

A CLASS OF MEROMORPHICALLY MULTIVALENT FUNCTIONS INVOLVING A LINEAR OPERATOR

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ABSTRACT. Let Σ_p denote the class of functions normalized by

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^{n-p} \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}),$$

which are analytic and p -valent in $\mathbb{U}_0 = \{z : 0 < |z| < 1\}$. Making use of a linear operator, which is defined here by means of the Hadamard product (or Convolution), we introduce and investigate a new subclass $\Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$ of Σ_p . Some interesting properties such as inclusion relationships, convolutions for this function class are obtained. The results presented here would provide extensions of these given in earlier works. Several other new results are also obtained.

2000 *Mathematics Subject Classification*: Primary 30C45.

Keywords: meromorphic functions, multivalent functions, convex univalent functions, linear operator, Hadamard product (or convolution), k -symmetric points, subordination between analytic functions.

1. INTRODUCTION AND PRELIMINARIES

Let Σ_p denote the class of functions $f(z)$ normalized by

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_n z^{n-p} \quad (p \in \mathbb{N} := \{1, 2, 3, \dots\}), \quad (1.1)$$

which are analytic in the *punctured* open unit disk $\mathbb{U}_0 = \{z : 0 < |z| < 1\}$ with a pole at $z = 0$.

A function $f(z) \in \Sigma_p$ is said to be in the class $\Sigma_p^*(\alpha)$ if it satisfies

$$\Re \left\{ -\frac{zf'(z)}{pf(z)} \right\} > \alpha \quad (z \in \mathbb{U} = \mathbb{U}_0 \cup \{0\})$$

for some $\alpha (\alpha < 1)$. Note that, for $0 \leq \alpha < 1$, $\Sigma_p^*(\alpha)$ is the class of meromorphically p -vealently starlike functions of order α in \mathbb{U} . Also we write $\Sigma_1^*(\alpha) = \Sigma^*(\alpha)$.

For functions $f(z) \in \Sigma_p$ given by (1.1) and $g(z) \in \Sigma_p$ given by

$$g(z) = z^{-p} + \sum_{n=1}^{\infty} b_n z^{n-p},$$

the Hadamard product (or convolution) $(f * g)(z)$ of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) := z^{-p} + \sum_{n=1}^{\infty} a_n b_n z^{n-p} =: (g * f)(z).$$

We now define the function $\varphi_p(a, c; z)$ by

$$\varphi_p(a, c; z) = z^{-p} + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} z^{n-p} \quad (z \in \mathbb{U}_0), \quad (1.2)$$

where

$$c \notin \{0, -1, -2, \dots\} \quad \text{and} \quad (x)_0 = 1, \quad (x)_n = x(x+1) \cdots (x+n-1) \quad (n \in \mathbb{N}).$$

Corresponding to the function $\varphi_p(a, c; z)$, the generalized hypergeometric function

$${}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z)$$

is defined by following infinite series:

$${}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \frac{z^n}{n!} \quad (z \in \mathbb{U}), \quad (1.3)$$

where

$$l \leq m + 1, \quad l, m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \quad \beta_j \notin \{0, -1, -2, \dots\} (j = 1, 2, \dots, m).$$

Now, we define

$$h_p(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) = z^{-p} \cdot {}_lF_m(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z).$$

Recently, Liu and Srivastava (see [7].) introduced a linear operator

$$H_p(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m) : \Sigma_p \rightarrow \Sigma_p$$

defined by following Hadamard product (or convolution):

$$H_p(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m)f(z) := h_p(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m; z) * f(z), \quad (1.4)$$

where

$$f(z) \in \Sigma_p, \quad l \leq m + 1, \quad l, m \in \mathbb{N}_0, \quad \beta_j \notin \{0, -1, -2, \dots\} (j = 1, 2, \dots, m).$$

If $f(z) \in \Sigma_p$ is given by (1.1), it follows from (1.3) and (1.4) that

$$H_p(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m)f(z) := z^{-p} + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n a_n}{(\beta_1)_n \cdots (\beta_m)_n n!} z^{n-p}. \quad (1.5)$$

In order to make the notation simple, we write

$$H_p^{l,m}(\alpha_1; \beta_1) := H_p(\alpha_1, \dots, \alpha_l; \beta_1, \dots, \beta_m) \quad (l \leq m + 1, l, m \in \mathbb{N}_0).$$

In particular, for $p = 1, l = 2, m = 1$ and $\alpha_1 = a, \alpha_2 = 1, \beta_1 = c \notin \{0, -1, -2, \dots\}$, we obtain the linear operator

$$L(\alpha_1, \beta_1)f(z) = H_1(a, 1; c)f(z) \quad (f(z) \in \Sigma_p),$$

which was introduced and studied earlier by Liu and Srivastava [9] and Yang [21].

Let \mathcal{P} be the class of analytic functions $h(z)$ with $h(0) = 1$, which are convex and univalent in \mathbb{U} and for which

$$\Re h(z) > 0 \quad (z \in \mathbb{U}).$$

For functions $f(z)$ and $g(z)$ analytic in \mathbb{U} . We say that the function $f(z)$ is *subordinate* to $g(z)$ in \mathbb{U} , and we write $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in \mathbb{U} such that

$$|w(z)| \leq |z| \quad \text{and} \quad f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

Furthermore, if the function $g(z)$ is univalent in \mathbb{U} , then

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Throughout our present investigation, we assume that

$$p, k \in \mathbb{N}, \quad l \leq m + 1, \quad l, m \in \mathbb{N}_0, \quad \beta_j \notin \{0, -1, -2, \dots\} (j = 1, \dots, m), \quad \varepsilon_k = \exp\left(\frac{2\pi i}{k}\right)$$

and

$$f_{p,k}^{l,m}(\alpha_1; \beta_1; z) = \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon_k^{jp} \left(H_p^{l,m}(\alpha_1; \beta_1) f \right) (\varepsilon_k^j z) = z^{-p} + \dots \quad (f(z) \in \Sigma_p). \quad (1.6)$$

Obviously, for $k = 1$, we have

$$f_{p,1}^{l,m}(\alpha_1; \beta_1; z) = H_p^{l,m}(\alpha_1; \beta_1) f(z).$$

Meromorphic (and analytic) functions with respect to k -symmetric points have been investigated by authors (see [13, 15-16, 19, 22-23 and the references therein]).

In this paper, making use of the linear operator $H_p^{l,m}(\alpha_1; \beta_1)$ and the above-mentioned principle of subordination between analytic functions, we introduce and investigate the following subclass of the class Σ_p :

Definition 1. A function $f(z) \in \Sigma_p$ is said to be in the class $\Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$ if it satisfies

$$-\frac{z \left(H_p^{l,m}(\alpha_1; \beta_1) f \right)'(z)}{p f_{p,k}^{l,m}(\alpha_1; \beta_1; z)} \prec h(z) \quad (z \in \mathbb{U}), \quad (1.7)$$

where

$$h(z) \in \mathcal{P} \quad \text{and} \quad f_{p,k}^{l,m}(\alpha_1; \beta_1; z) \neq 0$$

is defined by (1.6).

Remark 1. Note that

$$\Sigma_{p,1}^{2,1} \left(1, 1; 1; \frac{1 + (1 - 2\alpha)z}{1 - z} \right) = \Sigma_p^*(\alpha) \quad (0 \leq \alpha < 1).$$

For $l = 2, m = 1$ and $\alpha_1 = a, \alpha_2 = 1$ and $\beta_1 = c$, then $\Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$ reduces to the function class $\Sigma_{p,k}(a, c; h)$, which was introduced and investigated earlier by Srivastava et al. [15]. For some other recent investigations of meromorphic functions, see (for example) the works of [1-10, 14-15, 18, 21] and the references therein.

Let \mathcal{A} be the class of functions of form

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n,$$

which are analytic in \mathbb{U} . A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(\alpha)$ if it satisfies

$$\Re \left\{ \frac{z f'(z)}{f(z)} \right\} > \alpha \quad (z \in \mathbb{U}) \quad (1.8)$$

for some $\alpha (\alpha < 1)$. When $0 \leq \alpha < 1$, $\mathcal{S}^*(\alpha)$ is the class of starlike functions of order α in \mathbb{U} . A function $f(z) \in \mathcal{A}$ is said to be prestarlike of order α in \mathbb{U} if

$$\frac{z}{(1-z)^{2(1-\alpha)}} * f(z) \in \mathcal{S}^*(\alpha) \quad (\alpha < 1).$$

We denote this class by $\mathcal{R}(\alpha)$ (see [12]). It is clear that a function $f(z) \in \mathcal{A}$ is in the class $\mathcal{R}(0)$ if and only if $f(z)$ is convex univalent in \mathbb{U} and

$$\mathcal{R}\left(\frac{1}{2}\right) = \mathcal{S}^*\left(\frac{1}{2}\right).$$

In order to derive our results, we need the following lemmas.

Lemma 1. (Wang [18]) Let $f(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$. Then

$$-\frac{z \left(f_{p,k}^{l,m}(\alpha_1; \beta_1; z) \right)'}{p f_{p,k}^{l,m}(\alpha_1; \beta_1; z)} \prec h(z) \quad (z \in \mathbb{U}). \tag{1.9}$$

Remark 2. By Lemma 1 we see that, if $f(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$, then $f_{p,k}^{l,m}(\alpha_1; \beta_1; z) \in \Sigma_p^*(0)$.

Lemma 2. (Ruscheweyh [12]). Let $\alpha < 1$, $f(z) \in \mathcal{R}(\alpha)$ and $g(z) \in \mathcal{S}^*(\alpha)$. Then, for any analytic function $F(z)$ in \mathbb{U} ,

$$\frac{f * (gF)}{f * g}(\mathbb{U}) \subset \overline{co}(F(\mathbb{U})),$$

where $\overline{co}(F(\mathbb{U}))$ denotes the convex hull of $F(\mathbb{U})$.

In this paper, we aim at proving such results as inclusion relationships, convolution properties and some inequalities for the function class $\Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$. The results presented here would provide extensions of those given in a number of earlier works. Several other new results will also be obtained.

2. CONVOLUTION PROPERTIES

Theorem 3. Let $h(z) \in \mathcal{P}$ and $\alpha < 1$ satisfy

$$\Re h(z) < 1 + \frac{1-\alpha}{p} \quad (z \in \mathbb{U}). \tag{2.1}$$

If $f(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$,

$$g(z) \in \Sigma_p \quad \text{and} \quad z^{p+1}g(z) \in \mathcal{R}(\alpha) \quad (\alpha < 1), \quad (2.2)$$

then

$$(f * g)(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h).$$

Proof. Let $f(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$ and suppose that

$$w(z) = z^{p+1}f_{p,k}^{l,m}(\alpha_1; \beta_1; z). \quad (2.3)$$

Then

$$F(z) = -\frac{z \left(H_p^{l,m}(\alpha_1; \beta_1) f \right)'(z)}{p f_{p,k}^{l,m}(\alpha_1; \beta_1; z)} \prec h(z), \quad (2.4)$$

$w(z) \in \mathcal{A}$ and

$$\frac{zw'(z)}{w(z)} = \frac{z \left(f_{p,k}^{l,m}(\alpha_1; \beta_1; z) \right)'}{f_{p,k}^{l,m}(\alpha_1; \beta_1; z)} + p + 1. \quad (2.5)$$

It follows from Lemma 1 and (2.5) that

$$\frac{zw'(z)}{w(z)} \prec -ph(z) + p + 1 \quad (z \in \mathbb{U}). \quad (2.6)$$

In view of (2.1) and (2.6), we see that

$$\Re \left\{ \frac{zw'(z)}{w(z)} \right\} > \alpha \quad (z \in \mathbb{U}).$$

That is,

$$w(z) \in \mathcal{S}^*(\alpha) \quad (\alpha < 1). \quad (2.7)$$

Applying the definition of $H_p^{l,m}(\alpha_1; \beta_1)$ and properties of convolution, we have

$$z^{p+1} \left(H_p^{l,m}(\alpha_1; \beta_1)(f * g) \right) (\varepsilon_k^j z) = (z^{p+1}g(z)) * \left(z^{p+1} \left(H_p^{l,m}(\alpha_1; \beta_1)f \right) (\varepsilon_k^j z) \right) \quad (2.8)$$

$$(j = 0, 1, 2, \dots, k-1)$$

and

$$z^{p+2} \left(H_p^{l,m}(\alpha_1; \beta_1)(f * g) \right)'(z) = (z^{p+1}g(z)) * z^{p+2} \left(H_p^{l,m}(\alpha_1; \beta_1)f \right)'(z). \quad (2.9)$$

Making use of (1.6), (2.3), (2.4), (2.8) and (2.9), we deduce that

$$\begin{aligned}
 \Phi(z) &:= -\frac{z \left(H_p^{l,m}(\alpha_1; \beta_1)(f * g) \right)'(z)}{p(f * g)_{p,k}^{l,m}(\alpha_1; \beta_1; z)} \\
 &= -\frac{z \left(H_p^{l,m}(\alpha_1; \beta_1)(f * g) \right)'(z)}{\frac{p}{k} \sum_{j=0}^{k-1} \varepsilon_k^{jp} \left(H_p^{l,m}(\alpha_1; \beta_1)(f * g) \right) (\varepsilon_k^j z)} \\
 &= -\frac{(z^{p+1}g(z)) * \left(z^{p+2} \left(H_p^{l,m}(\alpha_1; \beta_1)f \right)'(z) \right)}{p(z^{p+1}g(z)) * \left(z^{p+1}f_{p,k}^{l,m}(\alpha_1; \beta_1; z) \right)} \\
 &= \frac{(z^{p+1}g(z)) * (w(z)F(z))}{(z^{p+1}g(z)) * w(z)} \quad (z \in \mathbb{U}) \tag{2.10}
 \end{aligned}$$

for $g(z) \in \Sigma_p$. Since $h(z)$ is convex univalent in \mathbb{U} , it follows from (2.2), (2.4), (2.7), (2.10) and Lemma 2 that $\Phi(z) \prec h(z)$. Hence we conclude that the function $(f * g)(z)$ belongs to $\Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$.

For $\alpha = 0$ and $\alpha = \frac{1}{2}$, Theorem 3 reduce to the following:

Corollary 4. *Let $h(z) \in \mathcal{P}$ and $g(z) \in \Sigma_p$ satisfy either of the following conditions:*

(i) $z^{p+1}g(z)$ is convex univalent in \mathbb{U} and

$$\Re h(z) < 1 + \frac{1}{p} \quad (z \in \mathbb{U})$$

or

(ii) $z^{p+1}g(z) \in \mathcal{S}^*(\frac{1}{2})$ and

$$\Re h(z) < 1 + \frac{1}{2p} \quad (z \in \mathbb{U}).$$

If $f(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$, then

$$(f * g)(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h).$$

Corollary 5. *Let $h(z) \in \mathcal{P}$ with*

$$\Re h(z) < 1 + \frac{1}{p} \quad (z \in \mathbb{U}).$$

If $f(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$, then the function $J_{p,\lambda}f(z) \in \Sigma_p$ defined by

$$J_{p,\lambda}f(z) := \frac{\lambda - p}{z^\lambda} \int_0^z t^{\lambda-1} f(t) dt \quad (\operatorname{Re} \lambda \geq 1 + p) \quad (2.11)$$

is also belongs to the class $\Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$, provided that $(J_{p,\lambda}f)_{p,k}^{l,m}(\alpha_1; \beta_1; z) \neq 0$, where $(J_{p,\lambda}f)_{p,k}^{l,m}(\alpha_1; \beta_1; z)$ is defined as in (1.6).

Proof. Let $f(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$ and $J_{p,\lambda}f(z)$ be defined by (2.11), where $h(z) \in \mathcal{P}$ and

$$\Re h(z) < 1 + \frac{1}{p} \quad (z \in \mathbb{U}).$$

Then

$$J_{p,\lambda}f(z) = \frac{\lambda - p}{z^\lambda} \int_0^z t^{\lambda-1} f(t) dt = (f * g)(z),$$

where

$$g(z) = z^{-p} + \sum_{n=1}^{\infty} \frac{\lambda - p}{\lambda + n - p} z^{n-p} \quad (z \in \mathbb{U}).$$

Since $\Re \lambda \geq 1 + p$, it follows from [11, Theorem 5] that $z^{p+1}g(z)$ is convex univalent in \mathbb{U} . Hence an application of Corollary 4 leads to $J_{p,\lambda}f(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$.

Corollary 6. Let $f(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$ and $h(z) \in \mathcal{P}$ satisfy

$$\Re h(z) < 1 + \frac{1}{p} \quad (z \in \mathbb{U}).$$

Then the function $\frac{F_\lambda(z)}{z^{p+1}}$ is also in the class $\Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)(|z| < r_0)$, where $F_\lambda(z)$ is defined by

$$F_\lambda(z) = (1 - \lambda) (z^{p+1} f(z)) + \lambda z (z^{p+1} f(z))' \quad (0 < \lambda \leq 1) \quad (2.12)$$

and

$$r_0 = \frac{1}{2\lambda + \sqrt{4\lambda^2 - 2\lambda + 1}}, \quad (2.13)$$

the radius r_0 is best possible.

On the other hand, if $\frac{F_\lambda(z)}{z^{p+1}} \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$, then $f(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$ for $z \in \mathbb{U}$.

Proof. For $0 < \lambda \leq 1$, $f(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$, we can write (2.12) as

$$F_\lambda(z) = (\Psi * (z^{p+1}f))(z),$$

where

$$\Psi(z) = (1 - \lambda) \frac{z}{1 - z} + \lambda \frac{z}{(1 - z)^2} = z + \sum_{n=2}^{\infty} (1 + (n - 1)\lambda) z^n \in \mathcal{A}. \quad (2.14)$$

It is well known that the function $\Psi(z)$ is convex for $|z| < r_0$, where r_0 is given by (2.13) and this radius is best possible. Applying Corollary 4, we see that $\frac{F_\lambda(z)}{z^{p+1}} \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)(|z| < r_0)$.

On the other hand, from (2.12), we have

$$f(z) = \frac{g(z)}{z^{p+1}} * \frac{F_\lambda(z)}{z^{p+1}},$$

where

$$g(z) = \sum_{n=1}^{\infty} \frac{1}{1 + (n - 1)\lambda} z^n \quad (z \in \mathbb{U}).$$

Since $g(z)$ is convex for $0 < \lambda \leq 1$ (see [11, Theorem 5]). Hence we have $f(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$ for $z \in \mathbb{U}$.

3. INCLUSION RELATIONSHIPS

Theorem 7. Let $\alpha'_1 \geq \alpha_1 > 0$, $h(z) \in \mathcal{P}$ and

$$\Re h(z) < 1 + \frac{\alpha'_1}{2p} \quad (z \in \mathbb{U}). \quad (3.1)$$

Then

$$\Sigma_{p,k}^{l,m}(\alpha'_1; \beta_1; h) \subset \Sigma_{p,k}^{l,m}(\alpha_1, \beta_1; h).$$

Proof. Let $\alpha'_1 \geq \alpha_1 > 0$ and $\varphi_p(\alpha_1, \alpha'_1; z)$ be defined by (1.2). Then

$$\psi(z) = z^{p+1} \varphi_p(\alpha_1, \alpha'_1; z) = z + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n}{(\alpha'_1)_n} z^{n+1} \in \mathcal{A} \quad (z \in \mathbb{U})$$

and

$$\frac{z}{(1 - z)^{\alpha'_1}} * \psi(z) = \frac{z}{(1 - z)^{\alpha_1}} \quad (z \in \mathbb{U}). \quad (3.2)$$

In view of $\alpha'_1 \geq \alpha_1 > 0$, it follows from (3.2) that

$$\frac{z}{(1-z)^{\alpha'_1}} * \psi(z) \in \mathcal{S}^* \left(1 - \frac{\alpha_1}{2}\right) \subset \mathcal{S}^* \left(1 - \frac{\alpha'_1}{2}\right),$$

which implies that

$$\psi(z) \in \mathcal{R} \left(1 - \frac{\alpha'_1}{2}\right). \tag{3.3}$$

For $f(z) \in \Sigma_{p,k}^{l,m}(\alpha'_1; \beta_1; h)$ and $h(z) \in \mathcal{P}$, $\Re h(z) < 1 + \frac{\alpha'_1}{2p}$, it follows from (3.3) and Theorem 3 that

$$(f * \varphi_p)(z) \in \Sigma_{p,k}^{l,m}(\alpha'_1; \beta_1; h),$$

that is

$$-\frac{z \left(H_p^{l,m}(\alpha'_1; \beta_1)(f * \varphi_p)\right)'(z)}{p(f * \varphi_p)_{p,k}^{l,m}(\alpha'_1; \beta_1; z)} \prec h(z) \quad (z \in \mathbb{U}). \tag{3.4}$$

Applying the definition of $H_p^{l,m}(\alpha_1; \beta_1)$ and the properties of convolution, we obtain

$$H_p^{l,m}(\alpha_1; \beta_1)f(z) = \varphi_p(\alpha_1, \alpha'_1; z) * H_p^{l,m}(\alpha'_1; \beta_1)f(z), \tag{3.5}$$

$$z \left(H_p^{l,m}(\alpha_1; \beta_1)f\right)'(z) = \varphi_p(\alpha_1, \alpha'_1; z) * \left(z \left(H_p^{l,m}(\alpha'_1; \beta_1)f\right)'(z)\right) \tag{3.6}$$

and

$$\begin{aligned} f_{p,k}^{l,m}(\alpha_1; \beta_1; z) &= \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon^{jp} \varphi_p(\alpha_1, \alpha'_1; z) * \left(H_p^{l,m}(\alpha'_1; \beta_1)f\right)(\varepsilon_k^j z) \\ &= \varphi_p(\alpha_1, \alpha'_1; z) * f_{p,k}^{l,m}(\alpha'_1, \beta_1; z). \end{aligned} \tag{3.7}$$

It follows from (3.4), (3.5), (3.6) and (3.7) that

$$-\frac{z \left(H_p^{l,m}(\alpha_1; \beta_1)f\right)'(z)}{p f_{p,k}^{l,m}(\alpha_1; \beta_1; z)} = -\frac{z \left(H_p^{l,m}(\alpha'_1; \beta_1)(f * \varphi_p)\right)'(z)}{p(f * \varphi_p)_{p,k}^{l,m}(\alpha'_1; \beta_1; z)} \prec h(z),$$

that is, $f(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$ and the proof of Theorem 7 is completed.

Theorem 8. Let $\beta'_1 \geq \beta_1 > 0$, $h(z) \in \mathcal{P}$ and

$$\Re h(z) < 1 + \frac{\beta'_1}{2p} \quad (z \in \mathbb{U}).$$

Then

$$\Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h) \subset \Sigma_{p,k}^{l,m}(\alpha_1, \beta'_1; h).$$

Proof. Let $\beta'_1 \geq \beta_1 > 0$ and

$$\psi_1(z) = z^{p+1} \varphi_p(\beta_1, \beta'_1; z) = z + \sum_{n=1}^{\infty} \frac{(\beta_1)_n}{(\beta'_1)_n} z^{n+1} \in \mathcal{A} \quad (z \in \mathbb{U}),$$

where $\varphi_p(\beta_1, \beta'_1; z)$ is defined by (1.2). In the same way as we have obtained (3.3) we get

$$\psi_1(z) \in \mathcal{R} \left(1 - \frac{\beta'_1}{2} \right). \quad (3.8)$$

For $f(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$, where $h(z) \in \mathcal{P}$ and

$$\Re h(z) < 1 + \frac{\beta'_1}{2p} \quad (z \in \mathbb{U}),$$

we have

$$H_p^{l,m}(\alpha_1; \beta'_1) f(z) = \varphi_p(\beta_1, \beta'_1; z) * H_p^{l,m}(\alpha_1; \beta_1) f(z), \quad (3.9)$$

$$z \left(H_p^{l,m}(\alpha_1; \beta'_1) f \right)'(z) = \varphi_p(\beta_1, \beta'_1; z) * \left(z \left(H_p^{l,m}(\alpha_1; \beta_1) f \right)'(z) \right) \quad (3.10)$$

and

$$f_{p,k}^{l,m}(\alpha_1; \beta'_1; z) = \varphi_p(\beta_1; \beta'_1; z) * f_{p,k}^{l,m}(\alpha_1, \beta_1; z). \quad (3.11)$$

Using (3.8)-(3.11) and the same arguments as in the proof of Theorem 7, we obtain

$$-\frac{z \left(H_p^{l,m}(\alpha_1; \beta'_1) f \right)'(z)}{p f_{p,k}^{l,m}(\alpha_1; \beta'_1; z)} = -\frac{z \left(H_p^{l,m}(\alpha_1; \beta_1) (f * \varphi_p) \right)'(z)}{p (f * \varphi_p)_{p,k}^{l,m}(\alpha_1; \beta_1; z)} \prec h(z),$$

that is, $f(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta'_1; h)$ and the proof of Theorem 8 is completed.

Corollary 9. Let $\alpha'_1 \geq \alpha_1 > 0, \beta'_1 \geq \beta_1 > 0, h(z) \in \mathcal{P}$ and

$$\Re h(z) < \min \left\{ 1 + \frac{\alpha'_1}{2p}, 1 + \frac{\beta'_1}{2p} \right\}.$$

Then

$$\Sigma_{p,k}^{l,m}(\alpha'_1; \beta_1; h) \subset \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h) \subset \Sigma_{p,k}^{l,m}(\alpha_1; \beta'_1; h).$$

Taking $\alpha_1 > 0, \beta_1 > 0, \alpha'_1 = \alpha_1 + 1$ and $\beta'_1 = \beta_1 + 1$ in Corollary 9, we have

Corollary 10. Let $\alpha_1 > 0, \beta_1 > 0, h(z) \in \mathcal{P}$ and

$$\Re h(z) < \min \left\{ 1 + \frac{\alpha_1 + 1}{2p}, 1 + \frac{\beta_1 + 1}{2p} \right\}.$$

Then

$$\Sigma_{p,k}^{l,m}(\alpha_1 + 1; \beta_1; h) \subset \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h) \subset \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1 + 1; h).$$

Remark 3. In [15, Theorem 1], Srivastava et al. only proved $\Sigma_{p,k}(a + 1, c; h) \subset \Sigma_{p,k}(a; c; h)$. Therefore above results improves and extends [15, Theorem 1].

Corollary 11. Let $\alpha'_1 \geq \alpha_1 > 0, \beta'_1 \geq \beta_1 > 0$ and

$$h(z) = \left(\frac{1 + Az}{1 + Bz} \right)^\gamma \quad (0 < \gamma \leq 1, -1 \leq B < A \leq 1; z \in \mathbb{U}). \quad (3.12)$$

If

$$\alpha'_1 \geq 2p \left(\left(\frac{1 - A}{1 - B} \right)^\gamma - 1 \right) \quad \text{and} \quad \beta'_1 \geq 2p \left(\left(\frac{1 - A}{1 - B} \right)^\gamma - 1 \right), \quad (3.13)$$

then

$$\Sigma_{p,k}^{l,m}(\alpha'_1; \beta_1; h) \subset \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h) \subset \Sigma_{p,k}^{l,m}(\alpha_1; \beta'_1; h).$$

Proof. The analytic function $h(z)$ defined by (3.12) is convex univalent in \mathbb{U} (see [20]), $h(0) = 1$, and $h(\mathbb{U})$ is symmetric with respect to the axis. Thus

$$0 \leq \left(\frac{1 - A}{1 - B} \right)^\gamma < \Re h(z) < \left(\frac{1 + A}{1 + B} \right)^\gamma \quad (z \in \mathbb{U}).$$

Hence we have the corollary by using (3.13) and Corollary 9.

4. SOME INEQUALITIES

Theorem 12. Let $f(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h), h(z) \in \mathcal{P}$ and $h(\mathbb{U})$ be symmetric with respect to the real axis. Then

$$\exp \left(p \int_0^1 \frac{1 - h(-\rho)}{\rho} d\rho \right) < \left| z^p f_{p,k}^{l,m}(\alpha_1; \beta_1; z) \right| < \exp \left(p \int_0^1 \frac{1 - h(\rho)}{\rho} d\rho \right) \quad (z \in \mathbb{U}). \quad (4.1)$$

Proof. Let $f(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$, $h(z) \in \mathcal{P}$, by Lemma 1, we obtain

$$-\frac{z(f_{p,k}^{l,m}(\alpha_1; \beta_1; z))'}{p f_{p,k}^{l,m}(\alpha_1; \beta_1; z)} \prec h(z) \quad (z \in \mathbb{U}). \quad (4.2)$$

Applying a result due to Suffridge [17, Theorem 3] to (4.2), we have

$$-\int_0^z \left(\frac{(f_{p,k}^{l,m}(\alpha_1; \beta_1; t))'}{f_{p,k}^{l,m}(\alpha_1; \beta_1; t)} + \frac{p}{t} \right) dt \prec p \int_0^z \frac{h(t) - 1}{t} dt$$

or

$$\log \frac{1}{z^p f_{p,k}^{l,m}(\alpha_1; \beta_1; z)} \prec p \int_0^1 \frac{h(\rho z) - 1}{\rho} d\rho. \quad (4.3)$$

Noting that the univalent function $h(z)$ maps the disk $|z| < \rho$ ($0 < \rho \leq 1$) onto a region which is convex and symmetric with respect to the real axis, we obtain

$$\int_0^1 \frac{h(-\rho) - 1}{\rho} d\rho < \Re \left(\int_0^1 \frac{h(\rho z) - 1}{\rho} d\rho \right) < \int_0^1 \frac{h(\rho) - 1}{\rho} d\rho \quad (z \in \mathbb{U}).$$

Consequently, the subordination (4.3) leads to

$$p \int_0^1 \frac{h(-\rho) - 1}{\rho} d\rho < \log \left| \frac{1}{z^p f_{p,k}^{l,m}(\alpha_1; \beta_1; z)} \right| < p \int_0^1 \frac{h(\rho) - 1}{\rho} d\rho,$$

which implies (4.1).

Corollary 13. Let $f(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$,

$$h(z) = \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}),$$

where $-1 \leq B < A \leq 1$ and $p(A - B) \leq 1$. Then

$$\Re \left\{ z^p f_{p,k}^{l,m}(\alpha_1; \beta_1; z) \right\}^{-\gamma} > \begin{cases} (1 - B)^{\frac{p(A-B)\gamma}{B}}, & (B \neq 0; z \in \mathbb{U}), \\ e^{-pA\gamma}, & (B = 0; z \in \mathbb{U}), \end{cases} \quad (4.4)$$

where $0 < \gamma \leq 1$.

Proof. In Theorem 12, let

$$h(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1; z \in \mathbb{U}).$$

It follows from (4.3) that

$$\left(z^p f_{p,k}^{l,m}(\alpha_1; \beta_1; z)\right)^{-1} \prec (1 + Bz)^{\frac{p(A-B)}{B}} = g_1(z) \quad (B \neq 0; z \in \mathbb{U}) \quad (4.5)$$

and

$$\left(z^p f_{p,k}^{l,m}(\alpha_1; \beta_1; z)\right)^{-1} \prec e^{pAz} \quad (B = 0; z \in \mathbb{U}). \quad (4.6)$$

For $-1 \leq B < A \leq 1, p(A - B) \leq 1$, the $g_1(\mathbb{U})$ is symmetric with respect to the real axis and $g_1(z)$ is convex univalent in \mathbb{U} because

$$\Re \left\{ 1 + \frac{z g_1''(z)}{g_1'(z)} \right\} = \Re \frac{1 + p(A - B)z}{1 + Bz} > 0 \quad (z \in \mathbb{U}).$$

Therefore, with the aid of the elementary $\Re(w^\gamma) \geq (\Re w)^\gamma (0 < \gamma \leq 1)$ and $\Re w > 0$, it follows from (4.5) and (4.6) that

$$\Re \left\{ \frac{1}{f_{p,k}^{l,m}(\alpha_1; \beta_1; z)} \right\}^\gamma \geq \left\{ \Re \frac{1}{f_{p,k}^{l,m}(\alpha_1; \beta_1; z)} \right\}^\gamma > \begin{cases} (1 - B)^{\frac{p(A-B)\gamma}{B}}, & (B \neq 0; z \in \mathbb{U}), \\ e^{-pA\gamma}, & (B = 0; z \in \mathbb{U}) \end{cases}$$

for $0 < \gamma \leq 1$. This proves (4.4).

Letting $A = 1 - 2\alpha(1 - \frac{1}{2p} \leq \alpha < 1), B = -1$ in Corollary 11, we have

Corollary 14. *If $f(z) \in \Sigma_p$ satisfies*

$$\Re \left\{ -\frac{z(H_{p,k}^{l,m}(\alpha_1; \beta_1)f)'(z)}{p f_{p,k}^{l,m}(\alpha_1; \beta_1; z)} \right\} > \alpha,$$

then

$$\Re \left\{ z^p f_{p,k}^{l,m}(\alpha_1; \beta_1; z) \right\}^{-1} > \frac{1}{4p(1-\alpha)\gamma},$$

where $0 < \gamma \leq 1$. The result is sharp for the function $f(z)$ given by

$$z^p f_{p,k}^{l,m}(\alpha_1; \beta_1; z) = (1 - z)^{2(1-\alpha)}.$$

Letting $k = l = 2, m = p = \gamma = \alpha_1 = \alpha_2 = \beta_1 = 1$ in Corollary 13, we have

Corollary 15. *If $f(z) \in \Sigma_1$ satisfies*

$$\Re \left\{ -\frac{zf'(z)}{f(z) - f(-z)} \right\} > \alpha,$$

where $\frac{1}{4} \leq \alpha < \frac{1}{2}$, then

$$\Re \left\{ \frac{1}{z(f(z) - f(-z))} \right\} > \frac{1}{4^{1-\alpha}}.$$

The result is sharp.

Corollary 16. *Let $f(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$, where*

$$h(z) = \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1 \text{ and } p(A - B) \leq 1; z \in \mathbb{U}).$$

Then

$$\left| \arg \left\{ \frac{1}{z^p f_{p,k}^{l,m}(\alpha_1; \beta_1; z)} \right\} \right| \leq \begin{cases} \frac{p(A-B)\gamma}{B} \arcsin B, & (B \neq 0; z \in \mathbb{U}), \\ pA, & (B = 0; z \in \mathbb{U}). \end{cases}$$

Finally, we prove an integral representation associated with the function class $\Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$.

Theorem 17. *Let $f(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$. Then*

$$f(z) = \left\{ -p \int_0^z \zeta^{-p-1} h(\omega_2(\zeta)) \cdot \exp \left\{ -p \int_0^\zeta \frac{h(\omega_1(\xi)) - 1}{\xi} d\xi \right\} d\zeta \right\} \\ * \left(\sum_{n=0}^{\infty} \frac{n!(\beta_1)_n \cdots (\beta_m)_n}{(\alpha_1)_n \cdots (\alpha_l)_n} z^{n-p} \right), \quad (4.7)$$

where $\omega_j(z) (j = 1, 2)$ are analytic in \mathbb{U} with

$$\omega_j(0) = 0 \quad \text{and} \quad |\omega_j(z)| \leq 1 \quad (z \in \mathbb{U}; j = 1, 2).$$

Proof. Let $f(z) \in \Sigma_{p,k}^{l,m}(\alpha_1; \beta_1; h)$. We find from (4.3) that

$$f_{p,k}^{l,m}(\alpha_1; \beta_1; h) = z^{-p} \exp \left\{ -p \int_0^z \frac{h(\omega_1(\xi)) - 1}{\xi} d\xi \right\}, \quad (4.8)$$

where $\omega_1(z)$ is analytic in \mathbb{U} and $\omega_1(0) = 1$. It follows from (1.7) and (4.8) that

$$\begin{aligned} \left(H_p^{l,m}(\alpha_1; \beta_1) f \right)'(z) &= -\frac{p}{z} f_{p,k}^{l,m}(\alpha_1; \beta_1; h) \cdot h(\omega_2(z)) \\ &= -pz^{-p-1} h(\omega_2(z)) \cdot \exp \left\{ -p \int_0^z \frac{h(\omega_1(\xi)) - 1}{\xi} d\xi \right\}, \end{aligned} \quad (4.9)$$

where $\omega_j(z) (j = 1, 2)$ are analytic in \mathbb{U} with

$$\omega_j(0) = 0 \quad \text{and} \quad |\omega_j(z)| \leq 1 \quad (z \in \mathbb{U}; j = 1, 2).$$

Upon integrating both sides of (4.9), we have

$$H_p^{l,m}(\alpha_1; \beta_1) f(z) = -p \int_0^z \zeta^{-p-1} h(\omega_2(\zeta)) \cdot \exp \left\{ -p \int_0^z \frac{h(\omega_1(\xi)) - 1}{\xi} d\xi \right\}. \quad (4.10)$$

In view of (1.5) and (4.10), we know that

$$z^{-p} + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n}{(\beta_1)_n \cdots (\beta_m)_n} \cdot \frac{a_{n-p}}{n!} z^{n-p} = -p \int_0^z \zeta^{-p-1} h(\omega_2(\zeta)) \cdot \exp \left\{ -p \int_0^z \frac{h(\omega_1(\xi)) - 1}{\xi} d\xi \right\}. \quad (4.11)$$

Thus, from (4.11), we easily arrive at (4.7).

Acknowledgements This work was partially supported by *the National Natural Science Foundation of China* (Grant No. 11171045)

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