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IDEAL CONVERGENT SEQUENCE SPACES DEFINED BY MUSIELAK-ORLICZ FUNCTION OVER N-NORMED SPACES

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Abstract. In the present paper we introduce sequence spaces using ideal convergence and Musielak-Orlicz function $\mathcal{M} = (M_k)$ over n-normed spaces and examine some properties of the resulting sequence spaces.

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1. Introduction and Preliminaries

The notion of ideal convergence was first introduced by P. Kostyrko [8] as a generalization of statistical convergence which was further studied in topological spaces by Das, Kostyrko, Wilczynski and Malik [2]. More applications of ideals can be seen in ([2], [3]). The concept of 2-normed spaces was initially developed by Gähler[4] in the mid of 1960's, while that of n-normed spaces one can see in Misiak[11]. As an interesting non linear generalization of a normed linear space which was subsequently studied by many others ([5],[17]) and references therein. Recently a lot of activities have been started to study sumability, sequence spaces and related topics in these non linear spaces (see [6],[18]). In particular Sahiner [18] combined these two concepts and investigated ideal sumability in these spaces and introduced certain sequence spaces using 2-norm.
We continue in this direction, by using Musielak-Orlicz function, generalized sequences and also ideals we introduce I-convergence of generalized sequences with respect to Musielak-Orlicz function in n-normed spaces.
Let $n \in \mathbb{N}$ and X be a real linear space of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

1. $||x_1, x_2, \cdots, x_n|| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;

2. $||x_1, x_2, \cdots, x_n||$ is invariant under permutation;

3. $||\alpha x_1, x_2, \cdots, x_n|| = |\alpha| \ ||x_1, x_2, \cdots, x_n||$ for any $\alpha \in \mathbb{R}$, and

4. $||x + x', x_2, \cdots, x_n|| \leq ||x, x_2, \cdots, x_n|| + ||x', x_2, \cdots, x_n||$

is called an n-norm on X, and the pair $(X, ||\cdot, \cdots, \cdot||)$ is called an n-normed space. For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $||x_1, x_2, \cdots, x_n||_E$ = the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$||x_1, x_2, \cdots, x_n||_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, ||\cdot, \cdots, \cdot||)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $||\cdot, \cdots, \cdot||_\infty$ on X^{n-1} defined by

$$||x_1, x_2, \cdots, x_{n-1}||_\infty = \max\{||x_1, x_2, \cdots, x_{n-1}, a_i|| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.
A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to converge to some $L \in X$ if

$$\lim_{k \to \infty} ||x_k - L, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, ||\cdot, \cdots, \cdot||)$ is said to be Cauchy if

$$\lim_{k,p \to \infty} ||x_k - x_p, z_1, \cdots, z_{n-1}|| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.
Let $(X, ||\cdot, \cdots, \cdot||)$ be a n-normed space. Recall that a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of X is called statistically convergent to $x \in X$ if the set $A(\epsilon) = \Big\{n \in \mathbb{N} : ||x_n - x|| \geq \epsilon\Big\}$ has natural density zero for each $\epsilon > 0$.
A family $\mathcal{I} \subset 2^Y$ of subsets of a non empty set Y is said to be an ideal in Yif

1. $\phi \in \mathcal{I}$

2. $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$

3. $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$,

while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$ see [5]. Given $\mathcal{I} \subset 2^{\mathbb{N}}$ be a non trivial ideal in $\mathbb{N}$. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be I-convergent to $x \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \Big\{n \in \mathbb{N} : ||x_n - x|| \geq \epsilon\Big\}$ belongs to $\mathcal{I}$ see [8].
Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$, for all $x \in X$,
2. $p(-x) = p(x)$, for all $x \in X$,
3. $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$,
4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda \ as\ n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0 \ as\ n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0 \ as\ n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [19], Theorem 10.4.2, P-183).
An orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.
Lindenstrauss and Tzafriri [9] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty, \text{ for some } \rho > 0\Big\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

It is shown in [9] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$, and for $L > 1$.
A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function see ([10],[14]). A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\},\ k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace

$h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for some } \ c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w : I_{\mathcal{M}}(cx) < \infty \ \text{ for all } \ c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\Big\{k > 0 : I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) : k > 0\Big\}.$$

The notion of difference sequence spaces was introduced by Kizmaz [7], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_o(\Delta)$. The notion was further generalized by Et and Colak [1] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_o(\Delta^n)$. Let m, n be non-negative integers, then for $Z = c, c_0$ and l_∞, we have sequence spaces

$$Z(\Delta_m^n) = \{x = (x_k) \in w : (\Delta_m^n x_k) \in Z\}$$

for $Z = c, c_0$ and l_∞ where $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m})$ and $\Delta_m^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_m^n x_k = \sum_{v=0}^{n} (-1)^v \begin{pmatrix} n \\ v \end{pmatrix} x_{k+mv}.$$

Taking $m = 1$, we get the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$ studied by Et and Colak [1]. Taking $m = n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz [7]. For more details about sequence spaces(see [12],[13],[15], [16])and references therein.

Let $\Lambda = (\lambda_n)$ be non-decreasing sequence of positive numbers tending to infinity such that $\lambda_{n+1} \geq \lambda_n + 1$, $\lambda_1 = 0$. Let I be an admissible ideal of $\mathbb{N}$, $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $(X, ||\cdot, \cdots, \cdot||)$ is a n-normed space. Further, suppose $p = (p_k)$ is a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. By $S(n - X)$ we denote the space of all sequences defined over $(X, ||\cdot, \cdots, \cdot||)$. Now we define the following sequence spaces in this paper:

$$W^I\Big(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big) =$$

$$\Big\{x \in S(n-X) : \forall\, \epsilon > 0,\ \Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[M_k\Big(||\frac{\Delta^m x_k - L}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}$$

$$\geq \epsilon\Big\} \in I \text{ for some } \rho > 0\ \ L \in X \text{ and each } z_1, \cdots, z_{n-1} \in X\Big\},$$

$$W_0^I\Big(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big) =$$

$$\Big\{x \in S(n-X) : \forall\, \epsilon > 0,\ \Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[M_k\Big(||\frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I$$

$$\text{for some } \rho > 0 \text{ and each } z_1, \cdots, z_{n-1} \in X\Big\},$$

$$W_\infty\Big(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big) =$$

$$\Big\{x \in S(n-X) : \exists\, K > 0 \text{ such that } \sup_{n \in \mathbb{N}} \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[M_k\Big(||\frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}$$

$$\leq K \text{ for some } \rho > 0 \text{ and each } z_1, \cdots, z_{n-1} \in X\Big\}$$

and

$$W_\infty^I\Big(\lambda, \mathcal{M}, \Delta^m, u, p, ||., \cdots, .||\Big) =$$

$$\Big\{x \in S(n-X) : \exists\, K > 0 \text{ such that } \Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[M_k\Big(||\frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}$$

$$\geq K\Big\} \in I \text{ for some } \rho > 0 \text{ and each } z_1, \cdots, z_{n-1} \in X\Big\}.$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $D = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \le \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The main purpose of this paper is to introduce some sequence spaces using ideal convergence for Musielak-Orlicz function $\mathcal{M} = (M_k)$ over n-normed spaces. We study some relevant algebraic and topological properties. Further some inclusion relations among these spaces are also examined.

2. Main Results

Theorem 1. *Let $\mathcal{M} = (M_k)$ be Musielak-Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers, $u = (u_k)$ be a sequence of strictly positive real numbers and I be an admissible ideal of $\mathbb{N}$. Then $W^I\Big(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big)$, $W_0^I\Big(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big)$, $W_\infty\Big(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big)$ and $W_\infty^I\Big(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big)$ are linear spaces over the real field $\mathbb{R}$.*

Proof. Let $x = (x_k), y = (y_k) \in W^I\Big(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big)$ and $\alpha, \beta \in \mathbb{R}$. Then there exist positive integers ρ_1 and ρ_2 such that

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[M_k\Big(||\frac{\Delta^m x_k - L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \ge \epsilon\Big\} \in I$$

and

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[M_k\Big(||\frac{\Delta^m y_k - L}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \ge \epsilon\Big\} \in I.$$

Since $||\cdot, \cdots, \cdot||$ is a n-norm and $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function.

$$\begin{aligned}
&\frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[M_k\Big(||\frac{\Delta^m(\alpha x_k + \beta y_k - L)}{|\alpha|\rho_1 + |\beta|\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&\le D\frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[\frac{\rho_1|\alpha|}{(|\alpha|\rho_1 + |\beta|\rho_2)} M_k\Big(||\frac{\Delta^m x_k - L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&+ D\frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[\frac{\rho_2|\beta|}{(|\alpha|\rho_1 + |\beta|\rho_2)} M_k\Big(||\frac{\Delta^m y_k - L}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&\le DF\frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[M_k\Big(||\frac{\Delta^m x_k - L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \\
&+ DF\frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[M_k\Big(||\frac{\Delta^m y_k - L}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k},
\end{aligned}$$

where $F = \max\Big[1, \Big(\frac{\rho_1|\alpha|}{(|\alpha|\rho_1+|\beta|\rho_2)}\Big)^H, \Big(\frac{\rho_2|\beta|}{(|\alpha|\rho_1+|\beta|\rho_2)}\Big)^H\Big]$. From the above inequality, we get $\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n} u_k\Big[M_k\Big(||\frac{\Delta^m(\alpha x_k+\beta y_k)-L}{|\alpha|\rho_1+|\beta|\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\}$

$$\subseteq \Big\{n \in \mathbb{N} : DF\frac{1}{\lambda_n}\sum_{k\in I_n} u_k\Big[M_k\Big(||\frac{\Delta^m x_k - L}{\rho_1}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\}$$
$$\cup \Big\{n \in \mathbb{N} : DF\frac{1}{\lambda_n}\sum_{k\in I_n} u_k\Big[M_k\Big(||\frac{\Delta^m y_k - L}{\rho_2}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \geq \frac{\epsilon}{2}\Big\}.$$

Two sets on the right hand side belong to I and this completes the proof.

Similarly, we can prove that $W_0^I\Big(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big)$, $W_\infty\Big(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big)$ and $W_\infty^I\Big(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big)$ are linear spaces.

Theorem 2. *Let $\mathcal{M} = (M_k)$ be Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. For any fixed $n \in \mathbb{N}$, $W_\infty\Big(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big)$ is a paranormed space with*

$$g_n(x) = \inf\Big\{\rho^{\frac{p_n}{H}} : \rho > 0 \ : \ \sup_k \frac{1}{\lambda_n}\sum_{k\in I_n} u_k\Big[M_k\Big(||\frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq 1, \quad \forall z_1, \cdots, z_{n-1} \in X\Big\}.$$

Proof. It is clear that $g_n(x) = g_n(-x)$. Since $M_k(0) = 0$, we get $\inf\{\rho^{\frac{p_n}{H}}\} = 0$ for $x = 0$ therefore, $g_n(0) = 0$. For $x = (x_k)$, $y = (y_k) \in W_\infty\Big(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big)$. Let

$$B(x) = \Big\{\rho > 0 : \sup_k \frac{1}{\lambda_n}\sum_{k\in I_n} u_k\Big[M_k\Big(||\frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq 1, \forall z_1, \cdots, z_{n-1} \in X\Big\},$$

$$B(y) = \Big\{\rho > 0 : \sup_k \frac{1}{\lambda_n}\sum_{k\in I_n} u_k\Big[M_k\Big(||\frac{\Delta^m y_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq 1, \forall z_1, \cdots, z_{n-1} \in X\Big\}.$$

Suppose $\rho_1 \in B(x)$ and $\rho_2 \in B(y)$. If $\rho = \rho_1 + \rho_2$, then we have

$$\sup_k \frac{1}{\lambda_n} \sum_{k \in I_n} u_k M_k\Big(||\frac{\Delta^m(x_k + y_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)$$

$$\begin{aligned} &\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big) \sup_k \frac{1}{\lambda_n} \sum_{k \in I_n} u_k M_k\Big(||\frac{\Delta^m x_k}{\rho_1}, z_1, \cdots, z_{n-1}||\Big) \\ &+ \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big) \sup_k \frac{1}{\lambda_n} \sum_{k \in I_n} u_k M_k\Big(||\frac{\Delta^m y_k}{\rho_2}, z_1, \cdots, z_{n-1}||\Big). \end{aligned}$$

Thus, $\sup_k \frac{1}{\lambda_n} \sum_{k \in I_n} u_k M_k\Big(||\frac{\Delta^m(x_k + y_k)}{\rho_1+\rho_2}, z_1, \cdots, z_{n-1}||\Big)^{p_k} \leq 1$ and

$$\begin{aligned} g_n(x+y) &\leq \inf\Big\{(\rho_1+\rho_2)^{\frac{p_n}{H}} : \rho_1 \in B(x),\ \rho_2 \in B(y)\Big\} \\ &\leq \inf\Big\{\rho_1^{\frac{p_n}{H}} : \rho_1 \in B(x)\Big\} + \inf\Big\{\rho_2^{\frac{p_n}{H}} : \rho_2 \in B(y)\Big\} \\ &= g_n(x) + g_n(y). \end{aligned}$$

Let $\sigma^s \to \sigma$ where $\sigma, \sigma^s \in \mathbb{C}$ and $g_n(x^s - x) \to 0$ as $s \to \infty$. We show that $g_n(\sigma^s x^s - \sigma x) \to 0$ as $s \to \infty$. For

$$B(x^s) = \Big\{\rho_s > 0 : \sup_k \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[M_k\Big(||\frac{\Delta^m(x^s_k)}{\rho_s}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq 1,$$

$$\forall z_1, \cdots, z_{n-1} \in X\Big\},$$

$$B(x^s - x) = \Big\{\rho'_s > 0 : \sup_k \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[M_k\Big(||\frac{\Delta^m(x^s_k - x_k)}{\rho'_s}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \leq 1,$$

$$\forall z_1, \cdots, z_{n-1} \in X\Big\}.$$

If $\rho_s \in B(x^s)$ and $\rho'_s \in B(x^s - x)$ then we observe that

$$
\begin{aligned}
&\tfrac{1}{\lambda_n}\sum_{k\in I_n} u_k M_k\Big(||\frac{\Delta^m(\sigma^s x_k^s-\sigma x_k)}{\rho_s|\sigma^s-\sigma|+\rho_s^{'}|\sigma|},z_1,\cdots,z_{n-1}||\Big)\\
&\leq \frac{1}{\lambda_n}\sum_{k\in I_n} u_k M_k\Big(||\frac{\Delta^m(\sigma^s x_k^s-\sigma x_k^s)}{\rho_s|\sigma^s-\sigma|+\rho_s^{'}|\sigma|},z_1,\cdots,z_{n-1}||\\
&+ ||\frac{(\sigma x_k^s-\sigma x_k)}{\rho_s|\sigma^s-\sigma|+\rho_s^{'}|\sigma|},z_1,\cdots,z_{n-1}||\Big)\\
&\leq \frac{|\sigma^s-\sigma|\rho_s}{\rho_s|\sigma^s-\sigma|+\rho_s^{'}|\sigma|}\frac{1}{\lambda_n}\sum_{k\in I_n} u_k M_k\Big(||\frac{(\Delta^m x_k^s)}{\rho_s},z_1,\cdots,z_{n-1}||\Big)\\
&+ \frac{|\sigma|\rho_s^{'}}{\rho_s|\sigma^s-\sigma|+\rho_s^{'}|\sigma|}\frac{1}{\lambda_n}\sum_{k\in I_n} u_k M_k\Big(||\frac{\Delta^m(x_k^s-x_k)}{\rho_s^{'}},z_1,\cdots,z_{n-1}||\Big).
\end{aligned}
$$

From the above inequality, it follows that

$$
\frac{1}{\lambda_n}\sum_{k\in I_n} u_k\Big(M_k\Big(||\frac{\Delta^m(\sigma^s x_k^s-\sigma x_k)}{\rho_s|\sigma^s-\sigma|+\rho_s^{'}|\sigma|},z_1,\cdots,z_{n-1}||\Big)\Big)^{p_k}\leq 1
$$

and consequently,

$$
\begin{aligned}
g_n(\sigma^m x^s-\sigma x) &\leq \inf\Big\{\Big(\rho_s|\sigma^s-\sigma|+\rho_s^{'}|\sigma|\Big)^{\frac{p_n}{H}}:\rho_s\in B(x^s),\rho_s^{'}\in B(x^s-x)\Big\}\\
&\leq (|\sigma^s-\sigma|)^{\frac{p_n}{H}}\inf\Big\{\rho^{\frac{p_n}{H}}:\rho_s\in B(x^s)\Big\}\\
&+ (|\sigma|)^{\frac{p_n}{H}}\inf\Big\{(\rho_s^{'})^{\frac{p_n}{H}}:\rho_s^{'}\in B(x^s-x)\Big\}\\
&\longrightarrow 0 \ \text{as}\ s\longrightarrow\infty.
\end{aligned}
$$

This completes the proof.

Theorem 3. *Let $\mathcal{M}=(M_k)$, $\mathcal{M}'=(M_k')$, $\mathcal{M}''=(M_k'')$ are Musielak-Orlicz functions. Then we have*
(i) $W_0^I\Big(\lambda,\mathcal{M}',\Delta^m,u,p,||\cdot,\cdots,\cdot||\Big)\subseteq W_0^I\Big(\lambda,\mathcal{M}\circ\mathcal{M}',\Delta^m,u,p,||\cdot,\cdots,\cdot||\Big)$ *provided that* $H_0=\inf p_k>0$.
(ii) $W_0^I\Big(\lambda,\mathcal{M}',\Delta^m,u,p,||\cdot,\cdots,\cdot||\Big)\cap W_0^I\Big(\lambda,\mathcal{M}'',\Delta^m,u,p,||\cdot,\cdots,\cdot||\Big)$

$$
\subseteq W_0^I\Big(\lambda,\mathcal{M}'+\mathcal{M}'',\Delta^m,u,p,||\cdot,\cdots,\cdot||\Big).
$$

Proof. (i) For given $\epsilon > 0$, first choose $\epsilon_0 > 0$ such that $\max\{\epsilon_0^H, \epsilon_0^{H_0}\} < \epsilon$. Now using the continuity of (M_k). Choose $0 < \delta < 1$ such that $0 < t < \delta$, this implies that $M_k(t) < \epsilon_0$. Let $x = (x_k) \in W_0\Big(\lambda, \mathcal{M}', \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big)$. Now from the definition

$$B(\delta) = \Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[M'_k\Big(||\frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \geq \delta^H\Big\} \in I.$$

Thus, if $n \notin B(\delta)$ then

$$\frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[M'_k\Big(||\frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} < \delta^H.$$

$$\Rightarrow \quad \sum_{k \in I_n} u_k \Big[M'_k\Big(||\frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} < \lambda_n \delta^H.$$

$$\Rightarrow \quad u_k \Big[M'_k\Big(||\tfrac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} < \delta^H \text{ for all } k \in I_n.$$

Thus, $u_k \Big[M'_k\Big(||\tfrac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big] < \delta$ for all $k \in I_n$. Hence,

$$u_k M_k\Big(M'_k\Big(||\frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big) < \epsilon_0 \ \forall \ k \in I_n$$

which consequently implies that

$$\begin{aligned} \sum_{k \in I_n} u_k \Big[M_k\Big(M'_k\Big(||\frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} &< \lambda_n \max\{\epsilon_0^H, \epsilon_0^{H_0}\} \\ &< \lambda_n \epsilon. \end{aligned}$$

Thus,

$$\frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[M_k\Big(M'_k\Big(||\frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} < \epsilon.$$

This shows that

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[M_k\Big(M'_k\Big(||\frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big)\Big]^{p_k} \geq \epsilon\Big\} \subset B(\delta)$$

and thus belongs to I. This proves the result.

(ii) Let $x = (x_k) \in W_0^I\Big(\lambda, \mathcal{M}', \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big) \cap W_0^I\Big(\lambda, \mathcal{M}'', \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big)$. Then the fact,

$$\frac{1}{\lambda_n} u_k \Big[(M'_k + M''_k)\Big(||\tfrac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}$$
$$\leq D \frac{1}{\lambda_n} u_k \Big[M'_k\Big(||\tfrac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} + D \frac{1}{\lambda_n} u_k \Big[M''_k\Big(||\tfrac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}$$

completes the proof of the theorem.

Theorem 4. *The sequence spaces* $W_0^I\Big(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big)$ *and* $W_\infty^I\Big(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big)$ *are solid.*

Proof. Let $x = (x_k) \in W_0^I\Big(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big)$, let (α_k) be a sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$. Then we have

$$\Big\{n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} u_k \Big[M_k\Big(||\frac{\Delta^m(\alpha_k x_k)}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k}\Big\}$$

$$\subset \Big\{n \in \mathbb{N} : \frac{C}{\lambda_n} \sum_{k \in I_n} u_k \Big[M_k\Big(||\frac{\Delta^m x_k}{\rho}, z_1, \cdots, z_{n-1}||\Big)\Big]^{p_k} \geq \epsilon\Big\} \in I,$$

where $C = \max\{1, |\alpha_k|^H\}$. Hence $(\alpha_k x_k) \in W_0^I\Big(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big)$ for all sequences of scalars α_k with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$ whenever $(x_k) \in W_0^I\Big(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot \cdots, \cdot||\Big)$.

Similarly, we can prove that $W_\infty^I\Big(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big)$ is a solid space.

Theorem 5. *The sequence spaces* $W_0^I\Big(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big)$ *and* $W_\infty^I\Big(\lambda, \mathcal{M}, \Delta^m, u, p, ||\cdot, \cdots, \cdot||\Big)$ *are monotone.*

Proof. It is obvious.

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