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# SANDWICH-TYPE THEOREMS FOR MULTIVALENT MEROMORPHIC FUNCTIONS ASSOCIATED WITH CERTAIN TRANSFORMS

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Abstract. In the present paper, the author investigates some subordination and superordination results for certain subclasses of multivalent meromorphic functions defined through the combinations and iterations of a meromorphic analogue of the Cho-Kwon-Srivastava operator for normalized analytic functions. Sandwich-type theorems for function belonging to these classes and some consequences are also obtained.

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## 1. Introduction and Definitions

Let $\sum_p$ denote the class of functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \quad (p \in \mathbb{N} = \{1, 2, 3, ....\}) \tag{1}$$

which are analytic and $p$-valent in the punctured unit disk $\mathbb{U}^* := \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$.

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ be the linear space of all analytic functions in the open unit disk $\mathbb{U}$ and let $\mathcal{H}[a, p]$ denote the subclass of $\mathcal{H}(\mathbb{U})$ consisting of functions of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + .... \quad (a \in \mathbb{C},\ p \in \mathbb{N}).$$

Let the functions $f$ and $g$ be members of the analytic function class $\mathcal{H}$. We say that the function $f$ is subordinate to $g$, written as $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists a Schwarz function $w$, which (by definition) is analytic in $\mathbb{U}$ with $w(0) =$

0 and $|w(z)| < 1$ such that$f(z) = g(w(z))$ $(z \in \mathbb{U})$. It follows from this definition that

$$f(z) \prec g(z) \Longrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

In particular, if the function $g$ is univalent in $\mathbb{U}$, then we have the following equivalence (see [1, 7, 8]):

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Now, we mention some definitions from the theory of differential subordination given by Miller and Mocanu [8, 9].

**Definition 1.** (see [8]) *Let $\phi : \mathbb{C}^2 \longrightarrow \mathbb{C}$ and let $h$ be univalent in $\mathbb{U}$. If $p$ is analytic in $\mathbb{U}$ and satisfies the following:*

$$\phi\left(p(z), zp'(z)\right) \prec h(z) \quad (z \in \mathbb{U}), \tag{2}$$

*then $p$ is called a solution of the first order differential subordination (2). The univalent function $q$ is called a* dominant *of the solutions of the differential subordination (2) or, more simply, a dominant if $p \prec q$ for every $p$ satisfying (2). An univalent dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants $q$ of (2) is said to be the best dominant.*

**Definition 2.** (see [9]) *Let $\varphi : \mathbb{C}^2 \longrightarrow \mathbb{C}$ and let $h$ be analytic in $\mathbb{U}$. If $p$ and $\varphi(p(z), zp'(z))$ are univalent in $\mathbb{U}$ and satisfy the differential superordination:*

$$h(z) \prec \varphi(p(z), zp'(z)) \quad (z \in \mathbb{U}), \tag{3}$$

*then $p$ is called a solution of the first order differential superordination (3). An analytic function $q$ is called a subordinant of the solutions of the differential superordination (3) or, more simply, a subordinant if $q \prec p$, for all $p$ satisfying (3). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants $q$ of (3) is said to be the best subordinant.*

**Definition 3.** (see [8], Definition 2.2b, p. 21; also see [9], Definition 2, p. 817 )
*We denote by $Q$ the class of functions $f$ that are analytic and injective on $\overline{\mathbb{U}} \setminus E(f)$, where*

$$E(f) = \left\{\zeta \in \partial\mathbb{U} : \lim_{z \longrightarrow \zeta} f(z) = \infty\right\},$$

*and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \setminus E(f)$.*

Let $f,\ g \in \sum_p$, where $f$ is given by (1) and the function $g$ is defined by

$$g(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p} \quad (p \in \mathbb{N}; z \in \mathbb{U}^*),$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$(f * g)(z) = \frac{z^p f(z) \star z^p g(z)}{z^p} = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p} z^{k-p} = (g * f)(z) \quad (z \in \mathbb{U}^*)$$

where $\star$ denotes the usual Hadamard product (or convolution ) of analytic functions.

Liu and Srivastava [6] defined the function $\phi_p(a, c; z)$ by

$$\phi_p(a, c; z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{k-p} \ (z \in \mathbb{U}^*;\ c \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ \mathbb{Z}_0^- := \{0, -1, -2\dots \cdots\}) \quad (4)$$

where $(\lambda)_n$ is the *Pochhammer symbol (or shifted factorial)* given by

$$(\lambda)_n := \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)\dots\dots(\lambda+n-1) & (n \in \mathbb{N}). \end{cases}$$

They defined the operator $\mathcal{L}(a, c) : \sum_p \longrightarrow \sum_p$ as

$$\mathcal{L}(a, c) f(z) = \phi_p(a, c; z) * f(z) \quad (z \in \mathbb{U}^*).$$

Corresponding to the function $\phi_p(a, c; z)$, Mishra et al. [10] ( see also [11, 12]) defined the function $\phi_p^\dagger(a, c; z)$, the generalized multiplicative inverse of $\phi_p(a, c; z)$ given by the relation

$$\phi_p(a, c; z) * \phi_p^\dagger(a, c; z) = \frac{1}{z^p(1-z)^{\lambda+p}} \quad (a, c \in \mathbb{C} \setminus \mathbb{Z}_0^-, \lambda > -p; z \in \mathbb{U}^*). \quad (5)$$

They defined the operator $\mathcal{L}_p^\lambda(a, c) : \sum_p \longrightarrow \sum_p$ as

$$\mathcal{L}_p^\lambda(a, c) f(z) = \phi_p^\dagger(a, c; z) * f(z) \quad (z \in \mathbb{U}^*). \quad (6)$$

Therefore, it follows from (5) and (6) that

$$\mathcal{L}_p^\lambda(a, c) f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} \frac{(\lambda+p)_k (c)_k}{(a)_k (1)_k} a_{k-p} z^{k-p} \quad (z \in \mathbb{U}^*). \quad (7)$$

Note that, the holomorphic analogue of the function $\phi_p^{\dagger}(a, c; z)$ and the corresponding transform is popularly known as the Cho-Kwon- Srivastava operator in literature (see[2, 13]).

For $f \in \sum_p$ given by (1), set

$$C^0 f(z) = f(z),$$

$$C^{(t,1)} f(z) = (1-t)f(z) + \frac{tz(-f(z))'}{p} = \frac{1}{z^p} + \sum_{k=1}^{\infty} \left(\frac{p-kt}{p}\right) a_{k-p} z^{k-p} := C^t f(z) \quad (t \geq 0)$$

and for $m = 2, 3 \cdots$

$$C^{(t,m)} f(z) = C^t \left(C^{(t,m-1)} f(z)\right) = \frac{1}{z^p} + \sum_{k=1}^{\infty} \left(\frac{p-kt}{p}\right)^m a_{k-p} z^{k-p} \ (z \in \mathbb{U}^*). \quad (8)$$

Similarly, the $n$-times *superimpositions* of the operator $\mathcal{L}_p^{\lambda}(a, c)$ is defined as follows;

$$\mathcal{L}_p^{(\lambda,0)}(a, c) f(z) = f(z)$$

and for $n = 1, 2, 3 \cdots$

$$\mathcal{L}_p^{(\lambda,n)}(a, c) f(z) = \mathcal{L}_p^{\lambda}(a, c) \left(\mathcal{L}_p^{(\lambda,n-1)}(a, c) f(z)\right) = \frac{1}{z^p} + \sum_{k=1}^{\infty} \left(\frac{(\lambda+p)_k (c)_k}{(a)_k (1)_k}\right)^n a_{k-p} z^{k-p}. \quad (9)$$

Note that for $n = 1$ and $p = 1$, we use the notation

$$\mathcal{L}_1^{(\lambda,1)}(a, c) f(z) = \mathcal{L}^{\lambda}(a, c) f(z).$$

Recently, Mishra et al. [10] ( see also [11, 12]) introduced and studied the operator

$$\mathcal{I}_{\lambda,p}^{n,m}(a, c) : \sum_p \longrightarrow \sum_p \quad (m, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, t \geq 0)$$

as the compsition of the operator $\mathcal{L}_p^{(\lambda,n)}(a, c)$ and $C^{(t,m)}$. Thus, for $f \in \sum_p$ given by (1), we have

$$\begin{aligned} \mathcal{I}_{\lambda,p}^{n,m}(a, c) f(z) &= \mathcal{L}_p^{(\lambda,n)}(a,\ c) \mathcal{C}^{(t,m)} f(z) \\ &= \frac{1}{z^p} + \sum_{k=1}^{\infty} \left(\frac{(\lambda+p)_k (c)_k}{(a)_k (1)_k}\right)^n \left(\frac{p-kt}{p}\right)^m a_{k-p} z^{k-p}, \quad (10) \end{aligned}$$

$$(m, n \in \mathbb{N}_0,\ \lambda > -p,\ t \geq 0;\ z \in \mathbb{U}^*)$$

.
The operator $\mathcal{I}^{n,m}_{\lambda,p}(a,c)$ generalizes several previously studied familiar operators and also provides meromorphic analogue for certain well known operators for analytic functions (see, for detail [10, 11]). Very recently, a similar operator for analytic functions has been studied by Srivastava et al. [18].
In the particular case $n = 1$, we use the notation

$$\mathcal{I}^{1,m}_{\lambda,p}(a,c)f(z) := \mathcal{I}^{m}_{\lambda,p}(a,c)f(z).$$

In the recent years, several authors obtained many interesting results involving various linear and non-linear operators associated with differential subordination and superordination (for detail, see [3, 4, 5, 15, 16, 17]).
The main object of the present paper is to obtain sufficient conditions for the functions $f \in \sum_p$ defined by using the operator $\mathcal{I}^{m}_{\lambda,p}(a,c)$ given by (10) such that sandwich relations of the form:

$$q_1(z) \prec \left(\frac{1}{z^p \mathcal{I}^{m}_{\lambda,p}(a,c)f(z)}\right)^{\alpha} \prec q_2(z),$$

holds good where $q_1$ and $q_2$ are given univalent functions in $\mathbb{U}$ with $q_1(0) = q_2(0) = 1$.

## 2. Preliminaries

To establish our results, we need the following:

**Lemma 1.** (see [14]) *Let $q$ be a convex univalent function in the open unit disk $\mathbb{U}$ and let $\psi \in \mathbb{C}$, $\gamma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ with $\Re\{1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{\gamma}\} > 0$. If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = q(0)$ and*

$$\psi p(z) + \gamma z p'(z) \prec \psi q(z) + \gamma z q'(z)$$

*then $p \prec q$ and $q$ is the best dominant.*

**Lemma 2.** (see [9]) *Let $q$ be convex univalent in the open unit disk $\mathbb{U}$ and $\gamma \in \mathbb{C}$ such that $\Re(\gamma) > 0$. If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, and $p(z) + \gamma z p'(z)$ is univalent in $\mathbb{U}$, then*

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z),$$

*then $q \prec p$ and $q$ is the best subordinant.*

**Lemma 3.** *Let $a$ and $c$ be complex numbers $(a, c \notin \mathbb{Z}_0^-)$, $n, m \in \mathbb{N}_0, t > 0, \lambda \in \mathbb{R}$ and $\lambda > -p$. Let $f \in \sum_p$. Then the following identities hold.*

$$z(\mathcal{I}_{\lambda,p}^{n,m}(a,c)f(z))' = \frac{p}{t}(1-t)\mathcal{I}_{\lambda,p}^{n,m}(a,c)f(z) - \frac{p}{t}\mathcal{I}_{\lambda,p}^{n,m+1}(a,c)f(z), \tag{11}$$

$$z(\mathcal{I}_{\lambda,p}^{m}(a,c)f(z))' = (a-1)\mathcal{I}_{\lambda,p}^{m}(a-1,c)f(z) - (a-1+p)\mathcal{I}_{\lambda,p}^{m}(a,c)f(z), \tag{12}$$

$$z(\mathcal{I}_{\lambda,p}^{m}(a,c)f(z))' = (\lambda+p)\mathcal{I}_{\lambda+1,p}^{m}(a,c)f(z) - (\lambda+2p)\mathcal{I}_{\lambda,p}^{m}(a,c)f(z), \tag{13}$$

$$z(\mathcal{I}_{\lambda,p}^{m}(a,c)f(z))' = c\mathcal{I}_{\lambda,p}^{m}(a,c+1)f(z) - (c+p)\mathcal{I}_{\lambda,p}^{m}(a,c)f(z). \tag{14}$$

*Proof.* These identities can be verified by considering series expansions of individual functions involved.

## 3. Main Results

Unless otherwise mentioned, we assume throughout the sequel that $t > 0, \lambda > -p, p \in \mathbb{N}, m \in \mathbb{N}_0, \eta \in \mathbb{C}^*$ and $0 < \alpha < 1$. The powers are considered as the principal one.
We prove the following.

**Theorem 4.** *Let $q$ be univalent in $\mathbb{U}$ and satisfies*

$$\Re\left\{1 + \frac{zq''(z)}{q'(z)} + \frac{\alpha}{\eta}\right\} > 0. \tag{15}$$

*Suppose $f \in \sum_p$ satisfies any one of the following subordination conditions:*

$$\left[1 - \frac{\eta p}{t}\right]\left(\frac{1}{z^p\mathcal{I}_{\lambda,p}^{m}(a,c)f(z)}\right)^{\alpha} + \frac{\eta p}{t} z^p \mathcal{I}_{\lambda,p}^{m+1}(a,c)f(z)\left(\frac{1}{z^p\mathcal{I}_{\lambda,p}^{m}(a,c)f(z)}\right)^{\alpha+1} \prec q(z) + \frac{\eta}{\alpha}zq'(z), \tag{16}$$

*or*

$$[1 + \eta(a-1)]\left(\frac{1}{z^p\mathcal{I}_{\lambda,p}^{m}(a,c)f(z)}\right)^{\alpha} - \eta(a-1)z^p\mathcal{I}_{\lambda,p}^{m}(a-1,c)f(z)\left(\frac{1}{z^p\mathcal{I}_{\lambda,p}^{m}(a,c)f(z)}\right)^{\alpha+1} \prec q(z) + \frac{\eta}{\alpha}zq'(z), \tag{17}$$

*or*

$$[1+\eta(\lambda+p)]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha}-\eta(\lambda+p)z^p\mathcal{I}^m_{\lambda+1,p}(a,c)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1}$$
$$\prec q(z)+\frac{\eta}{\alpha}zq'(z), \tag{18}$$

*or*

$$[1+\eta c]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha}-\eta c z^p\mathcal{I}^m_{\lambda,p}(a,c+1)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1}$$
$$\prec q(z)+\frac{\eta}{\alpha}zq'(z). \tag{19}$$

*Then*

$$\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha}\prec q(z) \tag{20}$$

*and $q$ is the best dominant of (20).*

*Proof.* Define the function $\phi(z)$ by

$$\phi(z)=\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha}\qquad(z\in\mathbb{U}^*). \tag{21}$$

Clearly, the function $\phi(z)$ is analytic in $\mathbb{U}$ and $\phi(0)=1$. Differentiating (21) logarithmically with respect to $z$ followed by applications of the identities (11) to (14) yield respectively

$$\frac{z\phi'(z)}{\phi(z)}=-\frac{p\alpha}{t}\left[1-\frac{\mathcal{I}^{m+1}_{\lambda,p}(a,c)f(z)}{\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right], \tag{22}$$

$$\frac{z\phi'(z)}{\phi(z)}=(a-1)\alpha\left[1-\frac{\mathcal{I}^{m}_{\lambda,p}(a-1,c)f(z)}{\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right], \tag{23}$$

$$\frac{z\phi'(z)}{\phi(z)}=(\lambda+p)\alpha\left[1-\frac{\mathcal{I}^{m}_{\lambda+1,p}(a,c)f(z)}{\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right], \tag{24}$$

and

$$\frac{z\phi'(z)}{\phi(z)}=c\alpha\left[1-\frac{\mathcal{I}^{m}_{\lambda,p}(a,c+1)f(z)}{\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right]. \tag{25}$$

Now, the subordination conditions (16) to (19) are equivalent to

$$\phi(z)+\frac{\eta}{\alpha}z\phi'(z)\prec q(z)+\frac{\eta}{\alpha}zq'(z). \tag{26}$$

The assertion of Theorem 4 now follows by an application of Lemma 1 with $\psi = 1$ and $\gamma = \frac{\eta}{\alpha}$. The proof of Theorem 4 is completed.

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ and $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$ $(0 < \gamma \leq 1)$ in Theorem 4, we have the following results (Corollaries 16 and 17 below.)

**Corollary 5.** *Let* $\Re\{\frac{1-Bz}{1+Bz} + \frac{\alpha}{\eta}\} > 0$ $(z \in \mathbb{U})$. *Suppose the function* $f \in \sum_p$ *satisfying any one of the following conditions:*

$$\left[1 - \frac{\eta p}{t}\right]\left(\frac{1}{z^p \mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha} + \frac{\eta p}{t} z^p \mathcal{I}^{m+1}_{\lambda,p}(a,c)f(z)\left(\frac{1}{z^p \mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1} \prec \frac{1+Az}{1+Bz} + \frac{\eta}{\alpha}\frac{(A-B)z}{(1+Bz)^2},$$

*or*

$$[1+\eta(a-1)]\left(\frac{1}{z^p \mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha} - \eta(a-1) z^p \mathcal{I}^{m}_{\lambda,p}(a-1,c)f(z)\left(\frac{1}{z^p \mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1} \prec \frac{1+Az}{1+Bz} + \frac{\eta}{\alpha}\frac{(A-B)z}{(1+Bz)^2},$$

*or*

$$[1+\eta(\lambda+p)]\left(\frac{1}{z^p \mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha} - \eta(\lambda+p) z^p \mathcal{I}^{m}_{\lambda+1,p}(a,c)f(z)\left(\frac{1}{z^p \mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1} \prec \frac{1+Az}{1+Bz} + \frac{\eta}{\alpha}\frac{(A-B)z}{(1+Bz)^2},$$

*or*

$$[1+\eta c]\left(\frac{1}{z^p \mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha} - \eta c z^p \mathcal{I}^{m}_{\lambda,p}(a,c+1)f(z)\left(\frac{1}{z^p \mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1} \prec \frac{1+Az}{1+Bz} + \frac{\eta}{\alpha}\frac{(A-B)z}{(1+Bz)^2}.$$

*Then*

$$\left(\frac{1}{z^p \mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha} \prec \frac{1+Az}{1+Bz} \tag{27}$$

*and* $\frac{1+Az}{1+Bz}$ *is the best dominant of (27).*

**Corollary 6.** *Let* $\Re\{\frac{1+2\gamma z+z^2}{1-z^2}+\frac{\alpha}{\eta}\}>0$ $(z\in\mathbb{U})$. *Suppose the function* $f\in\sum_p$ *satisfies any one of the following subordination conditions:*

$$\left[1-\frac{\eta p}{t}\right]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha}+\frac{\eta p}{t}z^p\mathcal{I}^{m+1}_{\lambda,p}(a,c)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1} \prec\left(\frac{1+z}{1-z}\right)^{\gamma}+\frac{2\gamma\eta}{\alpha}z\frac{(1+z)^{\gamma-1}}{(1-z)^{\gamma+1}},$$

*or*

$$[1+\eta(a-1)]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha}-\eta(a-1)z^p\mathcal{I}^m_{\lambda,p}(a-1,c)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1} \prec\left(\frac{1+z}{1-z}\right)^{\gamma}+\frac{2\gamma\eta}{\alpha}z\frac{(1+z)^{\gamma-1}}{(1-z)^{\gamma+1}},$$

*or*

$$[1+\eta(\lambda+p]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha}-\eta(\lambda+p)z^p\mathcal{I}^m_{\lambda+1,p}(a,c)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1} \prec\left(\frac{1+z}{1-z}\right)^{\gamma}+\frac{2\gamma\eta}{\alpha}z\frac{(1+z)^{\gamma-1}}{(1-z)^{\gamma+1}},$$

*or*

$$[1+\eta c]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha}-\eta cz^p\mathcal{I}^m_{\lambda,p}(a,c+1)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1} \prec\left(\frac{1+z}{1-z}\right)^{\gamma}+\frac{2\gamma\eta}{\alpha}z\frac{(1+z)^{\gamma-1}}{(1-z)^{\gamma+1}}.$$

*Then*

$$\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha}\prec\left(\frac{1+z}{1-z}\right)^{\gamma} \tag{28}$$

*and* $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best dominant of (28).*

Taking $p=t=1$ and $m=0$ in Theorem 4, we obtain the following results (Corollary 7 below).

**Corollary 7.** *Let* $q$ *be univalent in* $\mathbb{U}$ *and (15) holds. Suppose the function* $f\in\sum(\equiv\sum_1)$ *satisfies the following subordination:*

$$[1-\eta]\left(\frac{1}{z\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha}-\eta\frac{\left(\mathcal{L}^{\lambda}(a,c)f(z)\right)'}{z^{\alpha-1}}\left(\frac{1}{\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha+1}\prec q(z)+\frac{\eta}{\alpha}zq'(z),$$

*or*

$$[1+\eta(a-1)]\left(\frac{1}{z\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha} - \eta(a-1)\frac{\mathcal{L}^{\lambda}(a-1,c)f(z)}{z^{\alpha}}\left(\frac{1}{\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha+1} \prec q(z)+\frac{\eta}{\alpha}zq'(z),$$

*or*

$$[1+\eta(\lambda+1)]\left(\frac{1}{z\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha} - \eta(\lambda+1)\frac{\mathcal{L}^{\lambda+1}(a,c)f(z)}{z^{\alpha}}\left(\frac{1}{\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha+1} \prec q(z)+\frac{\eta}{\alpha}zq'(z),$$

*or*

$$[1+\eta c]\left(\frac{1}{z\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha} - \eta c\frac{\mathcal{L}^{\lambda}(a,c+1)f(z)}{z^{\alpha}}\left(\frac{1}{\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha+1} \prec q(z)+\frac{\eta}{\alpha}zq'(z).$$

*Then*

$$\left(\frac{1}{z\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha} \prec q(z) \tag{29}$$

*and $q(z)$ is the best dominant of (29).*

**Theorem 8.** *Let the function $q$ be univalent convex in $\mathbb{U}$. Further, let us assume that*

$$\Re(\eta)>0 \tag{30}$$

*and*

$$\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha} \in \mathcal{H}[q(0),1)]\cap Q.$$

*Suppose the function $f$ and $q$ satisfy any one of the following pair of conditions:*

$$\left[1-\frac{\eta p}{t}\right]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha} + \frac{\eta p}{t}z^p\mathcal{I}^{m+1}_{\lambda,p}(a,c)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1} \tag{31}$$

*is univalent in $\mathbb{U}$*

*and*

$$q(z)+\frac{\eta}{\alpha}zq'(z) \prec \left[1-\frac{\eta p}{t}\right]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha} + \frac{\eta p}{t}z^p\mathcal{I}^{m+1}_{\lambda,p}(a,c)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1}, \tag{32}$$

*or*

$$[1+\eta(a-1)]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha}-\eta(a-1)z^p\mathcal{I}^m_{\lambda,p}(a-1,c)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1} \tag{33}$$

*is univalent in* $\mathbb{U}$

*and*

$$q(z)+\frac{\eta}{\alpha}zq'(z)\prec[1+\eta(a-1)]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha}-\eta(a-1)z^p\mathcal{I}^m_{\lambda,p}(a-1,c)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1}, \tag{34}$$

*or*

$$[1+\eta(\lambda+p)]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha}-\eta(\lambda+p)z^p\mathcal{I}^m_{\lambda+1,p}(a,c)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1} \tag{35}$$

*is univalent in* $\mathbb{U}$

*and*

$$q(z)+\frac{\eta}{\alpha}zq'(z)\prec[1+\eta(\lambda+p)]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha}-\eta(\lambda+p)z^p\mathcal{I}^m_{\lambda+1,p}(a,c)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1}, \tag{36}$$

*or*

$$[1+\eta c]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha}-\eta cz^p\mathcal{I}^m_{\lambda,p}(a,c+1)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1} \tag{37}$$

*is univalent in* $\mathbb{U}$

*and*

$$q(z)+\frac{\eta}{\alpha}zq'(z)\prec[1+\eta c]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha}-\eta cz^p\mathcal{I}^m_{\lambda,p}(a,c+1)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1}. \tag{38}$$

*Then*

$$q(z) \prec \left( \frac{1}{z^p \mathcal{I}^m_{\lambda,p}(a,c) f(z)} \right)^{\alpha} \tag{39}$$

*and $q$ is the best dominant of (39).*

*Proof.* Differentiating logarithmically with respect to $z$ of the function

$$\phi(z) = \left( \frac{1}{z^p \mathcal{I}^m_{\lambda,p}(a,c) f(z)} \right)^{\alpha} \quad (z \in \mathbb{U}^*),$$

followed by application of the identities (11) to (14) give (22) to (25) respectively. Hence the subordination conditions (32), (34), (36) and (38) are equivalent to

$$q(z) + \frac{\eta}{\alpha} z q'(z) \prec \phi(z) + \frac{\eta}{\alpha} z \phi'(z).$$

The assertion (39) of Theorem 8 follows by an application of Lemma 2. The proof of Theorem 8 is thus completed.

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ and $q(z) = \left(\frac{1+z}{1-z}\right)^{\gamma}$ $(0 < \gamma \leq 1)$ in Theorem 8 we get the following results (Corollaries 9 and 10).

**Corollary 9.** *Assume that (30) holds and* $\left( \frac{1}{z^p \mathcal{I}^m_{\lambda,p}(a,c) f(z)} \right)^{\alpha} \in \mathcal{H}[1,1] \cap Q$. *Suppose the function $f \in \sum_p$ satisfies any one of the following pair of the conditions:*

$$\left[1 - \frac{\eta p}{t}\right] \left( \frac{1}{z^p \mathcal{I}^m_{\lambda,p}(a,c) f(z)} \right)^{\alpha} + \frac{\eta p}{t} z^p \mathcal{I}^{m+1}_{\lambda,p}(a,c) f(z) \left( \frac{1}{z^p \mathcal{I}^m_{\lambda,p}(a,c) f(z)} \right)^{\alpha+1}$$

*is univalent in* $\mathbb{U}$

*and*

$$\frac{1+Az}{1+Bz} + \frac{\eta}{\alpha} \frac{(A-B)z}{(1+Bz)^2} \prec \left[1 - \frac{\eta p}{t}\right] \left( \frac{1}{z^p \mathcal{I}^m_{\lambda,p}(a,c) f(z)} \right)^{\alpha} + \frac{\eta p}{t} z^p \mathcal{I}^{m+1}_{\lambda,p}(a,c) f(z) \left( \frac{1}{z^p \mathcal{I}^m_{\lambda,p}(a,c) f(z)} \right)^{\alpha+1}$$

*or*

$$[1 + \eta(a-1)] \left( \frac{1}{z^p \mathcal{I}^m_{\lambda,p}(a,c) f(z)} \right)^{\alpha} - \eta(a-1) z^p \mathcal{I}^m_{\lambda,p}(a-1,c) f(z) \left( \frac{1}{z^p \mathcal{I}^m_{\lambda,p}(a,c) f(z)} \right)^{\alpha+1}$$

*is univalent in* $\mathbb{U}$

*and*

$$\frac{1+Az}{1+Bz}+\frac{\eta}{\alpha}\frac{(A-B)z}{(1+Bz)^2}\prec[1+\eta(a-1)]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha} - \eta(a-1)z^p\mathcal{I}^m_{\lambda,p}(a-1,c)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1}$$

*or*

$$[1+\eta(\lambda+p)]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha}-\eta(\lambda+p)z^p\mathcal{I}^m_{\lambda+1,p}(a,c)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1}$$

*is univalent in* $\mathbb{U}$

*and*

$$\frac{1+Az}{1+Bz}+\frac{\eta}{\alpha}\frac{(A-B)z}{(1+Bz)^2}\prec[1+\eta(\lambda+p)]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha} - \eta(\lambda+p)z^p\mathcal{I}^m_{\lambda+1,p}(a,c)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1}$$

*or*

$$[1+\eta c]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha}-\eta cz^p\mathcal{I}^m_{\lambda,p}(a,c+1)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1}$$

*is univalent in* $\mathbb{U}$

*and*

$$\frac{1+Az}{1+Bz}+\frac{\eta}{\alpha}\frac{(A-B)z}{(1+Bz)^2}\prec[1+\eta c]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha}- \eta cz^p\mathcal{I}^m_{\lambda,p}(a,c+1)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1}.$$

*Then*

$$\frac{1+Az}{1+Bz}\prec\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha} \tag{40}$$

*and* $\frac{1+Az}{1+Bz}$ *is the best subordinant of (40).*

**Corollary 10.** *Assume that (30) holds and* $\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^\alpha \in \mathcal{H}[1,1]\cap Q$. *Suppose the function* $f \in \sum_p$ *satisfies any one of the following pair of the conditions:*

$$\left[1-\frac{\eta p}{t}\right]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^\alpha + \frac{\eta p}{t}z^p\mathcal{I}^{m+1}_{\lambda,p}(a,c)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1}$$

*is univalent in* $\mathbb{U}$

*and*

$$\left(\frac{1+z}{1-z}\right)^\gamma + \frac{2\eta\gamma}{\alpha}\frac{z(1+z)^{\gamma-1}}{(1-z)^{\gamma+1}} \prec \left[1-\frac{\eta p}{t}\right]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^\alpha + \frac{\eta p}{t}z^p\mathcal{I}^{m+1}_{\lambda,p}(a,c)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1}$$

*or*

$$[1+\eta(a-1)]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^\alpha - \eta(a-1)z^p\mathcal{I}^m_{\lambda,p}(a-1,c)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1}$$

*is univalent in* $\mathbb{U}$

*and*

$$\left(\frac{1+z}{1-z}\right)^\gamma + \frac{2\eta\gamma}{\alpha}\frac{z(1+z)^{\gamma-1}}{(1-z)^{\gamma+1}} \prec [1+\eta(a-1)]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^\alpha - \eta(a-1)z^p\mathcal{I}^m_{\lambda,p}(a-1,c)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1}$$

*or*

$$[1+\eta(\lambda+p)]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^\alpha - \eta(\lambda+p)z^p\mathcal{I}^m_{\lambda+1,p}(a,c)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1}$$

*is univalent in* $\mathbb{U}$

*and*

$$\left(\frac{1+z}{1-z}\right)^\gamma + \frac{2\eta\gamma}{\alpha}\frac{z(1+z)^{\gamma-1}}{(1-z)^{\gamma+1}} \prec [1+\eta(\lambda+p)]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^\alpha - \eta(\lambda+p)z^p\mathcal{I}^m_{\lambda+1,p}(a,c)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1}$$

*or*

$$[1+\eta c]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha}-\eta c z^p\mathcal{I}^m_{\lambda,p}(a,c+1)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1}$$

*is univalent in* $\mathbb{U}$

*and*

$$\left(\frac{1+z}{1-z}\right)^{\gamma}+\frac{2\eta\gamma}{\alpha}\frac{z(1+z)^{\gamma-1}}{(1-z)^{\gamma+1}}\prec[1+\eta c]\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha}-\eta c z^p\mathcal{I}^m_{\lambda,p}(a,c+1)f(z)\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1}.$$

*Then*

$$\left(\frac{1+z}{1-z}\right)^{\gamma}\prec\left(\frac{1}{z^p\mathcal{I}^m_{\lambda,p}(a,c)f(z)}\right)^{\alpha} \tag{41}$$

*and* $\left(\frac{1+z}{1-z}\right)^{\gamma}$ *is the best subordinant of (41).*

Taking $p=t=1$ and $m=0$ in Theorem 8, we obtain the following result (Corollary 11 below).

**Corollary 11.** *Let* $f\in\sum_p$ *and* $q$ *be univalent convex function in* $\mathbb{U}$ *satisfying the condition* $\Re(\eta)>0$ *and* $\left(\frac{1}{z\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha}\in\mathcal{H}[1,1]\cap Q$. *Suppose any one of the following pair of the conditions is satisfied:*

$$(1-\eta)\left(\frac{1}{z\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha}-\eta\frac{\left(\mathcal{L}^{\lambda}(a,c)f(z)\right)'}{z^{\alpha-1}}\left(\frac{1}{\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha+1}$$

*is univalent in* $\mathbb{U}$

*and*

$$q(z)+\frac{\eta}{\alpha}zq'(z)\prec(1-\eta)\left(\frac{1}{z\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha}-\eta\frac{\left(\mathcal{L}^{\lambda}(a,c)f(z)\right)'}{z^{\alpha-1}}\left(\frac{1}{\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha+1}$$

*or*

$$[1+\eta(a-1)]\left(\frac{1}{z\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha}-\eta(a-1)\frac{\mathcal{L}^{\lambda}(a-1,c)f(z)}{z^{\alpha}}\left(\frac{1}{\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha+1}$$

*is univalent in* $\mathbb{U}$,

*and*

$$q(z)+\frac{\eta}{\alpha}zq'(z) \prec [1+\eta(a-1)]\left(\frac{1}{z\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha}-\eta(a-1)\frac{\mathcal{L}^{\lambda}(a-1,c)f(z)}{z^{\alpha}}\left(\frac{1}{\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha+1}$$

*or*

$$[1+\eta(\lambda+1)]\left(\frac{1}{z\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha}-\eta(\lambda+1)\frac{\mathcal{L}^{\lambda+1}(a,c)f(z)}{z^{\alpha}}\left(\frac{1}{\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha+1}$$

*is univalent in* $\mathbb{U}$,

*and*

$$q(z)+\frac{\eta}{\alpha}zq'(z) \prec [1+\eta(\lambda+1)]\left(\frac{1}{z\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha}-\eta(\lambda+1)\frac{\mathcal{L}^{\lambda+1}(a,c)f(z)}{z^{\alpha}}\left(\frac{1}{\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha+1}$$

*or*

$$(1+\eta c)\left(\frac{1}{z\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha}-\eta c\frac{\mathcal{L}^{\lambda}(a,c+1)f(z)}{z^{\alpha}}\left(\frac{1}{\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha+1}$$

*is univalent in* $\mathbb{U}$,

*and*

$$q(z)+\frac{\eta}{\alpha}zq'(z) \prec (1+\eta c\left(\frac{1}{z\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha}-\eta c\frac{\mathcal{L}^{\lambda}(a,c+1)f(z)}{z^{\alpha}}\left(\frac{1}{\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha+1}.$$

*Then*

$$q(z) \prec \left(\frac{1}{z\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha} \tag{42}$$

*and* $q(z)$ *is the best subordinant of (42).*

Combining Theorem 4 and Theorem 8 we get the following sandwich theorem.

**Theorem 12.** *Let* $q_1$ *be univalent convex and* $q_2$ *be univalent in* $\mathbb{U}$. *Suppose* $q_1$ *and* $q_2$ *satisfy the conditions (30) and (15) respectively.*

*Further, assume that* $\left(\frac{1}{z^{p}\mathcal{I}^{m}_{\lambda,p}(a,c)f(z)}\right)^{\alpha} \neq 0 \in \mathcal{H}[q_1(0),1]\cap Q$.

*Suppose the function* $f \in \sum_p$ *satisfies any one of the following pair of conditions:*

$$\left[1-\frac{\eta p}{t}\right]\left(\frac{1}{z^{p}\mathcal{I}^{m}_{\lambda,p}(a,c)f(z)}\right)^{\alpha}+\frac{\eta p}{t}z^{p}\mathcal{I}^{m+1}_{\lambda,p}(a,c)f(z)\left(\frac{1}{z^{p}\mathcal{I}^{m}_{\lambda,p}(a,c)f(z)}\right)^{\alpha+1}$$

*is univalent in* $\mathbb{U}$

*and*

$$q_1(z)+\frac{\eta}{\alpha}zq_1'(z)\prec\left[1-\frac{\eta p}{t}\right]\left(\frac{1}{z^p\mathcal{I}_{\lambda,p}^m(a,c)f(z)}\right)^{\alpha}+$$

$$\frac{\eta p}{t}z^p\mathcal{I}_{\lambda,p}^{m+1}(a,c)f(z)\left(\frac{1}{z^p\mathcal{I}_{\lambda,p}^m(a,c)f(z)}\right)^{\alpha+1}\prec q_2(z)+\frac{\eta}{\alpha}zq_2'(z)$$

*or*

$$[1+\eta(a-1)]\left(\frac{1}{z^p\mathcal{I}_{\lambda,p}^m(a,c)f(z)}\right)^{\alpha}-\eta(a-1)z^p\mathcal{I}_{\lambda,p}^m(a-1,c)f(z)\left(\frac{1}{z^p\mathcal{I}_{\lambda,p}^m(a,c)f(z)}\right)^{\alpha+1}$$

*is univalent in* $\mathbb{U}$
*and*

$$q_1(z)+\frac{\eta}{\alpha}zq_1'(z)\prec[1+\eta(a-1)]\left(\frac{1}{z^p\mathcal{I}_{\lambda,p}^m(a,c)f(z)}\right)^{\alpha}-$$

$$\eta(a-1)z^p\mathcal{I}_{\lambda,p}^m(a-1,c)f(z)\left(\frac{1}{z^p\mathcal{I}_{\lambda,p}^m(a,c)f(z)}\right)^{\alpha+1}\prec q_2(z)+\frac{\eta}{\alpha}zq_2'(z)$$

*or*

$$[1+\eta(\lambda+p)]\left(\frac{1}{z^p\mathcal{I}_{\lambda,p}^m(a,c)f(z)}\right)^{\alpha}-\eta(\lambda+p)z^p\mathcal{I}_{\lambda+1,p}^m(a,c)f(z)\left(\frac{1}{z^p\mathcal{I}_{\lambda,p}^m(a,c)f(z)}\right)^{\alpha+1}$$

*is univalent in* $\mathbb{U}$
*and*

$$q_1(z)+\frac{\eta}{\alpha}zq_1'(z)\prec[1+\eta(\lambda+p)]\left(\frac{1}{z^p\mathcal{I}_{\lambda,p}^m(a,c)f(z)}\right)^{\alpha}-$$

$$\eta(\lambda+p)z^p\mathcal{I}_{\lambda+1,p}^m(a,c)f(z)\left(\frac{1}{z^p\mathcal{I}_{\lambda,p}^m(a,c)f(z)}\right)^{\alpha+1}\prec q_2(z)+\frac{\eta}{\alpha}zq_2(z)$$

*or*

$$[1+\eta c]\left(\frac{1}{z^p\mathcal{I}_{\lambda,p}^m(a,c)f(z)}\right)^{\alpha}-\eta cz^p\mathcal{I}_{\lambda,p}^m(a,c+1)f(z)\left(\frac{1}{z^p\mathcal{I}_{\lambda,p}^m(a,c)f(z)}\right)^{\alpha+1}$$

*is univalent in* $\mathbb{U}$

*and*

$$q_1(z) + \frac{\eta}{\alpha} z q_1'(z) \prec [1 + \eta c]) \left( \frac{1}{z^p \mathcal{I}_{\lambda,p}^m(a,c) f(z)} \right)^{\alpha} -$$

$$\eta c z^p \mathcal{I}_{\lambda,p}^m(a,c+1) f(z) \left( \frac{1}{z^p \mathcal{I}_{\lambda,p}^m(a,c) f(z)} \right)^{\alpha+1} \prec q_2(z) + \frac{\eta}{\alpha} z q_2(z)$$

*Then*

$$q_1(z) \prec \left( \frac{1}{z^p \mathcal{I}_{\lambda,p}^m(a,c) f(z)} \right)^{\alpha} \prec q_2(z)$$

*where $q_1$ and $q_2$ are the best subordinant and the best dominant respectively.*

Taking $p = t = 1$ and $m = 0$ in Theorem 12 we obtain the following result.

**Corollary 13.** *Let $q_1$ be univalent convex and $q_2$ be univalent in $\mathbb{U}$ satisfying the conditions (30) and (15) respectively. Let*

$$\left( \frac{1}{z \mathcal{L}^{\lambda}(a,c) f(z)} \right)^{\alpha} \neq 0 \in \mathcal{H}[q_1(0), 1] \cap Q.$$

*Suppose the function $f \in \sum_p$ satisfies any one of the following pair of conditions:*

$$[1 - \eta] \left( \frac{1}{z \mathcal{L}^{\lambda}(a,c) f(z)} \right)^{\alpha} - \eta \frac{\left(\mathcal{L}^{\lambda}(a,c) f(z)\right)'}{z^{\alpha-1}} \left( \frac{1}{\mathcal{L}^{\lambda}(a,c) f(z)} \right)^{\alpha+1}$$

*is univalent in $\mathbb{U}$*

*and*

$$\begin{aligned} q_1(z) + \frac{\eta}{\alpha} z q_1'(z) \quad &\prec \quad [1 - \eta] \left( \frac{1}{z \mathcal{L}^{\lambda}(a,c) f(z)} \right)^{\alpha} + \eta\alpha \frac{\left(\mathcal{L}^{\lambda}(a,c) f(z)\right)'}{z^{\alpha-1}} \left( \frac{1}{\mathcal{L}^{\lambda}(a,c) f(z)} \right)^{\alpha+1} \\ &\prec \quad q_2(z) + \frac{\eta}{\alpha} z q_2'(z) \end{aligned}$$

*or*

$$[1 + \eta(a-1)] \left( \frac{1}{z \mathcal{L}^{\lambda}(a,c) f(z)} \right)^{\alpha} - \eta(a-1) \frac{\mathcal{L}^{\lambda}(a-1,c) f(z)}{z^{\alpha}} \left( \frac{1}{\mathcal{L}^{\lambda}(a,c) f(z)} \right)^{\alpha+1}$$

*is univalent in $\mathbb{U}$*

*and*

$$q_1(z) + \frac{\eta}{\alpha} z q_1'(z) \prec [1 + \eta(a-1)] \left( \frac{1}{z \mathcal{L}^{\lambda}(a,c) f(z)} \right)^{\alpha} -$$

$$\eta(a-1) \frac{\mathcal{L}^{\lambda}(a-1,c) f(z)}{z^{\alpha}} \left( \frac{1}{\mathcal{L}^{\lambda}(a,c) f(z)} \right)^{\alpha+1} \prec q_2(z) + \frac{\eta}{\alpha} z q_2'(z)$$

*or*

$$[1+\eta(\lambda+1)]\left(\frac{1}{z\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha}-\eta(\lambda+1)\frac{\mathcal{L}^{\lambda+1}(a,c)f(z)}{z^{\alpha}}\left(\frac{1}{\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha+1}$$

*is univalent in* $\mathbb{U}$
*and*

$$q_1(z)+\frac{\eta}{\alpha}zq_1'(z)\prec[1+\eta(\lambda+1)]\left(\frac{1}{z\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha}-$$
$$\eta(\lambda+1)\frac{\mathcal{L}^{\lambda+1}(a,c)f(z)}{z^{\alpha}}\left(\frac{1}{\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha+1}\prec q_2(z)+\frac{\eta}{\alpha}zq_2'(z)$$

*or*

$$[1+\eta c]\left(\frac{1}{z\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha}-\eta c\frac{\mathcal{L}^{\lambda}(a,c+1)f(z)}{z^{\alpha}}\left(\frac{1}{\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha+1}$$

*is univalent in* $\mathbb{U}$
*and*

$$\begin{aligned}q_1(z)+\frac{\eta}{\alpha}zq_1'(z) &\prec [1+\eta c]\left(\frac{1}{z\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha}-\eta c\frac{\mathcal{L}^{\lambda}(a,c+1)f(z)}{z^{\alpha}}\left(\frac{1}{\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha+1}\\ &\prec q_2(z)+\frac{\eta}{\alpha}zq_2'(z)\end{aligned}$$

*Then*

$$q_1(z)\prec\left(\frac{1}{z\mathcal{L}^{\lambda}(a,c)f(z)}\right)^{\alpha}\prec q_2(z)$$

*where* $q_1$ *and* $q_2$ *are the best subordinant and the best dominant respectively.*

## References

[1] T. Bulboacă, *Differential Subordinations and Superordinations*, Recent Results, House of Scientific Book Publ., Cluj-Napoca., 2005.

[2] N. E. Cho, O. S. Kwon and H. M. Srivastava, *Inclusion relationships and argument properties for certain subclasses of multivalent functions associated with a family of linear operators*, J. Math. Anal. Appl. 292 (2004), 470-483.

[3] N. E. Cho and H. M. Srivastava, *A class of nonlinear integral operators preserving subordination and superordination*, Integral Transforms Spec. Funct. 18 (2007), 95-107.

[4] N. E. Cho, O. S. Kwon, S. Owa and H. M. Srivastava, *A class of integral operators preserving subordination and superordination for meromorphic functions*, Appl. Math. Comput. 193 (2007), 463-474.

[5] G. Murugusundaramoorthy and N. Magesh, *Differential subordinations and superordinations for analytic functions defined by the Dziok-Srivastava linear operator*, J. Inequal. Pure Appl. Math. 7(4) (2006), 1-9.

[6] J.- L. Liu and H. M. Srivastava, *A linear operator and associated families of meromorphically multivalent functions*, J. Math. Anal. Appl. 259 (2001), 566-581.

[7] S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J. 28 (1981), 157-171.

[8] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics, vol. 225, Marcel Dekker, New York and Basel, 2000.

[9] S. S. Miller and P. T. Mocanu, *Subordinants of differential superordinations*, Complex Var. Elliptic Equ. 48 (2003), 815–826.

[10] A. K. Mishra, T. Panigrahi and R. K. Mishra, *Subordiantion and inclusion theorems for subclasses of meromorphic functions with applications to electromagnetic cloaking*, Math. Comput. Modelling 57 (2013), 945-962.

[11] T. Panigrahi, *On Some Families of Analytic Functions Defined Through Subordination and Hypergeometric Functions*, Ph.D. Thesis, Berhampur University, Berhampur, 2011.

[12] T. Panigrahi, *Convolution properties of multivalent meromorphic functions associated with Cho-Kwon-Srivastava operator*, Southeast Asian Bull. Math. (to appear).

[13] J. Patel, N. E. Cho and H. M. Srivastava, *Certain subclasses of multivalent functions associated with a family of linear operators*, Math. Comput. Modelling 43 (2006), 320-338.

[14] T. N. Shanmugam, V. Ravichandran and S. Sivasubramanian, *Differential sandwich theorems for some subclasses of analytic functions*, Austral. J. Math. Anal. Appl. 3 (2006), 1-11.

[15] T. N. Shanmugam, S. Sivasubramanian and H. Silverman, *On sandwich theorems for some classes of analytic functions*, Internat. J. Math. Math. Sci. (2006), 1-13, Article ID 29684.

[16] T. N. Shanmugam, S. Sivasubramanian and H. M. Srivastava, *Differential sandwich theorems for certain subclasses of analytic functions involving multiplier transformations*, Integral Transforms Spec. Funct. 17, 12 (2006), 889-899.

[17] H. M. Srivastava, D.-G. Yang and N.-E. Xu, *Subordination for multivalent analytic functions associated with the Dziok-Srivastava operator*, Integral Transforms Spec. Funct. 20 (2009), 581-606.

[18] H. M. Srivastava, A. K. Mishra and S. N. Kund, *Certain classes of analytic functions associated with iterations of the Owa-Srivastava fractional derivative operator*, Southeast Asian Bull. Math. 37 (2013), 413-435.

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