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ON SOME DIFFERENTIAL INEQUALITIES WITH APPLICATIONS I

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Abstract. In this paper, we derive some interesting relations associated with some differential inequalities in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$. Some interesting applications of the main results are also obtained.

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1. Introduction and Preliminaries

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disc $\mathbb{U} = \{z : |z| < 1\}$. For n a positive integer and $a \in \mathbb{C}$,

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots\}$$

with $\mathcal{H}_0 \equiv \mathcal{H}[0, 1]$.
Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the unit disc $\mathbb{U}$.

Definition 1. *If f and g are two analytic functions in $\mathbb{U}$, we say that f is to subordinate to g, written symbolically as $f \prec g$, if there exists a Schwarz function w, which (by definition) is analytic in $\mathbb{U}$, with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(w(z))$, $z \in \mathbb{U}$.*

If the function g is univalent in $\mathbb{U}$, then we have the following equivalence (c.f [5, 6]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Definition 2. *Let Q denote the set of all functions q that are analytic and injective on $\partial\mathbb{U}\backslash E(q)$, where*

$$E(q) = \{\zeta \in \partial\mathbb{U} : \lim_{z\to\zeta} q(z) = \infty\},$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(q)$. Further, let the subclass of Q for $Q(0) \equiv a$ be denoted by $Q(a)$ and $Q(1) \equiv Q_1$.

To prove our results, we need the following results due to Miller and Mocanu [6]

Lemma 1. *[6, pp 24] Let $q \in Q$, with $q(0) = a$, and let $p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$ be analytic in $\mathbb{U}$ with $p(z) \not\equiv a$ and $k \geq 1$. If p is not subordinate to q, then there exists points $z_0 = r_0 e^{i\theta_0} \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}\backslash E(q)$ and $k \geq n \geq 1$ for $p(\mathbb{U}_{r_0}) \subset q(\mathbb{U})$,*
(i) $p(z_0) = q(\zeta_0)$
(ii)$z_0 p'(z_0) = k\zeta q'(\zeta_0)$.

Lemma 2. *[6, pp 26] Let $p \in \mathcal{H}[a, n]$, with $p(z) \not\equiv a$ and $k \geq 1$. If $z_0 \in \mathbb{U}$ and*

$$Re(p(z_0)) = min\{Re(p(z)) : |z| \leq |z_0|\},$$

then

$$z_0 p'(z_0) \leq -\frac{k}{2} \frac{|p(z_0) - a|^2}{Re(a - p(z_0))}.$$

Recently, Kanas et al.[4] have discussed certain results involving the harmonic mean which are supplemmaentary to the results involving the arithmetic and geometric means obtained in [3]. Attiya et al.[2] have obtained certain sufficient conditions for the Carathéodary functions in $\mathbb{U}$. Motivated by the recent work of Attiya [1] in the present paper, we obtain some interesting relations associated with some differential inequalities in $\mathbb{U}$. These relations extend and generalize earlier results. Some applications of the main results are also obtained.

2. Main Results

Unless and otherwise mentioned throughout the paper $\sigma \geq 0$, $0 \leq \beta \leq 1$, $a \in \mathbb{C}$ with $Re(a) > 0$ and all the powers are the principal ones.

Theorem 3. *Let $p(z)$ be an analytic function with $p(0) = 1$. Let $B(z)$ be a complex valued analytic function in the unit disc $\mathbb{U}$, for $0 \leq \beta \leq 1$ and $m \in [1, 2]$ if*

$$Re\left((1-\beta)[ap(z)]^m + \beta\left[ap(z) + \frac{zp'(z)B(z)}{p(z)}\right]\right) > -\frac{G}{Re(a)\sqrt{(2Re(a) + |B(z)|)|a|}}, \tag{1}$$

where

$$G = \left(|B(z)||a|^2 + 2Re(a)|a|^2 + Im(a)\sqrt{|B(z)|}\sqrt{(2Re(a) + |B(z)|)|a|}\right)\sqrt{|B(z)|}\beta, \tag{2}$$

then $Re(ap(z)) > 0$.

Proof. Let us define both $g(z)$ and $h(z)$ as follows

$$g(z) = ap(z)$$

and

$$h(z) = \frac{a + \bar{a}z}{1 - z} \ (Re(a) > 0),$$

where $g(z)$ and $h(z)$ are analytic functions in $\mathbb{U}$ with $g(0) = h(0) = a \in \mathbb{C}$ and $h(\mathbb{U}) = \{w : Re(w) > 0\}$.
Now, we suppose that $g(z) \not\prec h(z)$. Then by using Lemma 1 there exists $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}\backslash\{1\}$ such that

$$g(z_0) = h(\zeta_0) = i\gamma \text{ and } z_0 g'(z_0) = k\zeta_0 h'(\zeta_0)$$

Also, from Lemma 2 we obtain

$$z_0 g'(z_0) \leq \frac{-k|i\gamma - a|^2}{2Re(a)}, \ k \geq 1.$$

$$\begin{aligned} Re\left(\beta\left[ap(z_0) + \frac{z_0 p'(z_0)B(z_0)}{p(z_0)}\right]\right) &= Re\left(\beta\left[g(z_0) + \frac{z_0 g'(z_0)B(z_0)}{q(z_0)}\right]\right) \\ &= Re\left(\beta h(\zeta_0) + \frac{k\zeta_0 h'(z_0)\beta B(z_0)}{h(\zeta_0)}\right) \\ &\leq \left|i\gamma\beta - \frac{\beta}{2}\frac{|i\gamma - a|^2 B(z_0)}{Re(a)i\gamma}\right| \\ &\leq \left|i\gamma\beta\right| + \left|\frac{\beta}{2}\frac{|i\gamma - a|^2 B(z_0)}{Re(a)i\gamma}\right| \\ &= \beta\gamma + \frac{\beta}{2}\frac{[\gamma^2 - 2Im(a)\gamma + |a|^2]}{Re(a)\gamma}|B(z_0)|. \end{aligned} \tag{3}$$

Let $\chi(m) = h(\zeta_0)^m$ and $l = \{\chi(m) : m \in [1, 2]\}$, then l is an arc of the logarithmic spiral connecting the points $\chi(1) = h(\zeta_0)$ *and* $\chi(2) = h(\zeta_0)^2$, having

the property that it intersects every radial line at constant angle. We also observe that $arg(h(\zeta_0)^m) = m\ arg(h(\zeta_0)$, is an increasing function of m, so l lies in the closed half plane containing the origin , determined by the line of equation $Re(z) = Re(\chi(1)) = 0$.
Therefore

$$Re\big((1-\beta)[ap(z_0)]^m\big) = Re\big((1-\beta)h(\zeta_0)^m\big) \leq 0, for\ m \in [1,2]. \tag{4}$$

Combining (3) and (4) we have

$$Re\Bigg((1-\beta)[ap(z_0)]^m + \beta\bigg[ap(z_0) + \frac{z_0p'(z_0)B(z_0)}{p(z_0)}\bigg]\Bigg)$$
$$\leq \beta\gamma + \frac{\beta}{2}\frac{[\gamma^2 - 2Im(a)\gamma + |a|^2]}{Re(a)\gamma}|B(z_0)| = f(\gamma),$$

where $f(\gamma)$ is a function of γ,and also $f(\gamma)$ attains the maximum at

$$\gamma^* = -\frac{\sqrt{(2Re(a) + |B(z_0)|)|B(z_0)||a|}}{2Re(a) + |B(z_0)|}.$$

Therefore,

$$Re\Bigg((1-\beta)[ap(z_0)]^m + \beta\big(ap(z_0) + \frac{z_0p'(z_0)B(z_0)}{p(z_0)}\big)\Bigg) \leq f(\gamma^*)$$
$$= -\frac{G}{Re(a)\sqrt{(2Re(a) + |B(z)|)|a|}},$$

where G is defined by (2). This contradicts the hypothesis of the Theorem, therefore $g(z) \prec h(z)$ and hence $Re(ap(z)) > 0$.

Theorem 4. *Let $\lambda(z), Q(z)$ be complex valued functions defined in the unit disc, $\mathbb{U}$ such that $Re(\lambda(z) + ap(z)) \leq \sigma, Re(Q(z)\bar{a}) > 0$ and $m \in [1, 3/2]$. If $p(z)$ be an analytic function with $p(0) = 1$ and*

$$Re\Bigg(\big(1-\beta\big)\big(\lambda(z) + ap(z)\big)^m + \beta\big(p(z) + zp'(z)Q(z)\big)\Bigg) > \frac{N}{2Re(Q(z)\bar{a})|a|^2} \tag{5}$$

where

$$N = 2\sigma Re(Q(z)\bar{a})|a|^2 - 2\beta\sigma Re(Q(z)\bar{a})|a|^2 + 2\beta\sigma Re(a)Re(Q(z)\bar{a}) \tag{6}$$
$$+Im(a)^2\beta Re(a-\sigma) + 2Im(a)^2\beta Re(Q(z)\bar{a}) - \beta Re(a-\sigma)Re(Q(z)\bar{a})^2 \ with\ (Re(a) > \sigma),$$

then

$$Re(ap(z)) > \sigma.$$

Proof. Let us define both $q(z)$ and $h(z)$ as follows

$$q(z) = ap(z)$$

and

$$h(z) = \frac{a-(2\sigma-\bar{a})z}{1-z} \quad (Re(a) > \sigma),$$

where $q(z)$ and $h(z)$ are analytic in $\mathbb{U}$ with $q(0) = h(0) = a$ and $h(\mathbb{U}) = \{w : Re(w) > \sigma\}$. Suppose $q(z) \not\prec h(z)$ then by Lemma 1 there exists a $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}\backslash\{1\}$ such that

$$q(z_0) = h(\zeta_0) = \sigma + i\gamma \text{ and } z_0q'(z_0) = k\zeta_0 h'(\zeta_0) \;\; (k \geq 1).$$

Also, from Lemma 2 we have

$$z_0q'(z_0) \leq -\frac{|\sigma+i\gamma-a|^2}{2Re(a-\sigma)}, \; k \geq 1.$$

Therefore,

$$\begin{aligned} &Re\bigg((1-\beta)(\lambda(z_0)+ap(z_0))^m + \beta(p(z_0)+z_0p'(z_0)Q(z_0))\bigg) \\ &= Re\bigg((1-\beta)(\lambda(z_0)+q(z_0))^m + \beta\bigg(\frac{q(z_0)}{a} + z_0q'(z_0)Q(z_0)\frac{1}{a}\bigg)\bigg) \\ &= Re\bigg((1-\beta)(\lambda(z_0)+h(\zeta_0))^m + \beta\bigg(\frac{h(\zeta_0)}{a} + k\zeta_0h'(\zeta_0)Q(z_0)\frac{1}{a}\bigg)\bigg). \end{aligned}$$

Now

$$\begin{aligned} &Re\bigg(\beta\bigg(\frac{h(\zeta_0)}{a} + k\zeta_0h'(\zeta_0)Q(z_0)\frac{1}{a}\bigg)\bigg) \\ &\qquad \leq Re\bigg(\beta\frac{(\sigma+i\gamma)}{a} - k\beta\frac{|\sigma+i\gamma-a|^2}{2Re(a-\sigma)}\frac{Q(z_0)}{a}\bigg) \\ &\qquad \leq Re\bigg(\beta\frac{(\sigma+i\gamma)}{a} - \beta\frac{|\sigma+i\gamma-a|^2}{2Re(a-\sigma)}\frac{Q(z_0)}{a}\bigg) \\ &= \frac{\beta(\sigma Re(a)+\gamma Im(a))}{|a|^2} - \frac{\beta[|a|^2-2\sigma Re(a)+\sigma^2+\gamma^2-2\gamma Im(a)]Re(Q(z_0)\bar{a})}{2Re(a-\sigma)|a|^2}. \end{aligned} \tag{7}$$

Consider,

$$(\lambda(z_0)+h(\zeta_0))^m = (\lambda_x+i\lambda_y+\sigma+i\gamma)^m = (\lambda_x+\sigma+i(\lambda_y+\gamma))^m, \text{ for } z_0 \in \mathbb{U} \text{ and } \zeta_0 \in \partial\mathbb{U}\backslash\{1\}.$$

Now due to symmetry we discuss the following two cases.
Case I: If $arg(\lambda(z_0) + h(\zeta_0)) \in [\pi/2, \pi]$ i.e.,$(Re(\lambda(z_0) + h(\zeta_0)) \leq 0)$, then for $m \in [1, 3/2]$

$$arg(\lambda(z_0) + h(\zeta_0))^m \in \big[\pi/2, 3\pi/2\big] \text{ which implies } Re(\lambda(z_0) + h(\zeta_0))^m \leq 0 \leq \sigma.$$

Case II: For $0 < Re(\lambda(z_0) + h(\zeta_0)) \leq \sigma$, let

$$E(m) = (\lambda(z_0) + h(\zeta_0))^m,\ m \in [1, 3/2] \text{ and define } l = \big\{E(m) : m \in [\pi/2, 3\pi/2]\big\}.$$

Then l is a logarithmic spiral, which connects $E(1)$ and $E(3/2)$ and is bounded by the line of equation $ReE(1) = Re(\lambda(z_0)+h(\zeta_0)) \leq \sigma$, therefore we have $Re(E(m)) \leq \sigma, m \in [\pi/2, 3\pi/2]$.
Hence, from the above cases we obtain the following inequality

$$Re\bigg((1-\beta)(\lambda(z_0) + h(\zeta_0))^m\bigg) \leq (1-\beta)\sigma. \tag{8}$$

Combining (7) and (8) we obtain

$$\begin{aligned} &Re\bigg((1-\beta)(\lambda(z_0) + ap(z_0)) + \beta(p(z_0) + z_0 p'(z_0)Q(z_0))\bigg) \\ &\leq (1-\beta)\sigma + \frac{\beta(\sigma Re(a) + \gamma Im(a))}{|a|^2} - \frac{\beta[|a|^2 - 2\sigma Re(a) + \sigma^2 + \gamma^2 - 2\gamma Im(a)]Re(Q(z_0)\bar{a})}{2Re(a-\sigma)|a|^2} \\ &= g(\gamma), \end{aligned}$$

where $g(\gamma)$ is a function of γ and it attains its maximum at

$$\gamma^{'} = \frac{Im(a)(\beta Re(a-\sigma) + Re(Q(z_0)\bar{a}))}{Re(Q(z_0)\bar{a})}.$$

Therefore, we obtain

$$Re\bigg((1-\beta)(\lambda(z_0) + ap(z_0)) + \beta(p(z_0) + z_0 p'(z_0)Q(z_0))\bigg) \leq g(\gamma^{'}) = \frac{N}{Re(Q(z)\bar{a})|a|^2},$$

where N is defined by (6), and hence we obtain a contradiction to (5). Therefore $q(z) \prec h(z)$ and $Re(ap(z)) > \sigma$.

Theorem 5. *If $p(z)$ be an analytic function in $\mathbb{U}$ with $p(0) = 1$ and*

$$Re\bigg(ap(z)^2 + zp'(z)\bigg) > \frac{G}{2|a|^2 Re(a)(2Re(a-\sigma)+1)Re(a-\sigma)}, \tag{9}$$

where

$$G = 4Re(a-\sigma)^2|a|^2\sigma^2 + 2Re(a)Re(a-\sigma)[|a|^2(2\sigma - Re(a)) \\ +Re(a)^2(Re(a)(2\sigma - Re(a)) - \sigma^2)], \tag{10}$$

then

$$Re(ap(z)) > \sigma.$$

Proof. Let us define both $q(z)$ and $h(z)$ as follows

$$q(z) = ap(z)$$

and

$$h(z) = \frac{a - (2\sigma - \bar{a})z}{1-z} \quad (Re(a) > \sigma),$$

where $q(z)$ and $h(z)$ are analytic in $\mathbb{U}$ with $q(0) = h(0) = a$ and $h(\mathbb{U}) = \{w : Re(w) > \sigma\}$. Suppose $q(z) \not\prec h(z)$ then by Lemma 1 there exists a $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}\backslash\{1\}$ such that

$$q(z_0) = h(\zeta_0) = \sigma + i\gamma \text{ and } z_0q'(z_0) = k\zeta_0h'(\zeta_0) \quad (k \geq 1).$$

Also, from Lemma 2 we have

$$z_0q'(z_0) \leq -\frac{|\sigma + i\gamma - a|^2}{2Re(a-\sigma)}, \ k \geq 1.$$

Next,

$$\begin{aligned} Re\left(ap(z_0)^2 + z_0p'(z_0)\right) &= Re\left(\frac{h(\zeta_0)^2}{a} + \frac{k\zeta_0h'(\zeta_0)}{a}\right) \\ &\leq Re\left(\frac{(\sigma+i\gamma)^2}{a} - \frac{k|\sigma+i\gamma|^2}{2Re(a-\sigma)a}\right) \\ &\leq Re\left(\frac{(\sigma+i\gamma)^2}{a}\right) - \frac{|\sigma+i\gamma|^2}{2Re(a-\sigma)}Re\left(\frac{1}{a}\right) \\ &= \mathcal{P}\gamma^2 + \mathcal{Q}\gamma + \mathcal{R} = \mathcal{W}(\gamma) \end{aligned} \tag{11}$$

where

$$\mathcal{P} = \frac{-1}{|a|^2}\left[Re(a) + \frac{Re(a)}{2Re(a-\sigma)}\right],$$

$$\mathcal{Q} = \frac{Im(a)}{|a|^2}\left[2\sigma + \frac{Re(a)}{2Re(a-\sigma)}\right]$$

and

$$\mathcal{R} = \frac{1}{|a|^2}\left[\sigma^2 Re(a) - \frac{(\sigma^2 - 2Re(a) + |a|^2)Re(a)}{2Re(a-\sigma)}\right].$$

We can see that the function $\mathcal{W}(\gamma)$ has a maximum at γ_1, given by

$$\gamma_1 = Im(a)\left[\frac{Re(a) + 2Re(a-\sigma)}{Re(a) + 2Re(a)Re(a-\sigma)}\right].$$

Hence, we have

$$Re(ap(z_0)^2 + z_0 p'(z_0)) \leq \mathcal{W}(\gamma_1) = \frac{G}{2|a|^2 Re(a)(2Re(a-\sigma)+1)Re(a-\sigma)},$$

where G is defined by (10). Hence, we obtain a contradiction to our assumption (9). Therefore, $q(z) \prec h(z)$ and hence $Re(ap(z)) > \sigma$.

3. Applications and Examples

In this section, we will discuss some of the applications of the Theorem discussed in the earlier section to the univalent function theory.
Letting $a = 1, m = 1$ and $B(z) = 1$ in the Theorem 3, we have the following Corollary

Corollary 6. *If $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and*

$$Re\left((1-\beta)p(z) + \beta\left(p(z) + \frac{zp'(z)}{p(z)}\right)\right) > -\sqrt{3}\beta,$$

then $Re(p(z)) > 0$.

Letting $p(z) = \frac{zf'(z)}{f(z)}$ in the above Corollary, we get

Corollary 7. *If $f \in \mathcal{A}$ satisfies $f(z) \neq 0$ in $0 < |z| < 1$ and*

$$Re\left((1-\beta)\frac{zf'(z)}{f(z)} + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > -\sqrt{3}\beta$$

then $Re\left(\frac{zf'(z)}{f(z)}\right) > 0$.

By taking $p(z) = \left(\frac{f(z)}{z}\right)^{\mu}$, $(\mu > 0)$ in Corollary 6 we obtain the following result

Corollary 8. *If $f \in \mathcal{A}$ satisfies $\left(\frac{f(z)}{z}\right)^{\mu} \neq 0$ and*

$$Re\left[(1-\beta)\left(\frac{f(z)}{z}\right)^{\mu} + \beta\left(\left(\frac{f(z)}{z}\right)^{\mu} - \mu\left(1 - \frac{zf'(z)}{f(z)}\right)\right] > -\sqrt{3}\beta$$

then $Re\left(\left(\frac{f(z)}{z}\right)^{\mu}\right) > 0.$

Letting $p(z) = \frac{z^2 f'(z)}{f^2(z)}$ and $\beta = 1$ in Corollary 6 we have

Corollary 9. *If* $Re\left(\frac{z^2 f'(z)}{f^2(z)} + \frac{(zf(z))''}{f'(z)} - \frac{2zf'(z)}{f(z)}\right) > -\sqrt{3}$ *then* $Re\left(\frac{z^2 f'(z)}{f^2(z)}\right) > 0.$

Putting $a = 1$ *and* $\sigma = 0$ in Theorem 5, we obtain the following Corollary

Corollary 10. *If* $Re(p(z)^2 + zp'(z)) > -1/2$ *then* $Re(p(z)) > 0.$

Letting $p(z) = \frac{zf'(z)}{f(z)}$ in the above Corollary, we have a result due to Nunokawa et. al [8, Theorem 1, p. 2910].
Letting $p(z) = f'(z)$ in the above Corollary we obtain

Corollary 11. *If $f \in \mathcal{A}$ satisfies $Re(f'(z)^2 + zf''(z)) > -1/2$ then $Re(f'(z)) > 0$, hence f is univalent in $\mathbb{U}$.*

Remark 1. *(1) Putting $\beta = 1$ in Theorem 4, we have a result due to Attiya[1, Theorem 2.1, p. 2].*
(2) Putting $Q(z) = \delta(\delta > 0)$ and $\beta = 1$ in Theorem 4, we have a Corollary due to Attiya[1, Corollary 3.1, p. 5].
(3) Putting $Q(z) = 1$ and $\beta = 1$ in Theorem 4, we have another Corollary due to Attiya[1, Corollary 3.2, p. 5].
(4) Putting $a = e^{i\lambda}(|\lambda| < \pi/2)$ and $\sigma = 0$ in Theorem 4, we have due to Kim and Cho[7, Theorem 2.1].
(5) Putting $a = e^{i\lambda}(|\lambda| < \pi/2)$, $Q(z) = \delta(\delta > 0)$ and $\sigma = 0$ in Theorem 4, we have a Corollary due to Kim and Cho[7, Corollary 1].
(6) Putting $a = e^{i\lambda}(|\lambda| < \pi/2)$, $Q(z) = 1$ and $\sigma = 0$ in Theorem 4, we have result due to Kim and Cho[7, Corollary 2].
(7) Putting $\beta = 1$ and $\sigma = 0$ in Theorem 4, we have a Theorem due to Attiya and Nasr[2, Theorem 2.1, p. 2].
(8) Putting $\beta = 1$, $Q(z) = \delta(\delta > 0)$ and $\sigma = 0$ in Theorem 4, we have a Corollary due to Attiya and Nasr[2, Corollary 3.1, p. 7].
(9) Putting $\beta = 1$, $Q(z) = 1$ and $\sigma = 0$ in Theorem 4, we have a result due to Attiya and Nasr[2, Corollary 3.2, p. 7].

Finally, we give an example of the Corollary 11. The function

$$h(z) = z + \frac{z^2}{6}$$

maps the unit circle onto the following domain,

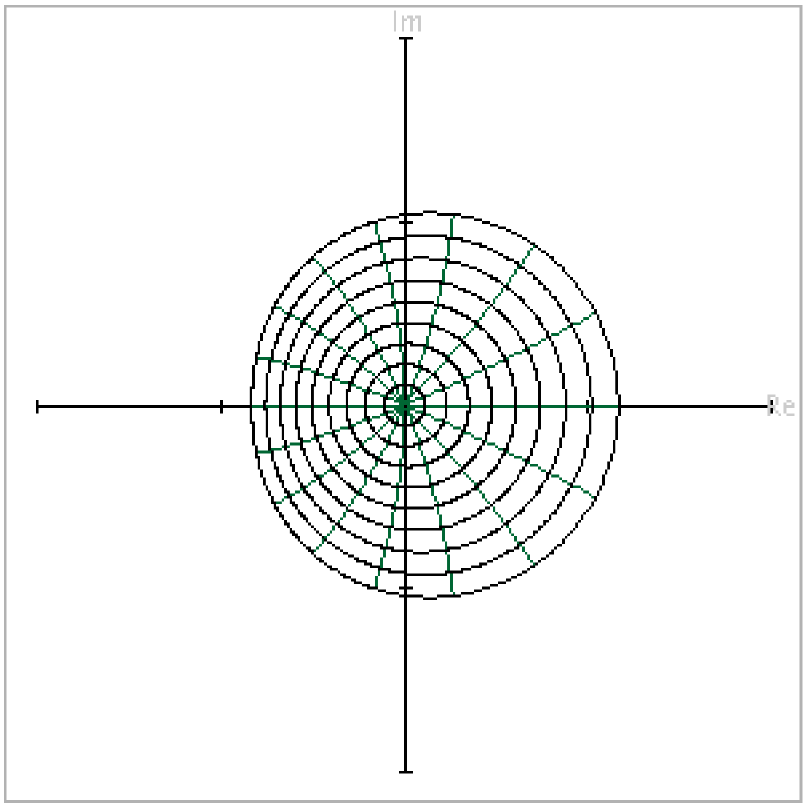

Figure 1: Image of $h(z) = z + \frac{z^2}{6}$.

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