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ON SOME PROBABILISTIC AND INFORMATIONAL CHARACTERIZATIONS OF THE WEIBULL DISTRIBUTION

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Abstract. This distribution was proposed by Weibull,W. in 1930. Such a distribution is a member of the family of extreme value distributions. The mathematical properties of this distribution have been studied in detail by Gupta and Kundu in 2001 and by Gupta and Raqab in 2009. The aim of this paper is to establish some probabilistic and informational properties for such a distribution.

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1. Introduction

Weibull distribution was originally proposed by Weibull (1939), a Swedish physicist, and he used it to represent the distribution of the breaking strength of materials [16]. This distribution is a member of the family of extreme value distributions which are the limit distributions of the smallest or greatest value, respectively in a sample with sample size $n \longrightarrow \infty$. The Weibull distributions includes the exponential and the Rayleigh distribution as special cases. The usefulness and applications of these distributions are seen in various areas including reliability, renewal theory and branching processes. Also, in recent years the Weibull distribution becoming very popular distribution widely used for analyzing lifetime data.

2. Some probabilistic properties of the Weibull distribution

Definition 2.1. ([3],[4]) *A random variable X follows a two-parameter Weibull distribution with the shape parameter α and scale parameter λ , respectively if its probability density function, denoted by $f(x;\alpha,\lambda)$, is as follows :*

$$f(y;\alpha,\lambda)=\begin{cases}\alpha\lambda(\lambda x)^{\alpha-1}e^{-(\lambda x)^{\alpha}}, & \alpha>0,\lambda>0,\, x>0\\ 0, & x\le 0.\end{cases}\tag{2.1}$$

Remark 2.1. Using a change of variables as

$$t=(\lambda x)^{\alpha}\Rightarrow \lambda x=t^{\frac{1}{\alpha}}\Longrightarrow dx=\frac{1}{\alpha\lambda}t^{\frac{1-\alpha}{\alpha}}dt\ and\ t\in(0,\infty),\tag{2.2}$$

we get that such a function satisfies the conditions

$$\begin{cases}1^0 & f(x;\alpha,\lambda)>0,\ \alpha>0,\lambda>0,\ x>0\\ 2^0 & \int\limits_{\mu}^{\infty}f(x;\alpha,\lambda)dx=1\end{cases}\tag{2.3}$$

and the corresponding **distribution function** $F(x;\alpha,\lambda)$ has the following expression

$$F(x;\alpha,\lambda)=P(X\le x)=\int\limits_0^x f(t;\alpha,\lambda)dt=\begin{cases}1-e^{-(\lambda x)^{\alpha}}, & \alpha>0,\lambda>0,x>0\\ 0, & x\le 0.\end{cases}\tag{2.4}$$

Lemma 2.1. *The probability density function* (1.1) *is* **log − convex** *if* $0<\alpha\le 1$ *and* **log − concave** *if* $\alpha\ge 1$.

Proof. Indeed, using (2.1), we can obtain the following relations

$$\log_e f(x;\alpha,\lambda)=\log(\alpha\lambda)+(\alpha-1)\log(1-e^{-\lambda x})-\lambda x\tag{2.5}$$

$$\frac{d}{dx}\left[\log_e f(x)\right]=\frac{f\,'(x)}{f(x)}=\frac{\lambda(\alpha-1)e^{-\lambda x}}{1-e^{-\lambda x}}-\lambda\tag{2.5a}$$

$$\frac{d^2}{dx^2}\left[\log_e f(x)\right]=-(\alpha-1)\underbrace{\left\{\frac{\lambda^2e^{-\lambda x}}{1-e^{-\lambda x}}+\left[\frac{\lambda e^{-\lambda x}}{1-e^{-\lambda x}}\right]^2\right\}}_{>0}\tag{2.5b}$$

which, evidently, imply inequalities as

$$\frac{d^2}{dx^2}\left[\log_e f(x)\right]\begin{cases}\ge 0 & if\quad 0<\alpha\le 1\\ \le 0 & if\quad \alpha\ge 1,\end{cases}\tag{2.6}$$

and hence the above conclusions of the lemma follow.

Lemma 2.2. *If X follows a two − parameter Weibull distribution with probability density function* (2.1), *then a new random variabile Y, defined as*

$$Y=X+b,\tag{2.7}$$

will follows a three−parameter Weibull distribution with the probability density function as

$$g(y;\alpha,\lambda,b)=\begin{cases}\alpha\lambda[\lambda(y-b)]^{\alpha-1}e^{-[\lambda(y-b)]^{\alpha}}, & \alpha>0,\lambda>0,\ y>b\\ 0, & y\le b,\end{cases} \tag{2.8}$$

where : α *is the shape parameter*, λ *is the scale parameter*, *and* b *is the location parameter*.

Proof. The distribution function of the random variable Y, denoted by $G(y;\alpha,\lambda,b)$, can be obtained as follows

$$\begin{aligned} G(y;\alpha,\lambda,b) &= P(Y\le y) && (2.9)\\ &= P(X+b\le y)=P(X\le y-b) && (2.9a)\\ &= F(y-b)=\int_0^{y-b} f(x;\alpha,\lambda)dx\ (see\ (1.4)) && (2.9b)\\ &= 1-e^{-[\lambda(y-b)]^{\alpha}},\alpha>0,\lambda>0,y>b, \end{aligned}$$

that is,

$$G(y;\alpha,\lambda,b)=P(Y<y)=\begin{cases}1-e^{-[\lambda(y-b)]^{\alpha}},\alpha>0,\lambda>0, & y>b\\ 0, & y\le b\end{cases} \tag{2.10}$$

with the property

$$\frac{dG(y;\alpha,\lambda,b)}{dy}=g(y;\alpha,\lambda,b). \tag{2.10a}$$

Theorema 2.1. *If* X *follows a two−Weibull distribution with the probability density functions* (2.1), *then its* **the** k^{th} **moment** *has the following form*

$$E(X^k)=\frac{k}{\alpha\lambda^k}\Gamma\left(\frac{k}{\alpha}\right),\alpha>0,\lambda>0,\ for\ k\ge 1. \tag{2.11}$$

Proof. Using the definition of **the** k^{th} **moment of a distribution** as well as the probability density function (1.1), we obtain a first form as

$$E(X^k) = \int_0^\infty x^k f(x;\alpha,\lambda)dx = \int_0^\infty x^k \alpha\lambda(\lambda x)^{\alpha-1} e^{-(\lambda x)^\alpha} dx \tag{2.12}$$

$$= \frac{\alpha\lambda}{\lambda^k}\int_0^\infty (\lambda x)^{k+\alpha-1} e^{-(\lambda x)^\alpha} dx \tag{2.12a}$$

or a second form as

$$E(X^k) = \frac{\alpha\lambda}{\lambda^k}\int_0^\infty y^{\frac{k+\alpha-1}{\alpha}} e^{-y} \frac{1}{\alpha\lambda} y^{\frac{1}{\alpha}-1} dy \tag{2.13}$$

$$= \frac{1}{\lambda^k}\int_0^\infty y^{\frac{k}{\alpha}} e^{-y} dy \tag{2.13a}$$

$$= \frac{k}{\alpha\lambda^k}\Gamma\left(\frac{k}{\alpha}\right), \alpha > 0, \lambda > 0, k \geq 1, \tag{2.13b}$$

if we have used the change of variables (2.2) as well as the Gamma function, that is

$$\Gamma(a) = \int_0^\infty s^{a-1} e^{-s} ds, a > 0. \tag{2.14}$$

3. Fisher information and Weibull distribution

In the next, we will consider a family of probability density functions as $\{f(x;\theta) : \theta \in D_\theta\}$, associated to a continuous random variable X, defined on the probability space (Ω, K, P), where $D_\theta \subset \mathbb{R}^k$, $(k \geq 1)$. The *parameter space*, D_θ, must to be either an open interval of the real line $\mathbb{R}$, if $k = 1$ or an open subset of $k-$*dimensional Euclidian space* $\mathbb{R}^k$, *if* $k > 1$.

Remark 3.1. We supposed that such a probability density function satisfies the following regularity conditions (which are known as the Fisher information regularity conditions **FIRCs)** [12]**,** namely:

R$_1$) *The set* $\{x : f(x,\theta) > 0\}$ *is the same for all* $x \in \Omega$ ($\Omega -$ *an open interval on the real line*) *and all* $\theta \in D_\theta$;

R$_2$) $\frac{\partial}{\partial\theta_i}[f(x,\theta)]$ *exists for all* $x \in \Omega$, *all* $\theta \in D_\theta$ *and all* $i = \overline{1,k}$;

$\mathbf{R}_3)$ $\frac{\partial}{\partial\theta_i}\left[\int_A f(x,\theta)dx\right] = \int_A \frac{\partial}{\partial\theta_i}[f(x,\theta)]dx$ *for any* $A, A \subset K$, *all* $\theta \in D_\theta$ *and all* $i = \overline{1,k}$;

$\mathbf{R}_4)$ $\int_A \frac{\partial}{\partial\theta_i\partial\theta_j}\left[f(x,\theta)\right]dx < \infty$ *for any* $A, A \subset K$, *all* $\theta \in D_\theta$ *and all* $i = \overline{1,k}$.

Definition 3.1. ([3], [4]) *If the function* $f : \mathbb{R} \to \mathbb{R}$ *is a differentiable and strictly positive density for the random variable* X *then , the* **Fisher** $f-$**score in the direction** x (*or with respect to translation parameter*)

$$U_f = U_X : \mathbb{R} \to \mathbb{R} \tag{3.1}$$

is defined by the relation

$$U_f(x;\theta) = U_X(x;\theta) := \frac{\partial \ln f(x;\theta)}{\partial x}. \tag{3.2}$$

Lemma 3.1. ([8], [9]) *Under the above FIRCs, the score function has the following properties* :

$$\mathbb{E}[U_f(X;\theta)] = \mathbb{E}\left[\frac{\partial \ln f(X;\theta)}{\partial x}\right] = 0, (\textit{the first Bartlett identity}) \tag{3.3}$$

$$Var\left[U(X;\theta)\right] = Var\left[\frac{\partial \ln f(X;\theta)}{\partial x}\right] \tag{3.4}$$

$$= \mathbb{E}\left\{\left[\frac{\partial \ln f(X;\theta)}{\partial x}\right]^2\right\}$$

$$= -\mathbb{E}\left[\frac{\partial^2 \log f(X;\theta)}{\partial x^2}\right], (\textit{the second Bartlett identity}). \tag{3.5}$$

Using the above lemma, we obtain the next definitions.

Definition 3.2. ([3], [4], [8]) **The Fisher information measure**, *in the direction* x (*or with respect to translation parameter*) , *contained in the random variable*

X, denoted by $I_f(X)$ *or* $I_X(x)$*, is defined as*

$$I_f(x) = I_X(x) := \int_{-\infty}^{+\infty} \left[\frac{\partial \ln f(x;\theta)}{\partial x}\right]^2 f(x;\theta)dx \tag{3.6}$$

$$= \mathbb{E}\left\{\left[\frac{\partial \ln f(X;\theta)}{\partial x}\right]^2\right\} \tag{3.7}$$

$$= -\int_{-\infty}^{+\infty} \left[\frac{\partial^2 \ln f(x;\theta)}{\partial x^2}\right] f(x;\theta)dx$$

$$= -\mathbb{E}\left\{\left[\frac{\partial^2 \ln f(X;\theta)}{\partial x^2}\right]\right\}. \tag{3.8}$$

Definition 3.3.[4], [8]) *The quantity* $I_f(\theta)$*, defined as*

$$I_f(\theta) := E_\theta\left\{\left[\frac{\partial \ln f(\theta;X)}{\partial \theta}\right]^2\right\} = \int_{-\infty}^{+\infty} \left[\frac{\partial \ln f(\theta;x)}{\partial \theta}\right]^2 f(\theta;x)dx \tag{3.9}$$

$$= -\int_{-\infty}^{+\infty} \left[\frac{\partial^2 \ln f(\theta;x)}{\partial \theta^2}\right] f(\theta;x)dx = -E_\theta\left\{\left[\frac{\partial^2 \ln f(\theta;X)}{\partial \theta^2}\right]\right\}, \tag{3.10}$$

where $\theta \in D_\theta \subset \mathbb{R}$*, represents the* **Fisher information measure about** θ *(that is, to respect with the univariate unknown parameter* θ*) that is contained in* X.

Theorem 3.1. *If* Y *is a random variables with the three −parametres Weibull distribution then, their probability density functions, with shape parameter* $\alpha > 0$*, scale parameter* $\lambda > 0$ *and with the location parameter* $b > 0$*, has the form* (2.8)*, then the associated* **Fisher's information,** *with respect to* y*, denoted by* $I_g(y)$*, has the form*

$$I_g(y;\alpha,\lambda,b) = -E\left[\frac{d}{dy}U_g(Y;\alpha,\lambda,b)\right] \tag{3.11}$$

$$= (\alpha-1)^2\lambda^2\Gamma(\frac{\alpha-2}{\alpha}),\ for\ \alpha\ > 2 \tag{3.12}$$

where $U_g(y;\alpha,\lambda,b)$ *is* **the score function**, *with respect to* y*, associated to this random variable, according to*

$$U_g(y;\alpha,\lambda,b) := \frac{d}{dy}\left[\log_e\ g(y;\alpha,\lambda,b)\right]. \tag{3.13}$$

Proof. Using (2.8), we get

$$\begin{aligned}\log_e\ g(y;\alpha,\lambda,b) &= \log_e\{\alpha\lambda[\lambda(y-b)]^{\alpha-1}e^{-[\lambda(y-b)]^\alpha}\}\\ &= \log_e(\alpha\lambda)+(\alpha-1)[\log_e\lambda(y-b)]-[\lambda(y-b)]^\alpha \end{aligned}\tag{3.14}$$

and the score function, with respect to y, can be obtained as

$$U_g(y;\alpha,\lambda,b) = \frac{d}{dy}\left[\log_e\ g(y;\alpha,\lambda,b)\right] \tag{3.15}$$

$$= \frac{\alpha-1}{y-b}-\alpha\lambda[\lambda(y-b)]^{\alpha-1} \tag{3.15a}$$

and its first derivative, with respect to y, has the forms as

$$\frac{d}{dy}\left[U_g(y;\alpha,\lambda,b)\right] = \frac{d^2}{dy^2}\left[\log\ g(y;\alpha,\lambda,b)\right] \tag{3.16}$$

$$= -\frac{\alpha-1}{(y-b)^2}-\alpha\lambda^2(\alpha-1)[\lambda(y-b)]^{\alpha-2}. \tag{3.16a}$$

Now, using (3.16a), as well as the definition of Fisher information, with respect to y, we get

$$\begin{aligned}I_g(y;\alpha,\lambda,b) &= -E\left\{\frac{d^2}{dy^2}\left[\log_e\ g(Y;\alpha,\lambda,b)\right]\right\}\\ &= E\left[\frac{\alpha-1}{(Y-b)^2}\right]+E\left\{\alpha\lambda^2(\alpha-1)[\lambda(Y-b)]^{\alpha-2}\right\}\\ &= I_1+I_2,\end{aligned}$$

where the integrals $I_1$and I_2 are represented by the expressions as

$$I_1 = E\left[\frac{\alpha-1}{(Y-b)^2}\right] \tag{3.18}$$

$$= \alpha(\alpha-1)\lambda^3\int_b^\infty[\lambda(y-b)]^{\alpha-3}e^{-[\lambda(y-b)]^\alpha}dy \tag{3.18a}$$

and

$$I_2 = E\left\{\alpha\lambda^2(\alpha-1)[\lambda(Y-b)]^{\alpha-2}\right\} = \tag{3.19}$$

$$= \alpha\lambda^2(\alpha-1)\alpha\lambda\int\limits_b^{\infty}[\lambda(y-b)]^{2\alpha-3}e^{-[\lambda(y-b)]^{\alpha}}dy, \tag{3.19a}$$

respectively.

Using the change of variables (2.2) then, for the above integrals I_1 and I_2, we get

$$I_1 = (\alpha-1)\lambda^2\int\limits_0^{\infty}t^{-\frac{2}{\alpha}}e^{-t}dt$$

$$= (\alpha-1)\lambda^2\underbrace{\int\limits_0^{\infty}t^{\frac{\alpha-2}{\alpha}-1}e^{-t}dt}_{\Gamma\left(\frac{\alpha-2}{\alpha}\right)}$$

that is,

$$I_1 = (\alpha-1)\lambda^2\Gamma(\frac{\alpha-2}{\alpha}),\ \alpha > 2 \tag{3.20}$$

and

$$I_2 = \alpha\lambda^2(\alpha-1)\int\limits_0^{\infty}t^{\frac{\alpha-2}{\alpha}}e^{-t}dt$$

$$= \alpha\lambda^2(\alpha-1)\underbrace{\int\limits_0^{\infty}t^{\left(\frac{\alpha-2}{\alpha}+1\right)-1}e^{-t}dt}_{\Gamma\left(\frac{\alpha-2}{\alpha}+1\right)}$$

that is,

$$I_2 = \lambda^2(\alpha-1)(\alpha-2)\Gamma\left(\frac{\alpha-2}{\alpha}\right), \tag{3.21}$$

respectively.

Now, using last relations (3.20) and (3.21), **the Fisher information with respect to** y, $I_g(y;\alpha,\lambda)$, takes the final form, namely

$$I_g(y;\alpha,\lambda) = -E\left\{\frac{d^2}{dy^2}\left[\log f(Y;\alpha,\lambda,b)\right]\right\}$$
$$= (\alpha-1)^2\lambda^2\Gamma(\frac{\alpha-2}{\alpha}),\ for\ \alpha\ > 2, \tag{3.22}$$

that is, an expression that not depends of the location parameter b.

Corollary 3.1. *If Y is a random variables with the three – parametres Weibull distribution that has the probability density functions* (2.8) *then the random variable*

$$X = Y - b \tag{3.23}$$

follows a two – parameters Weibull distribution with the probability density function (2.1) *and its* **Fisher information**,*with respect to* x, *denoted by* $I_f(x;\alpha,\lambda)$, *satisfies the relation*

$$I_f(x;\alpha,\lambda) = I_g(y;\alpha,\lambda) = (\alpha-1)^2\lambda^2\Gamma(\frac{\alpha-2}{\alpha}),\ for\ \alpha\ > 2, \tag{3.24}$$

that is, **the Fisher information is translation invariant**.

Proof. This very important property of the Fisher information follows if we have in view the forms of the score functions associated with the probability density functions $g(y;\alpha,\lambda,b)$ and $f(x;\alpha,\lambda)$, that is

$$U_g(y;\alpha,\lambda,b) = \frac{d}{dy}\left[\log_e\ g(y;\alpha,\lambda,b)\right]$$
$$= \frac{\alpha-1}{y-b} - \alpha\lambda[\lambda(y-b)]^{\alpha-1}, \tag{3.24a}$$

and

$$U_f(x;\alpha,\lambda) = \frac{d}{dx}\left[\log_e\ f(x;\alpha,\lambda)\right]$$
$$= \frac{\alpha-1}{x} - \alpha\lambda(\lambda x)^{\alpha-1}, \tag{3.24b}$$

respectively.

4. Fisher′s information measure in terms of the hazard and survival functions

In **the survival analysis** literatures the cumulative distribution function is known as **the failure function** but, in this case, the random variable T must to be a **non-negative random variable** representing **the waiting time until the occurrence of an event** and the function $F(t) \equiv P(T \leq t)$ represents just the probability that such a event has occurred by duration t. Also, in **the survival analysis**, an another important role is played by **the survival function,** denoted as

$$S(t) = P(T > t) = \int_t^{\infty} f(x)dx = 1 - F(t),\ t > 0. \tag{4.1}$$

Such a function gives **the probability** that **the event of interest has not occurred by duration** t. We can observe that

$$f(t) = \lim_{\triangle t \longrightarrow 0^+} \frac{P(t < T \leq t + \triangle t)}{\triangle t} = \lim_{\triangle t \longrightarrow 0^+} \frac{P\{T \in (t, t + \triangle t]\}}{\triangle t} \tag{4.2}$$

$$= \frac{\partial F(t)}{\partial t} = -\frac{\partial S(t)}{\partial t}, \tag{4.3}$$

where $\triangle t$ is a very small "infinitesimal" interval of time.

The fact that **the survival function** (or **the right-side cumulative distribution function**) $S(t)$ and **the failure function** $F(t)$, corresponding to a probability density function $f(t)$, imply that they are each probabilities and we have the following properties:

$$\begin{cases} 0 \leq S(t) \leq 1 & (4.4a) \\ S(0) = 1 & (4.4b) \\ \lim_{t \longrightarrow \infty} S(t) = 0 & (4.4c) \\ \frac{\partial S(t)}{\partial t} = -f(t) < 0 & (4.4d) \end{cases}, \tag{4.4}$$

where (4.4d) and (4.4c) can be interpreted as boundary conditions for $S(t)$. Thus, for instance, $S(\infty) = 0$ signifies that: given enough time the proportion surviving of a element goes down to zero.

Remark 4.1.An alternative characterization of the distribution of T is given by **the hazard function (**or **instantaneous rate of occurrence of the event or the hazard rate),** denoted by $h(t)$, which can be defined using the following succession of relations

$$h(t) = \lim_{dt \longrightarrow 0} \frac{P(t < T \leq t + dt \mid T > t)}{dt} = \tag{4.5}$$

$$= \lim_{dt \longrightarrow 0} \frac{\dfrac{P[(t < T \leq t + dt) \cap (T > t)]}{P(T > t)}}{dt} \tag{4.5a}$$

$$\{but\ because\ (t < T \leq t + dt) \cap (T > t) = (t < T \leq t + dt)\}$$

$$= \lim_{dt \longrightarrow 0} \frac{\dfrac{P(t < T \leq t + dt)}{S(t)}}{dt} \tag{4.5b}$$

$$= \lim_{dt \longrightarrow 0} \frac{P(t < T \leq t + dt)}{dtS(t)} \tag{4.5c}$$

$$= \frac{1}{S(t)} \lim_{dt \longrightarrow 0} \frac{P(t < T \leq t + dt)}{dt} = \frac{1}{S(t)} \underbrace{\lim_{dt \longrightarrow 0} \frac{F(t + dt) - F(t)}{dt}}_{= f(t)} \tag{4.5d}$$

$$= \frac{f(t)}{S(t)}, \tag{4.5e}$$

that is, we obtain that

$$h(t) = \frac{f(t)}{S(t)},\ for\ all\ t\ such\ that\ S(t) > 0, \tag{4.6}$$

where in the relation (4.5), the numerator, $P(t < T < t + dt \mid T > t)$ is the conditional probability that the event of interest will occur in the interval $(t, t+dt)$, given that it has not occurred before and the denominator dt represents the width of the interval $(t, t + dt)$. In others words, **the hazard function** can be defined as the risk of immediate failure or the rate of event occurrence per unit of time.

Remark 4.2. The above definitions imply very important relations, namely

$$\frac{dS(t)}{dt} = -\frac{dF(t)}{dt} = -f(t), \tag{4.7}$$

$$h(t) = \frac{f(t)}{S(t)} = -\frac{d\left[\log_e S(t)\right]}{dt}. \tag{4.8}$$

Remark 4.3. Using the definition relation of the hazard function (4.6) as well as the probability density function of the form $f(t; \theta)$ where $t > 0, \theta \in D_\theta$, we obtain

$$f(t; \theta) = h(t; \theta).S(t; \theta), t > 0, \theta \in D_\theta. \tag{4.9}$$

Theorem 4.1.*If the probability density function of the random variable* T *has the form* $f(t;\theta)$, *where* $t>0, \theta\in D_\theta$ *then, the score functions corresponding to the survival and hazard functions, denoted by* $U_S(t;\theta)$ *and* $U_h(t;\theta)$ *satisfy the following relation*

$$U_S(t;\theta)=\frac{\partial\log_e S(t;\theta)}{\partial t}=-\frac{\partial\log_e h(t;\theta)}{\partial t}=-U_h(t;\theta); t>0, \theta=(\alpha,\lambda)\in D_\theta \tag{4.10}$$

Proof. From (4.9) , we get

$$\log_e f(t;\theta)=\log_e h(t;\theta)+\log_e S(t;\theta), t>0, \theta\in D_\theta \tag{4.11}$$

as well as

$$\underbrace{\frac{\partial\log_e f(t;\alpha,\lambda)}{\partial t}}_{U_f(t;\theta)}=\underbrace{\frac{\partial\log_e h(t;\alpha,\lambda)}{\partial t}}_{U_h(t;\theta)}+\underbrace{\frac{\partial\log_e S(t;\alpha,\lambda)}{\partial t}}_{U_S(t;\theta)}, \tag{4.12}$$

that is, we have

$$U_f(t;\theta)=U_h(t;\theta)+U_S(t;\theta),\ t>0, \theta\in D_\theta. \tag{4.13}$$

Now, using the property that was mentioned in the relation (3.3), that is

$$\mathbb{E}[U_f(T;\theta)]=\mathbb{E}\left[\frac{\partial\ln f(T;\theta)}{\partial t}\right]=0, (The\ first\ Bartlett\ identity), \tag{4.14}$$

from (4.13), we get the relation

$$0=E[U_f(T;\theta)]=E[U_h(T;\theta)]+E[U_S(T;\theta)] \tag{4.15}$$

which proofs even the relation (4.10).

Theorem 4.2. *If* T *is a random variables with the two−parametres Weibull distribution that has the probability density functions* (2.1) *then its* **Fisher information** *with respect to* t, *denoted by* $I_f(t;\alpha,\lambda)$, *has the form*

$$I_f(t;\alpha,\lambda)=I_h(t;\alpha,\lambda)+I_S(t;\alpha,\lambda), \tag{4.16}$$

where the Fisher informations, with respect to t, *corresponding to hazard and survival functions have forms as*

$$\begin{aligned} I_h(t;\alpha,\lambda) &= I_h(t) = -E\left[\frac{\partial^2 \log_e h(T;\alpha,\lambda)}{\partial t^2}\right] \\ &= (\alpha-1)\,\lambda^2\Gamma\left(\frac{\alpha-2}{\alpha}\right), for\ \lambda>0,\ \alpha\ >2, \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} I_S(t;\alpha,\lambda) &= I_S(t) = -E\left[\frac{\partial^2 \log_e S(T;\alpha,\lambda)}{\partial t^2}\right] \\ &= (\alpha-1)(\alpha-2)\lambda^2\Gamma\left(\frac{\alpha-2}{\alpha}\right), for\ \lambda>0,\ \alpha\ >2, \end{aligned} \tag{4.18}$$

respectively.

Proof. Using the relations (2.1), (2.4), (4.1) and (4.6), namely

$$f(t;\alpha,\lambda) = \ \alpha\lambda(\lambda t)^{\alpha-1}e^{-(\lambda t)^{\alpha}},\quad \alpha>0,\lambda>0,\, t>0 \qquad , \tag{4.19}$$

$$F(t;\alpha,\lambda) = 1-e^{-(\lambda t)^{\alpha}}, t>0,\alpha>0,\lambda>0, \tag{4.20}$$

$$S(t;\alpha,\lambda) = e^{-(\lambda t)^{\alpha}}, t>0,\alpha>0.\lambda>0 \tag{4.21}$$

and

$$h(t;\alpha,\lambda) = \frac{f(t;\alpha,\lambda)}{S(t;\alpha,\lambda)} = \alpha\lambda(\lambda t)^{\alpha-1}, \alpha>0.\lambda>0, t>0, \tag{4.22}$$

respectively, we obtain:

$$\log_e h(t;\alpha,\lambda) = \log_e \alpha + \log_e \lambda + (\alpha-1)\log_e(\lambda t) \tag{4.22a}$$

$$U_h(t) = \frac{\partial \log_e h(t;\alpha,\lambda)}{\partial t} = (\alpha-1)\frac{1}{t}, \alpha>0, t>0, \tag{4.22b}$$

$$\log_e S(t;\alpha,\lambda) = -\,(\lambda t)^{\alpha} \tag{4.21a}$$

$$U_S(t) = \frac{\partial \log_e S(t;\alpha,\lambda)}{\partial t} = -\alpha\lambda(\lambda t)^{\alpha-1}, \alpha>0,\lambda>0, t>0. \tag{4.21b}$$

Then, using the change of variables

$$z = (\lambda t)^{\alpha} \Longrightarrow \lambda t = z^{\frac{1}{\alpha}} \Longrightarrow dt = \frac{1}{\alpha\lambda} z^{\frac{1}{\alpha}-1} dz, z \in (0, \infty), \tag{4.23}$$

we obtain

$$\begin{aligned} E\left[U_h(T) + U_S(T)\right] &= E[\frac{\alpha-1}{T}] - E[\alpha\lambda(\lambda T)^{\alpha-1}] \qquad (4.24) \\ &= \alpha(\alpha-1)\lambda^2 \int_0^{\infty} (\lambda t)^{\alpha-2} e^{-(\lambda t)^{\alpha}} dt \\ &\quad -\alpha^2\lambda^2 \int_0^{\infty} (\lambda t)^{2\alpha-2} e^{-(\lambda t)^{\alpha}} dt \\ &= (\alpha-1)\lambda \int_0^{\infty} z^{\frac{-1}{\alpha}} e^{-z} dz - \alpha\lambda \int_0^{\infty} z^{\frac{\alpha-1}{\alpha}} e^{-z} dz \\ &= (\alpha-1)\lambda\Gamma\left(\frac{\alpha-1}{\alpha}\right) - \alpha\lambda\Gamma\left(\frac{2\alpha-1}{\alpha}\right) \\ &= \underbrace{(\alpha-1)\lambda\Gamma\left(\frac{\alpha-1}{\alpha}\right)}_{E[U_h(T)]} - \underbrace{\lambda\left(\alpha-1\right)\Gamma\left(\frac{\alpha-1}{\alpha}\right)}_{E[U_S(T)]} \\ &= 0, \end{aligned}$$

that is, the equality

$$E\left[U_h(T) + U_S(T)\right] = E[U_h(T)] + E[U_S(T)] = 0 \tag{4.24a}$$

put in evidence the property (4.15) from the Theorem 4.1, that is, we have

$$E[U_h(T)] = -E[U_S(T)] = (\alpha-1)\lambda\Gamma\left(\frac{\alpha-1}{\alpha}\right), \quad \alpha > 0, \lambda > 0, t > 0. \tag{4.24b}$$

Also, using the relation (4.22b), we get

$$\begin{aligned} \frac{\partial U_h(t)}{\partial t} &= \frac{\partial^2 \log_e h(t;\alpha,\lambda)}{\partial t^2} = \frac{\partial}{\partial t}\left[(\alpha-1)\frac{1}{t}\right] \\ &= -\frac{\alpha-1}{t^2} \qquad (4.25) \end{aligned}$$

and, again, using the change of variables (4.23), we obtain

$$\begin{aligned} I_h(t) &= -E\left[\frac{\partial^2 \log_e h(T;\alpha,\lambda)}{\partial t^2}\right] = E\left(\frac{\alpha-1}{T^2}\right) \\ &= \alpha\,(\alpha-1)\,\lambda^3 \int\limits_0^\infty (\lambda t)^{\alpha-3} e^{-(\lambda t)^\alpha} dt \\ &= (\alpha-1)\,\lambda^2 \underbrace{\int\limits_0^\infty z^{\frac{\alpha-2}{\alpha}-1} e^{-z} dz}_{\Gamma\left(\frac{\alpha-2}{\alpha}\right)}, \end{aligned}$$

that is, for the Fisher information with respect to t of the hazard function $h(t;\alpha,\lambda)$, we obtain an expression as

$$I_h(t) = (\alpha-1)\,\lambda^2 \Gamma\left(\frac{\alpha-2}{\alpha}\right), \alpha > 0, \lambda > 0, t > 0. \tag{4.26}$$

Analogous, using the relation(4.22b), we obtain

$$\begin{aligned} I_S(t) &= -E\left[\frac{\partial^2 \log_e S(T;\alpha,\lambda)}{\partial t^2}\right] && (4.27) \\ &= -E\left[\frac{\partial U_S(t;\alpha,\lambda)}{\partial t}\right] && (4.27a) \\ &= E\left[\alpha(\alpha-1)\lambda^2(\lambda T)^{\alpha-2}\right] \\ &= \alpha^2(\alpha-1)\lambda^3 \int\limits_0^\infty (\lambda t)^{2\alpha-3} e^{-(\lambda t)^\alpha} dt \\ &= \alpha(\alpha-1)\lambda^2 \underbrace{\int\limits_0^\infty z^{\left(\frac{\alpha-2}{\alpha}+1\right)-1} e^{-z} dz}_{\Gamma\left(\frac{\alpha-2}{\alpha}+1\right)} \end{aligned}$$

that is, the Fisher information with respect to t of the survival function $S(t;\alpha,\lambda)$ has the form

$$I_S(t) = (\alpha-1)(\alpha-2)\lambda^2 \Gamma\left(\frac{\alpha-2}{\alpha}\right). \tag{4.28}$$

Finally, the Fisher information with respect to t of the probability density function $f(t;\alpha,\lambda)$,we obtain

$$I_f(t) = I_h(t) + I_S(t) = (\alpha - 1)^2 \lambda^2 \Gamma\left(\frac{\alpha - 2}{\alpha}\right), (see\ (3.12) \quad (4.29)$$

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