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MOCANU AND ŞERB TYPE UNIVALENCE CRITERIA FOR SOME GENERAL INTEGRAL OPERATORS

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Abstract. In this paper we derive univalence criteria for two general integral operators defined by analytic functions in the open unit disk, using the univalence criteria given by Pascu, respectively a lemma given by Mocanu and Şerb.

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1. Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

normalized by $f(0) = f'(0) - 1 = 0$, which are analytic in the open unit disk, $U = \{z \in \mathbb{C} : |z| < 1\}$.

We denote $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are univalent in U.

One of the topics in geometric function theory is the study of univalence of the integral operators. In the last decade, some general integral operators, defined as a family of integral operators, using more than one analytic function in their definition, have been studied with respect to their univalence (see for example, the works [2], [3] and [16], and many other recent paper as [5], [8], [19]).

In this paper, the univalence study is focused on the following general integral operators:

$$T_n(z) = \left\{ \beta \int_0^z u^{\beta-1} \left(f_1'(u)\right)^{\gamma_1} ... \left(f_n'(u)\right)^{\gamma_n} du \right\}^{\frac{1}{\beta}}, \tag{1}$$

$$B_n(z) = \left\{ \beta \int_0^z u^{\beta-1} \left(\frac{f_1(u)}{u} \right)^{\mu_1} ... \left(\frac{f_n(u)}{u} \right)^{\mu_n} \left(g_1'(u) \right)^{\eta_1} ... \left(g_n'(u) \right)^{\eta_n} du \right\}^{\frac{1}{\beta}}, \quad (2)$$

β, γ_j, μ_j, η_j complex numbers, $\beta \neq 0$, $f_j, g_j \in \mathcal{A}$, $j = \overline{1,n}$.

Remark 1.1. (i) *The integral operator T_n, introduced by Breaz and Breaz in the paper [3] is a general integral operator of Pfaltzgraff type which extends also the operator introduced by Pescar and Owa([18]), derived from (1), for $n = 1$. This operator has been studied with respect to its univalence, in many papers (see for example [16] and [17]).*

(ii) *Let's consider also the integral operator*

$$H_n(z) = \left\{ \beta \int_0^z u^{\beta-1} \left(\frac{f_1(u)}{u} \right)^{\gamma_1} ... \left(\frac{f_n(u)}{u} \right)^{\gamma_n} du \right\}^{\frac{1}{\beta}}. \quad (3)$$

The integral operator H_n, introduced by Breaz and Breaz in the paper [2] is a general integral operator of Kim-Merkes type, which extends also the operator introduced in [14], by Pascu and Pescar, derived from (3), for $n = 1$.

Thus, the integral operator B_n, introduced here by the formula (2), can be considered as an extension of both H_n (for $\eta_j = 0, j = \overline{1,n}$) and T_n (for $\mu_j = 0, j = \overline{1,n}$).

Moreover, if in the definition of B_n, we take $g = f$, we obtain the general integral operator given by Frasin ([6]), from which, if we take $n = 1$, we can derive further the operator given by Ovesea ([12]). Also, some different versions of this operator, B_n, were studied in other papers as for example [5] and [8].

Here, we obtain new conditions of univalence for these two general integral operators, T_n and B_n, by applying the improvement of Becker univalence criteria, obtained by Pascu in the paper [13]. Also, a lemma given by Mocanu and Şerb in the paper [11], will be used to get some parts of the results.

2. Preliminaries

In what follows we present the lemmas which are used in the proofs of the main results and also other lemmas which we refere to.

Lemma 2.1. *(Mocanu and Şerb, [11].) Let* $M_0 = 1,5936...$ *the positive solution of equation*

$$(2-M)e^M = 2. \tag{4}$$

If $f \in \mathcal{A}$ *and*

$$\left|\frac{f''(z)}{f'(z)}\right| \leq M_0, \quad z \in U, \tag{5}$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1, \quad z \in U. \tag{6}$$

The edge M_0 *is sharp.*

Remark 2.1. *It can be noticed that Lemma 2.1 constitutes a criteria for a function to be in a subclass of starlike function, hence it is also a criteria of starlikeness and consequently, a criteria of univalence.*

Lemma 2.2. *(Pascu, [13].) Let* α *be a complex number,* $\mathrm{Re}\alpha > 0$ *and the function* $f \in \mathcal{A}$. *If*

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|z\frac{f''(z)}{f'(z)}\right| \leq 1, \tag{7}$$

for all $z \in U$*, then for every complex number* β*,* $\mathrm{Re}\beta \geq \mathrm{Re}\alpha$*, the function*

$$F_\beta(z) = \left[\beta\int_0^z u^{\beta-1}f'(u)du\right]^{\frac{1}{\beta}} \tag{8}$$

is regular and univalent in U.

Lemma 2.3. *(Kudriasov, [7].) Let* f *be a regular function in* U*,* $f(z) = z + a_2z^2 + ...$ *If*

$$\left|\frac{f''(z)}{f'(z)}\right| \leq K, \quad z \in U, \tag{9}$$

for all $z \in U$*, where* $K \cong 3,05$*, the function* f *is univalent in* U.

Remark 2.2. *The constant* K *was obtained by Kudriasov as a solution of the equation,* $8\left[x\left(x-2\right)^3\right]^{\frac{1}{2}} - 3\left(4-x\right)^2 = 12$. *The Kudriasov result is not sharp, but the maximum value* M *for which the condition (9) implies univalence is proved to be* $M \in [K, \pi]$*, since the function* $f\left(z\right) = e^{\lambda \cdot z}$ *is univalent if and only if* $|\lambda| \leq \pi$ *(see [10]).*

Lemma 2.4. *(Mocanu, [9].) Let be $f \in \mathcal{A}$. If*

$$\left|\frac{f''(z)}{f'(z)}\right| \leq M, \quad z \in U, \tag{10}$$

where $M \cong 2,83$, then f is starlike in U and the result is sharp.

Remark 2.3. *The constant M was obtained by Mocanu as $M = \sqrt{1+y_0^2}$, where y_0 is the smallest positive root of the equation, $y\sin y + \cos y = \frac{1}{e}$. The same criteria of starlikeness (and consequently, criteria of univalence) was obtained by Anisiu and Mocanu in the paper [1], using different methods of proving.*

3. Main results

Theorem 3.1. *Let α, γ_j be complex numbers, $j = \overline{1,n}$, $\mathrm{Re}\alpha > 0$ and the functions $f_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + ...$, $j = \overline{1,n}$, $n \in \mathbb{N} - \{0\}$, M a positive real number.*

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \leq M, \quad z \in U, \; j = \overline{1,n} \tag{11}$$

and

$$|\gamma_1| + |\gamma_2| + ... + |\gamma_n| \leq \frac{(2\mathrm{Re}\alpha + 1)^{\frac{2\mathrm{Re}\alpha+1}{2\mathrm{Re}\alpha}}}{2M} \tag{12}$$

then for every complex number β, $\mathrm{Re}\beta \geq \mathrm{Re}\alpha$, the integral operator T_n belongs to the class $\mathcal{S}$.

Proof. We consider the function

$$t_n(z) = \int_0^z \left(f_1'(u)\right)^{\gamma_1} ... \left(f_n'(u)\right)^{\gamma_n} du \tag{13}$$

which is regular in U and $t_n(0) = t_n'(0) - 1 = 0$.

After some calculus we have the following evaluation for the expression involved in the hypothesis of Lemma 2.2,

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zt_n''(z)}{t_n'(z)}\right| \leq \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}|z|\left[|\gamma_1|\left|\frac{f_1''(z)}{f_1'(z)}\right| + ... + |\gamma_n|\left|\frac{f_n''(z)}{f_n'(z)}\right|\right], \tag{14}$$

for all $z \in U$.

On the other hand it can be proved that

$$\max_{|z|\in[0,1]} \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}|z| = \frac{2}{(2\mathrm{Re}\alpha+1)^{\frac{2\mathrm{Re}\alpha+1}{2\mathrm{Re}\alpha}}}. \tag{15}$$

Hence, if we apply hypothesis conditions (11), (12) and also (15) in the formula (14), the condition of Lemma 2.2 is satisfied, consequently $T_n \in \mathcal{S}$.

Remark 3.1. *If in Theorem 3.1, we take different values for the positive constant M, we can obtain also some information about the functions f_j, $j=\overline{1,n}$, not only about the integral operator. Thus:*

- **(i)** *For $M = K \cong 3.05$ (Kudriasov constant), we have that the functions f_j, $j=\overline{1,n}$ are univalent (see Lemma 2.3). Thus, Theorem 3.1 extends the result obtained by us, using Kudriasov constant in the paper [17].*
- **(ii)** *For $M \cong 2.83$ (Mocanu constant), f_j, $j=\overline{1,n}$ are starlike and consequently univalent (see Lemma 2.4).*
- **(iii)** *For $M = M_0 \cong 1.5936$ (Mocanu and Şerb constant), the functions f_j, $j=\overline{1,n}$ belongs to some special class of starlike functions, consequently they are univalent (see Lemma 2.1).*

Corollary 3.2. *Let α, γ_j be complex numbers, $j=\overline{1,n}$, $0 \le \mathrm{Re}\alpha \le 1$ and the functions $f_j \in \mathcal{A}$, $f_j(z) = z + a_{2j}z^2 + ...$, $j=\overline{1,n}$, $n \in \mathbb{N}-\{0\}$, M the positive real number. If*

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \le M, \quad z \in U, \ j=\overline{1,n} \tag{16}$$

and

$$|\gamma_1| + |\gamma_2| + ... + |\gamma_n| \le \frac{(2\mathrm{Re}\alpha+1)^{\frac{2\mathrm{Re}\alpha+1}{2\mathrm{Re}\alpha}}}{2M} \tag{17}$$

then the integral operator K_n defined by

$$K_n(z) = \int_0^z \left(f_1'(u)\right)^{\gamma_1} ... \left(f_n'(u)\right)^{\gamma_n} du \tag{18}$$

is in the class $\mathcal{S}$.

Proof. We take $\beta = 1$.

Remark 3.2. *The general integral operator of Pfaltzgraff type K_n was introduced by Breaz et al. in the paper [4].*

Theorem 3.3. *Let α, μ_j, η_j, be complex numbers,* $\mathrm{Re}\alpha > 0$, $f_j, g_j \in \mathcal{A}$, $j = \overline{1,n}$, $n \in \mathbb{N} - \{0\}$, *M a positive real number and $M_0 = 1,5936...$ the positive solution of equation*

$$(2-M)e^M = 2. \tag{19}$$

If

$$\left|\frac{f_j''(z)}{f_j'(z)}\right| \le M_0, \quad z \in U, j = \overline{1,n}, \tag{20}$$

$$\left|\frac{g_j''(z)}{g_j'(z)}\right| \le M, \quad z \in U, j = \overline{1,n}, \tag{21}$$

and

$$\sum_{j=1}^{n} |\mu_j| + M \cdot \sum_{j=1}^{n} |\eta_j| \le \mathrm{Re}\alpha, \tag{22}$$

then $f_j \in \mathcal{S}$, $j = \overline{1,n}$ and for every complex number β, $\mathrm{Re}\beta \ge \mathrm{Re}\alpha$, *we have $B_n \in \mathcal{S}$.*

Proof. We apply Lemma 2.2 for the regular function

$$b_n(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{f_j(u)}{u}\right)^{\mu_j} \prod_{j=1}^{n} \left(g_j'(u)\right)^{\eta_j} du. \tag{23}$$

After some derivative calculus, we get

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zb_n''(z)}{b_n'(z)}\right| \le \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha} \sum_{j=1}^{n} \left[|\mu_j| \left|\frac{zf_j'(z)}{f_j(z)} - 1\right| + |\eta_j|\,|z| \left|\frac{g_j''(z)}{g_j'(z)}\right|\right], \tag{24}$$

for all $z \in U$.

Applying all hypothesis conditions and further, Lemma 2.1, for f_j, $j = \overline{1,n}$, we have

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zb_n''(z)}{b_n'(z)}\right| \le 1, \tag{25}$$

which according to Lemma 2.2 implies that $B_n \in \mathcal{S}$.

Remark 3.3. **(i)** *For $\eta_j = 0, j = \overline{1,n}$, we get the same univalence criteria for the integral operator H_n, recalled in the Remark 1.1 (ii), as it was obtained in [15].*

(ii) *For $\mu_j = 0, j = \overline{1,n}$, we get a new univalence criteria for the integral operator T_n, based on the condition $\sum_{j=1}^{n} |\eta_j| \le \frac{\mathrm{Re}\alpha}{M}$ and the Remark 3.1 could be also reiterated.*

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