

$$|w_2 - tw_1^2| + (1+t)|w_1|^2 \le 1 \quad (-1 < t \le 0) \tag{5}$$

and

$$|w_2 - tw_1^2| + (1-t)|w_1|^2 \le 1 \quad (0 \le t < 1) \tag{6}$$

Lemma 2. *[6] (see also [11]) If $w \in \Omega$, then, for any complex number t,*

$$|w_2 - tw_1^2| \le \max\{1; |t|\}$$

and the result is sharp for the functions given by $w(z) = z^2$ or $w(z) = z$.

2. The Fekete-Szegö Inequality

We begin with the following result for the class of functions in $\mathcal{S}_g^\alpha(\varphi)$.

Theorem 3. *Let $g(z)$ be given by (2) with b_2, b_3 non-zero real numbers. Assume that $\alpha \ge 0$ and $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. If $f \in \mathcal{S}_g^\alpha(\varphi)$, then for any real number μ*

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{B_1}{2(2\alpha+1)|b_3|}\left(\frac{B_2}{B_1} - \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} - \frac{2\mu(2\alpha+1)B_1 b_3}{(1+\alpha)^2 b_2^2}\right) & \text{if } \mu \le \sigma_1; \\ \frac{B_1}{2(2\alpha+1)|b_3|} & \text{if } \sigma_1 \le \mu \le \sigma_2; \\ \frac{B_1}{2(2\alpha+1)|b_3|}\left(\frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)B_1 b_3}{(1+\alpha)^2 b_2^2} - \frac{B_2}{B_1}\right) & \text{if } \mu \ge \sigma_2, \end{cases} \tag{7}$$

where

$$\sigma_1 := \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)B_1 b_3}\left(\frac{B_2}{B_1} - \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} - 1\right)$$

and

$$\sigma_2 := \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)B_1 b_3}\left(1 + \frac{B_2}{B_1} - \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2}\right).$$

The inequality (7) is sharp.

Further, when $\sigma_1 < \mu < \sigma_2$ the above result can be improved as follows: Let

$$\sigma_3 := \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)B_1 b_3}\left(\frac{B_2}{B_1} - \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2}\right).$$

If $\sigma_1 < \mu \leq \sigma_3$, *then*

$$|a_3 - \mu a_2^2| + \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)B_1|b_3|}\left(1 - \frac{B_2}{B_1} + \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)B_1 b_3}{(1+\alpha)^2 b_2^2}\right)|a_2|^2 \leq \frac{B_1}{2(2\alpha+1)|b_3|}$$

and if $\sigma_3 \leq \mu < \sigma_2$, *then*

$$|a_3 - \mu a_2^2| + \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)B_1|b_3|}\left(1 + \frac{B_2}{B_1} - \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} - \frac{2\mu(2\alpha+1)B_1 b_3}{(1+\alpha)^2 b_2^2}\right)|a_2|^2 \leq \frac{B_1}{2(2\alpha+1)|b_3|}.$$

Proof. Since $f \in \mathcal{S}_g^\alpha(\varphi)$, there exits an analytic function $w(z) = w_1 z + w_2 z^2 + \cdots \in \Omega$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$1 + \frac{z(f*g)'(z))}{(f*g)(z)} + \frac{z(f*g)''(z)}{(f*g)'(z)} - \frac{(1-\alpha)z^2(f*g)''(z) + z(f*g)'(z)}{(1-\alpha)z(f*g)'(z) + \alpha(f*g)(z)} = \varphi(w(z)). \quad (8)$$

A calculation shows that

$$\frac{z((f*g)'(z))}{(f*g)(z)} = 1 + a_2 b_2 z + [2a_3 b_3 - a_2^2 b_2^2]z^2 + \ldots,$$

$$1 + \frac{z(f*g)''(z)}{(f*g)'(z)} = 1 + 2a_2 b_2 z + [6a_3 b_3 - 4a_2^2 b_2^2]z^2 + \cdots$$

and

$$\frac{(1-\alpha)z^2(f*g)''(z) + z(f*g)'(z)}{(1-\alpha)z(f*g)'(z) + \alpha(f*g)(z)} = 1 + (2-\alpha)a_2 b_2 z + [(6-4\alpha)a_3 b_3 - (\alpha-2)^2 a_2^2 b_2^2]z^2 + \cdots.$$

Substituting these values in (8), we obtain

$$(1+\alpha)a_2 b_2 = B_1 w_1 \quad (9)$$

and

$$2(2\alpha+1)a_3 b_3 + (\alpha^2 - 4\alpha - 1)a_2^2 b_2^2 = B_1 w_2 + B_2 w_1^2. \quad (10)$$

By using (9) and (10), we have

$$a_3 - \mu a_2^2 = \frac{B_1}{2(2\alpha+1)b_3}[w_2 - t w_1^2], \quad (11)$$

where

$$t := -\frac{B_2}{B_1} + \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)B_1b_3}{(1+\alpha)^2b_2^2}. \tag{12}$$

If $t \leq -1$, then

$$-\frac{B_2}{B_1} + \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)B_1b_3}{(1+\alpha)^2b_2^2} \leq -1,$$

which implies

$$\mu \leq \frac{(1+\alpha)^2b_2^2}{2(2\alpha+1)B_1b_3}\left(\frac{B_2}{B_1} - \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} - 1\right) := \sigma_1.$$

Now an application of Lemma 1 gives

$$|a_3-\mu a_2^2| \leq \frac{B_1}{2(2\alpha+1)|b_3|}\left(\frac{B_2}{B_1} - \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} - \frac{\mu(2\alpha+1)B_1b_3}{(1+\alpha)^2b_2^2}\right) \quad (\mu \leq \sigma_1),$$

which is nothing but the first part of assertion (7).

Next, if $t \geq 1$, then

$$-\frac{B_2}{B_1} + \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)B_1b_3}{(1+\alpha)^2b_2^2} \geq 1.$$

Which implies

$$\mu \geq \frac{(1+\alpha)^2b_2^2}{2(2\alpha+1)B_1b_3}\left(1 + \frac{B_2}{B_1} - \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2}\right) =: \sigma_2,$$

applying Lemma 1, we have

$$|a_3-\mu a_2^2| \leq \frac{B_1}{2(2\alpha+1)|b_3|}\left(\frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} + \frac{\mu(2\alpha+1)B_1b_3}{(1+\alpha)^2b_2^2} - \frac{B_2}{B_1}\right) \quad (\mu \geq \sigma_2),$$

which is essentially the third part of assertion (7).

Finally if $-1 \leq t \leq 1$, then

$$-1 \leq -\frac{B_2}{B_1} + \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)B_1b_3}{(1+\alpha)^2b_2^2} \leq 1.$$

Which shows that $\sigma_1 \leq \mu \leq \sigma_2$. Thus by an application of Lemma 1, we obtain

$$|a_3-\mu a_2^2| \leq \frac{B_1}{2(2\alpha+1)|b_3|} \quad (\sigma_1 \leq \mu \leq \sigma_2)$$

which is the second part of assertion (7). The sharpness of the result is a direct consequence of Lemma 1.

Further when $\sigma_1 < \mu < \sigma_2$ the above result can be improved as follows:
If $-1 < t \leq 0$, then

$$-1 < -\frac{B_2}{B_1} + \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)B_1 b_3}{(1+\alpha)^2 b_2^2} \leq 0$$

which implies that $\sigma_1 < \mu \leq \sigma_3$, where

$$\sigma_3 := \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)B_1 b_3}\left(\frac{B_2}{B_1} - \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1+\alpha)^2}\right).$$

Now using (5), (11) and (12), we have

$$\frac{2(2\alpha+1)b_3}{B_1}|a_3 - \mu a_2^2| + \left(1 - \frac{B_2}{B_1} + \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)B_1 b_3}{(1+\alpha)^2 b_2^2}\right)|w_1|^2 \leq 1. \tag{13}$$

Substituting the value of w_1^2 from (9) in (13) and simplifying, we have

$$|a_3 - \mu a_2^2| + \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)B_1|b_3|}\left(1 - \frac{B_2}{B_1} + \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)B_1 b_3}{(1+\alpha)^2 b_2^2}\right)|a_2|^2 \leq \frac{B_1}{2(2\alpha+1)|b_3|} \quad (\sigma_1 < \mu \leq \sigma_3).$$

Further if $0 \leq t < 1$, then $\sigma_3 \leq \mu < \sigma_2$. Now a similar computation using (6), (9) (11) and (12) gives

$$|a_3 - \mu a_2^2| + \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)B_1|b_3|}\left(1 + \frac{B_2}{B_1} - \frac{(\alpha^2 - 4\alpha - 1)B_1}{(1+\alpha)^2} - \frac{2\mu(2\alpha+1)B_1 b_3}{(1+\alpha)^2 b_2^2}\right)|a_2|^2 \leq \frac{B_1}{2(2\alpha+1)|b_3|}.$$

This completes the proof.

Remark 3. *If we set $\alpha = 1$ and $g(z) = z/(1-z)$ in Theorem 3, then we have the result [9, Theorem 3] of Ma and Minda.*

Remark 4. *By setting $\alpha = 0$ and $g(z) = z/(1-z)$ in Theorem 3, we obtain the result of Murugusundaramoorthy et al. [10, Corollary 2.2].*

Using Lemma 2 and equation (11), we deduce the following:

Theorem 4. *Let $g(z)$ be given by (2) with b_2, b_3 non zero real numbers. Assume that $\alpha \geq 0$ and $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$. If $f \in \mathcal{S}_g^\alpha(\varphi)$, then for any complex number μ*

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{2(2\alpha+1)|b_3|} \max\left\{1; \left|\frac{2\mu(2\alpha+1)B_1 b_3}{(1+\alpha)^2 b_2^2} + \frac{(\alpha^2-4\alpha-1)B_1}{(1+\alpha)^2} - \frac{B_2}{B_1}\right|\right\}.$$

From Theorem 3, we deduce the following result:

Corollary 5. *Let $g(z)$ be given by (2) with b_2, b_3 non zero real numbers. Assume that $\alpha \geq 0$ and $-1 \leq D < C \leq 1$. If $f \in \mathcal{S}_g^\alpha((1+Cz)/(1+Dz))$, then for any real number μ*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{D-C}{2(2\alpha+1)|b_3|}\left(D + \frac{(\alpha^2-4\alpha-1)(C-D)}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)(C-D)b_3}{(1+\alpha)^2 b_2^2}\right) & \text{if } \mu \leq \sigma_1; \\ \frac{C-D}{2(2\alpha+1)|b_3|} & \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\ \frac{C-D}{2(2\alpha+1)|b_3|}\left(D + \frac{(\alpha^2-4\alpha-1)(C-D)}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)(C-D)b_3}{(1+\alpha)^2 b_2^2}\right) & \text{if } \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)(D-C)b_3}\left(1 + D + \frac{(\alpha^2-4\alpha-1)(C-D)}{(1+\alpha)^2}\right)$$

and

$$\sigma_2 := \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)(C-D)b_3}\left(1 - D - \frac{(\alpha^2-4\alpha-1)(C-D)}{(1+\alpha)^2}\right).$$

The result is sharp.

The above result can be improved when $\sigma_1 < \mu < \sigma_2$ as follows:
Let

$$\sigma_3 := \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)(D-C)b_3}\left(D + \frac{(\alpha^2-4\alpha-1)(C-D)}{(1+\alpha)^2}\right).$$

If $\sigma_1 < \mu \leq \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)(C-D)|b_3|}\left(1 + D + \frac{(\alpha^2-4\alpha-1)(C-D)}{(1+\alpha)^2} + \frac{2\mu(2\alpha+1)(C-D)b_3}{(1+\alpha)^2 b_2^2}\right)|a_2|^2 \leq \frac{C-D}{2(2\alpha+1)|b_3|}$$

and if $\sigma_3 \leq \mu < \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{(1+\alpha)^2 b_2^2}{2(2\alpha+1)(C-D)|b_3|}\left(1 - D - \frac{(\alpha^2-4\alpha-1)(C-D)}{(1+\alpha)^2} - \frac{2\mu(2\alpha+1)(C-D)b_3}{(1+\alpha)^2 b_2^2}\right)|a_2|^2 \leq \frac{C-D}{2(2\alpha+1)|b_3|}.$$

By taking $D = -1$ and $C = 1$ in the above Corollary 5, we obtain the following:

Example 1. *Let $\alpha \geq 0$ and $g(z)$ be given by (2) with b_2, b_3 non zero real numbers. If $f \in \mathcal{S}_g^\alpha(\frac{1+z}{1-z})$, then for any real number μ*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{(1+\alpha)^2|b_3|}\left(\frac{3+10\alpha-\alpha^2}{2\alpha+1} - \frac{4\mu b_3}{b_2^2}\right) & \text{if } \mu \leq \sigma_1; \\ \frac{1}{(2\alpha+1)|b_3|} & \text{if } \sigma_1 \leq \mu \leq \sigma_2; \\ \frac{1}{(1+\alpha)^2|b_3|}\left(\frac{\alpha^2-10\alpha-3}{2\alpha+1} + \frac{4\mu b_3}{b_2^2}\right) & \text{if } \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 := \frac{(1+4\alpha-\alpha^2)b_2^2}{2(2\alpha+1)b_3} \text{ and } \sigma_2 := \frac{(3\alpha+1)b_2^2}{(2\alpha+1)b_3}.$$

The result can be improved when $\sigma_1 \leq \mu \leq \sigma_2$ as follows: Let

$$\sigma_3 := \frac{(3+10\alpha-\alpha^2)b_2^2}{4(2\alpha+1)b_3}.$$

If $\sigma_1 \leq \mu \leq \sigma_3$, then

$$|a_3 - \mu a_2^2| + \frac{b_2^2}{2|b_3|}\left(\frac{\alpha^2-4\alpha-1}{2\alpha+1} + \frac{2\mu b_3}{b_2^2}\right)|a_2|^2 \leq \frac{1}{(2\alpha+1)|b_3|}$$

and if $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{b_2^2}{|b_3|}\left(\frac{3\alpha+1}{2\alpha+1} - \frac{\mu b_3}{b_2^2}\right)|a_2|^2 \leq \frac{1}{(2\alpha+1)|b_3|}.$$

The result is sharp.

Remark 5. *If we take $g(z) = z + \sum_{n=2}^{\infty} n^m z^n \quad (m \in \{0, 1, 2, 3, \ldots\})$ in Example 1 it reduces to the result [14, Theorem 2] of Răducanu.*

Taking $\varphi(z) = (1 + Cz)/(1 + Dz), -1 \leq D < C \leq 1$ in Theorem 4, we deduce the following result:

Corollary 6. *Let $\alpha \geq 0$ and $g(z)$ be given by (2) with b_2, b_3 non zero real numbers. If $f \in \mathcal{S}_g^\alpha\left(\frac{1+Cz}{1+Dz}\right)$, then for any complex number μ*

$$|a_3 - \mu a_2^2| \leq \frac{C-D}{2(2\alpha+1)|b_3|} \max\left\{1; \left|\frac{2\mu(2\alpha+1)(C-D)b_3}{(1+\alpha)^2 b_2^2} + \frac{(\alpha^2-4\alpha-1)(C-D)}{(1+\alpha)^2} + D\right|\right\}.$$

Remark 6. *If we take* $g(z) = z + \sum_{n=2}^{\infty} n^m z^n$, $D = -1$ *and* $C = 1$ *in the above Corollary 6, we have the following result [14, Theorem 3] of Răducanu:*

Let $\alpha \geq 0$. *If* $f \in \mathcal{HS}_m^*(\alpha)$, *then for any complex number* μ

$$|a_3 - \mu a_2^2| \leq \frac{1}{3^m(1+2\alpha)} \max\left\{1; \frac{\left|2^{2m-1}(\alpha^2 - 10\alpha - 3) + 2.3^m(1+2\alpha)\mu\right|}{2^{2m-1}(1+\alpha)^2}\right\}.$$

Remark 7. *If we set* $D = -1, C = 1$ *and* $g(z) = z/(1-z)$ *in Corollary 6, then for* $\alpha = 0$, *we have the following result [6, Theorem 1](see also [16]):*

Let $f \in \mathcal{S}^*$. *Then for any complex number* μ

$$|a_3 - \mu a_2^2| \leq \max\{1; |4\mu - 3|\}.$$

Setting $\alpha = 1, D = -1, C = 1$ *and* $g(z) = z/(1-z)$ *in Corollary 6, we obtain the following result [6, Corollary 1] due to Keogh and Merkes: Let* $f \in \mathcal{K}$, *then for any complex number* μ

$$|a_3 - \mu a_2^2| \leq \max\left\{\frac{1}{3}; |\mu - 1|\right\}.$$

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