Acta Universitatis Apulensis No. 41/2015
ISSN: 1582-5329 pp. 131-140
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2015.41.10

CONVERGENCE OF WAVELET GALERKIN METHOD FOR FREDHOLM INTEGRAL EQUATION OF THE FIRST KIND

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Abstract. In this work, we study about Fredholm integral equation of the first kind which is one of the most important ill-posed problems. There are several methods such as projection methods, regularization method and moment method for solving integral equations of the first kind. In this paper, we use Galerkin method as one of the projection methods with wavelet basis to discretize the equation. In this process, solution of Fredholm integral equation is found by solving the generated system of equations. Convergence of the method and a bound for condition number of the system is also presented.

2010 *Mathematics Subject Classification*: 45B05, 65L60, 34K28.

Keywords: Fredholm integral equation, First kind, Wavelet basis, Galerkin method.

1. Introduction

Fredholm integral equation of the first kind is one of the ill-posed problems which is appeared in many engineering fields. Consider the following Fredholm integral equation of the first kind:

$$\int_a^b k(s,t)f(t)dt = g(s), \quad -\infty < a \le s \le b < \infty$$

where $k(s,t)$ and $g(s)$ are known functions and $f(t)$ is an unknown function to be determined. We suppose that the equation has a solution in $L^2[a,b]$. If we define an operator $\mathcal{K}$ as following

$$\mathcal{K}(f(t)) = \int_a^b k(s,t)f(t)dt, \quad \mathcal{K} : X \to Y,$$

where X, Y are normed spaces, then we have the following definition:

Definition 1. *Let $\mathcal{K} : X \to Y$ be an operator from a normed space X into a normed space Y, the equation*

$$\mathcal{K}(f) = g \tag{1}$$

is called well posed if $\mathcal{K}$ is onto, one-to-one and the inverse operator $\mathcal{K}^{-1} : Y \to X$ is continuous. Otherwise the equation is called ill posed [4].

According to this definition we may distinguish three type of ill posedness [3]:

1. If $\mathcal{K}$ is not onto, then (1) is not solvable for all $g \in Y$. (non existence)
2. If $\mathcal{K}$ is not one-to-one, then (1) may have more than one solution. (non uniqueness)
3. If $\mathcal{K}^{-1}$ exist but is not continuous, then the solution f of (1) dose not depend continuously on the data g. (instability)

Discussion on stability of solution of the Fredholm integral equation of the first kind needs to determine the inverse of integral operator, but this is impossible in many cases, so that we prefer to discuss on convergence of numerical methods instead of stability of them. In this paper, we present a theorem about convergence of Galerkin method with wavelet basis for integral equation of the first kind.

2. Discreatization by Galerkin Method

Consider the first kind Fredholm integral equation of the form

$$\int_a^b k(s,t)f(t)dt = g(s), \quad -\infty < a \leq s \leq b < \infty. \tag{2}$$

For numerical solving of (2) we should choose a finite dimensional family of functions which the exact solution can be estimated by them. Methods that use this strategy are called projection methods, because the exact solution of equation is projected to a finite dimensional space. One of the most famous projection methods, is Galerkin method.

For introducing this method we write the equation (1) in the operator form

$$\mathcal{K}f = g.$$

We choose a sequence of finite dimensional subspaces $X_n \subset L^2[a,b]$ for $n \geq 1$, with X_n having dimension d_n.

Assume that X_n has a basis $\{\phi_1, \phi_2, ..., \phi_d\}$ with $d \equiv d_n$ for notational simplicity and f_n is a function belongs to X_n, so that we can write it as $f_n(t) = \sum_{j=1}^{d} c_j \phi_j(t)$. By substituting in (2) we have

$$\begin{aligned} r_n(s) &= \int_a^b k(s,t) f_n(t) dt - g(s) \\ &= \int_a^b k(s,t) \sum_{j=1}^{d} c_j \phi_j(t) dt - g(s), \quad a \le s \le b \end{aligned}$$

where r_n is called the residual when using $f \approx f_n$. In the operator form we have

$$r_n = \mathcal{K} f_n - g.$$

In Galerkin method, require r_n to satisfy

$$(r_n, \phi_i) = 0, \quad i = 1, ..., d,$$

which $(\cdot,\cdot)$ shows the inner product $(x, y) = \int_a^b x(t)\overline{y(t)}dt$ for $L^2[a, b]$.

This yields the linear system

$$\sum_{j=1}^{d} c_j (\mathcal{K}\phi_j, \phi_i) = (g, \phi_i), \quad i = 1, 2, ..., d. \tag{3}$$

Now we should discuss about solution of the above system. For this result we define orthogonal projection operator as $P_n f = \sum_{i=1}^{d} (f, \psi_i)\psi_i$ where $\{\psi_1, \psi_2, ..., \psi_d\}$ is an orthonormal basis that can be create by using Gram-Schmidt process from elementary basis $\{\phi_1, \phi_2, ..., \phi_d\}$.

By using $P_n f$ we will have the following problem

$$\|f - P_n f\| = \min_{z \in X_n} \|f - z\|. \tag{4}$$

If we show that the above problem has unique solution and this solution is $P_n f$ then the system of linear (3) have unique solution. Since $P_n f$ is an orthogonal projection operator, so we have

$$\begin{gathered} \|f\|^2 = \|f - P_n f\|^2 + \|P_n f\|^2 \\ ((I - P_n)f, P_n g) = 0, \quad \forall f, g \in L^2(\mathbb{R}) \\ \|f - z\|^2 = \|f - P_n f\|^2 + \|P_n f - z\|^2 \quad z \in X_n \end{gathered}$$

from the latest result we can conclude that (4) has a unique solution and the solution is $P_n f$. (See [1] for more details.)

3. Wavelet basis

The basic construction of wavelet basis is based on scaling function ϕ. The basis are generated by the scaling function $\phi(t)$ as

$$\phi_{jk}(t) = 2^{-j/2}\phi(2^{-j}t - k), \quad t \in \mathbb{R}, \quad j,k \in \mathbb{Z}$$

where j, k are scaling and shifting parameters, respectively. The scaling function is satisfied the following equation

$$\phi(t) = \sum_{n=n_0}^{n_1} a_n \phi(2t - n), \tag{5}$$

This equation is called refinement equation where $[n_0, n_1]$ is the support of the scaling function $\phi(t)$. So when the scaling function has compact support then there is a finite number of nonzero a_n. Then, the mother wavelet basis of $L^2(\mathbb{R})$ space defined as

$$\psi_{jk}(t) = 2^{-j/2}\psi(2^{-j}t - k), \quad t \in \mathbb{R}, \quad j,k \in \mathbb{Z}$$

where j, k are scaling and shifting parameters. For mother wavelet basis we have the following refinement equation

$$\psi(t) = \sum_n (-1)^n a_{1-n} \phi(2t - n)$$

where the a_ns are the coefficient in (5). For every $f \in L^2(\mathbb{R})$ there is a unique expansion

$$f(t) = \sum_{j,k \in \mathbb{Z}} (f, \psi_{jk}) \psi_{jk}(t)$$

which converges in the L^2-norm (see [2] for more details).

4. Convergence of Galerkin method by wavelets

Here we introduce a theorem and a lemma to show the convergence of the method.

Theorem 1. *Consider the integral equation of the first kind*

$$\int_a^b k(s,t) f(t) dt = g(s), \quad -\infty < a \le s \le b < \infty$$

assume that $k(s,t)$ is continuous on the square $[a,b]^2$,and that the solution of the equation belong to $C^\alpha[a,b]$ for some $\alpha > \frac{1}{2}$, (C^α is Holder continuous space of order

α), also assume that the scaling function ϕ is Holder continuous of order $r > \alpha$ and assume that the support of ϕ is $[N_1, N_2]$ $(N_1, N_2 \in \mathbb{Z})$ also assume that for a positive integer J, S_J is a set of all integers where ϕ_{-Jk} is nonzero. Now by using projection operator $P_J^{num}(f)(t) = \sum_{k\in S_J} \alpha_{Jk}^{num}\phi_{-Jk}(t)$ and Galerkin method we have system of linear equation $A_J X = b_J$ where

$$A_J = \left[\int_a^b \int_a^b k(s,t)\phi_{-Jk}(t)\phi_{-Jk'}(s)dtds\right]_{k,k'\in S_J}$$

$$b_J = \left[\int_a^b g(s)\phi_{-Jk'}(s)ds\right]_{k'\in S_J}$$

$$X^T = [\alpha_{Jk}^{num}]_{k\in S_J}$$

if A is invertible, then

$$\sup_{t\in[a,b]}\left|f(t) - \sum_{k\in S_J}\alpha_{Jk}^{num}\phi_{-Jk}(t)\right| \le c2^{-J\alpha}\big(1 + 2^J\|A_J^{-1}\|_\infty\big).$$

Proof. Let $f(t) \simeq P_J^{num}(f)(t) = \sum_{k\in S_J}\alpha_{Jk}^{num}\phi_{-Jk}(t)$ and

$$P_J(f)(t) = \sum_{k\in S_J}\alpha_{Jk}\phi_{-Jk}(t).$$

Now, if we substitute the approximation of $f(t)$ with wavelet basis in the integral equation, then the right hand side of integral equation is exchanged by a new function that we denote it by $\hat{g}(s)$, such that,

$$g(s) = \int_a^b k(s,t)P_J^{num}(f)(t)dt \tag{6}$$

$$\hat{g}(s) = \int_a^b k(s,t)P_J(f)(t)dt. \tag{7}$$

If we solve (6), we determine the $\{\alpha_{Jk}^{num}, k \in S_J\}$ by

$$(\alpha_{Jk}^{num})_{k\in S_J} = A_J^{-1}\left(\left[\int_a^b g(s)\phi_{-Jk}(s)ds\right]_{k\in S_J}\right)$$

and if we solve (7); we determine the $\{\alpha_{Jk}, k \in S_J\}$ by

$$(\alpha_{Jk})_{k\in S_J} = A_J^{-1}\left(\left[\int_a^b \hat{g}(s)\phi_{-Jk}(s)ds\right]\right)_{k\in S_J}.$$

Consequently we have

$$\begin{aligned}\sup_{k\in S_J}|\alpha_{Jk}-\alpha_{Jk}^{num}| &\leq \|A_J^{-1}\|_\infty \sup_{k\in S_J}\left|\int_a^b \big(\hat{g}(s)-g(s)\big)\phi_{-Jk}(s)ds\right| \\ &\leq \|A_J^{-1}\|_\infty \sup_{s\in[a,b]}|\hat{g}(s)-g(s)| \sup_{k\in S_J, s\in[a,b]}|\phi_{-Jk}(s)|\cdot(b-a)\end{aligned} \quad (8)$$

The scaling function ϕ has compact support, so ϕ is bounded, and

$$\sup_{k\in S_J, s\in[a,b]}|\phi_{-Jk}(s)| = \sup_{k\in S_J, s\in[a,b]}|2^{J/2}\phi(2^Js-k)| \leq 2^{J/2}M_1.$$

Now, with substituting the above bound in the (8) we have:

$$\sup_{k\in S_J}|\alpha_{Jk}-\alpha_{Jk}^{num}| \leq \|A_J^{-1}\|_\infty 2^{J/2}M_1(b-a)\sup_{s\in[a,b]}|\hat{g}(s)-g(s)|. \quad (9)$$

For finding a bound for $\sup_{s\in[a,b]}|\hat{g}(s)-g(s)|$, we need to estimate the $\hat{g}(s)$. For this we have $g(s)=\int_a^b k(s,t)f(t)dt$, so that

$$\int_a^b k(s,t)\left[f(t)-P_J(f)(t)\right]dt + \int_a^b k(s,t)P_J(f)(t)dt = g(s),$$

$$\int_a^b k(s,t)P_J(f)(t)dt = g(s) - \int_a^b k(s,t)\left[f(t)-P_J(f)(t)\right]dt,$$

$$\hat{g}(s) = g(s) - \int_a^b k(s,t)\left(f(t)-P_J(f)(t)\right)dt,$$

then

$$\begin{aligned}\sup_{s\in[a,b]}|\hat{g}(s)-g(s)| &= \sup_{s\in[a,b]}\left|\int_a^b k(s,t)\left(f(t)-P_J(f)(t)\right)dt\right| \\ &\leq (b-a)\sup_{s,t\in[a,b]}\Big(\big|k(s,t)\big|\cdot\big|f(t)-P_J(f)(t)\big|\Big),\end{aligned}$$

Let $M=\sup_{s,t\in[a,b]}|k(s,t)|$ and from [5] we have $|f(t)-P_J(f)(t)|\leq c_f 2^{-J\alpha}$, then

$$\sup_{s,t\in[a,b]}|\hat{g}(s)-g(s)| \leq Mc_f(b-a)2^{-J\alpha} = M_2(b-a)2^{-J\alpha},$$

and with substituting this bound in the inequality (9), we have

$$\sup_{k\in S_J}|\alpha_{Jk}-\alpha_{Jk}^{num}| \leq \|A_J^{-1}\|_\infty M_1M_2(b-a)^2 2^{(1/2-\alpha)J}.$$

Also we need to determine a bound for $|P_J(f)(t) - P_J^{num}(f)(t)|$, hence we have:

$$\begin{aligned}|P_J(f)(t) - P_J^{num}(f)(t)| &= \left|\sum_{k\in S_J}(\alpha_{Jk} - \alpha_{Jk}^{num})\phi_{-Jk}(t)\right| \\ &\le \|\alpha_{Jk} - \alpha_{Jk}^{num}\|_\infty \sup_{t\in[a,b]}\sum_{k\in S_J}|\phi_{-Jk}(t)| \\ &\le \|\alpha_{Jk} - \alpha_{Jk}^{num}\|_\infty 2^{J/2}\sup_{t\in[a,b]}\sum_{k\in S_J}|\phi(2^J t - k)| \\ &= \|\alpha_{Jk} - \alpha_{Jk}^{num}\|_\infty 2^{J/2}(N_2 - N_1 + 1)M_1\end{aligned}$$

Now, with using this inequalities the result of the theorem can be determined. So

$$\begin{aligned}\|f(t) - P_J^{num}(f)(t)\|_\infty &\le \|f(t) - P_J(f)(t)\|_\infty + \|P_J(f)(t) - P_J^{num}(f)(t)\|_\infty \\ &\le c_f 2^{-J\alpha} + \|A_J^{-1}\|_\infty M_1^2 M_2(b-a)^2 2^{-J(\alpha-1)}(N_2 - N_1 + 1).\end{aligned}$$

Then

$$\|f(t) - P_J^{num}(f)(t)\| \le 2^{-J\alpha}\Big(c_f + M_1^2 M_2(b-a)^2(N_2 - N_1 + 1)2^J\|A_J^{-1}\|_\infty\Big)$$

where $C = \max\{c_f, M_1^2 M_2(b-a)^2(N_2 - N_1 + 1)\}$. Hence $\|f(t) - P_J^{num}(f)(t)\|_\infty \le C2^{-J\alpha}\{1 + 2^J\|A_J^{-1}\|_\infty\}$ and the proof is completed.

The only weak point of this theorem is that the error bound of the scheme contains $\|A_J^{-1}\|_\infty$, hence, in the following lemma by using an extra condition we will found a bound for $\|A_J^{-1}\|_\infty$ and condition number of A_J.

Lemma 2. *Consider the previous theorem and assume that there exists $J' > J$ such that*

$$\|2^{-J'}A_J - I_{S_J}\| = M_{J'} < 1$$

where $\|\cdot\|$ is the maximum norm of rows and I_{S_J} is an identity matrix of order $|S_J|$. Then $\|A_J^{-1}\| \le \frac{2^{-J'}}{1-M_{J'}}$ and also

$$cond(A_J) \le \frac{2^{(J-J')}}{1 - M_{J'}}.$$

Proof. First we determine a bound for $\|A_J^{-1}\|$, let

$$\|B_{J'}\| = \|2^{-J'}A_J - I_{S_J}\| = M_{J'} < 1,$$

from geometric series theorem (see e.g. [1]) we have

$$\|(I+B_{J'})^{-1}\| \leq \frac{1}{1-\|B_{J'}\|},$$

and from

$$\begin{aligned}\|(I_{S_J}+B_{J'})^{-1}\| &= \|(I_{S_J}+2^{-J'}A_J-I_{S_J})^{-1}\| \\ &= 2^{J'}\|A_J^{-1}\| \\ &\leq \frac{1}{1-\|B_{J'}\|}\end{aligned}$$

we have

$$\|A_J^{-1}\| \leq \frac{2^{-J'}}{1-M_{J'}}.$$

Now, we need a bound for $\|A_J\|$.

$$\begin{aligned}\|A_J\| = \max_{k'\in S_J}\sum_{k\in S_J}&\left|\int_a^b\int_a^b k(s,t)\phi_{-Jk}(s)\phi_{-Jk'}(t)dsdt\right| \\ = &\left(\int_a^b\int_a^b |k(s,t)|^2dsdt\right)^{\frac{1}{2}}\max_{k'\in S_J}\left(\int_a^b|\phi_{-Jk'}(t)|^2dt\right)^{\frac{1}{2}}\sum_{k\in S_J}\left(\int_a^b|\phi_{-Jk}(s)|^2ds\right)^{\frac{1}{2}}\end{aligned} \tag{10}$$

Since $k(s,t)$ is continuous on the square $[a,b]^2$, then

$$M_1 = \left(\int_a^b\int_a^b |k(s,t)|^2dsdt\right)^{1/2} \tag{11}$$

is finite. Consequently,

$$\begin{aligned}\max_{k'\in S_J}\left(\int_a^b|\phi_{-Jk'}(t)|^2dt\right)^{1/2} &\leq (b-a)^{1/2}\left[\sup_{t\in[a,b]}\left(\left|2^{J/2}\phi(2^Jt-k')\right|^2\right)\right]^{1/2} \\ &\leq (b-a)^{1/2}2^{J/2}\big(\sup_{t\in[a,b]}|\phi(t)|^2\big)^{1/2} \\ &\leq 2^{J/2}M_2\end{aligned} \tag{12}$$

$$\begin{aligned}
\sum_{k\in S_J}\left(\int_a^b |\phi_{-Jk}(s)|^2 ds\right)^{1/2} &\leq \sum_{k\in S_J}\left((b-a)^{1/2}\Big(\sup_{s\in[a,b]}|\phi_{-Jk}(s)|^2\Big)^{1/2}\right) \\
&= (b-a)^{1/2}\sum_{k\in S_J}\Big[\sup_{s\in[a,b]}\Big(\big|2^{J/2}\phi(2^J s-k)\big|^2\Big)\Big]^{1/2} \\
&= (b-a)^{1/2}2^{J/2}(N_2-N_1+1)\Big(\sup_{s\in[a,b]}|\phi(s)|^2\Big)^{1/2} \\
&= 2^{J/2}M_3 \qquad (13)
\end{aligned}$$

with substituting (11),(12) and (13) in (10), we have

$$\|A_J\| \leq 2^J M_1 M_2 M_3 = 2^J M,$$

hence,

$$cond(A_J) = \|A_J\|\cdot\|A_J^{-1}\| \leq M\frac{2^{(J-J')}}{1-M_{J'}}$$

and the proof is completed.

5. Conclusion

In this paper, first we presented Galerkin method, then by using wavelet basis which were satisfied in the conditions of Theorem 1 we convert the integral equation of the first kind to a linear system of equation. From Theorem 1 the convergence of this method was granted and condition number of system of linear equations was estimated by Lemma 1.

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