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ON THE COMPOSITION OF TWO STARLIKE FUNCTIONS

Olga Engel

Abstract. In this paper we determine a subclass of the class of starlike functions S^*, which is denoted by S^{**} and which has the property that the composition of two functions from S^{**} is in the class S^*. The basic tool of our research is the differential subordination theory.

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1. Introduction

Let $U(r) = \{z \in \mathbb{C} : |z| < r\}$ be the a disk in the complex plane $\mathbb{C}$, centered at zero, and $U = U(1)$. We denote by $\mathcal{A}$ the class of the functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

defined in U. Let r be a real number with $r \in (0, 1]$.

We say that f is starlike in $U(r)$ if $f : U(r) \to \mathbb{C}$ is univalent and $f(U(r))$ is a starlike domain in $\mathbb{C}$ with respect to origin. It is well-known that $f \in \mathcal{A}$ is starlike in $U(r)$ if and only if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad \text{for all } z \in U(r).$$

It is also known that the composition of two starlike functions generally is not starlike (if the composition can be done). In this paper we determined conditions which imply that the composition of two starlike functions is also starlike. This problem seems to be new, we did not find in the literature results regarding this question.

We define the class S^{**} by the equality

$$S^{**}=\left\{f\in\mathcal{A}:\left|1+\frac{zf''(z)}{f'(z)}\right|<\sqrt{\frac{5}{4}},\ z\in U\right\}. \tag{1}$$

We will prove in the followings that $S^{**}\subset S^{*}$, and if

$$f,g\in S^{**},\ \text{ then }\ f\circ g \text{ is starlike in a disk } U(r_0).$$

2. Preliminaries

The following definitions and lemmas are necessary to prove our main results.

Definition 1. *[1][2] Let f and g be analytic functions in U. We say that the function f is subordinate to the function g, if there exist a function w, which is analytic in U and for which $w(0)=0$, $|w(z)|<1$ for $z\in U$, such that $f(z)=g(w(z))$, for all $z\in U$. We denote by $\prec$ the subordination relation.*

Definition 2. *[1][2] Let Q be the class of analytic functions q in U which has the property that are analytic and injective on $\overline{U}\backslash E(q)$, where*

$$E(q)=\{\zeta\in\partial U:\lim_{z\longrightarrow\zeta}q(z)=\infty\},$$

and are such that $q'(\zeta)\neq 0$ for $\zeta\in\partial U\backslash E(q)$.

Lemma 1. *[Miller-Mocanu] Let $q\in Q$, with $q(0)=a$, and let $p(z)=a+a_nz^n+\dots$ be analytic in U with $p(z)\not\equiv a$ and $n\geq 1$. If $p\not\prec q$, then there are two points $z_0=r_0e^{i\theta_0}\in U$, and $\zeta_0\in\partial U\backslash E(q)$ and a real number $m\in[n,\infty)$ for which $p(U_{r_0})\subset q(U)$,*
(i) $p(z_0)=q(\zeta_0)$
(ii) $z_0p'(z_0)=m\zeta_0q'(\zeta_0)$
(iii) $\operatorname{Re}\frac{z_0p''(z_0)}{p'(z_0)}+1\geq m\operatorname{Re}\left(\frac{\zeta_0q''(\zeta_0)}{q'(\zeta_0)}+1\right)$.

The following result is a particular case of Lemma 1.

Lemma 2. *[Miller-Mocanu] Let $p(z)=1+a_nz^n+\dots$ be analytic in U with $p(z)\not\equiv 1$ and $n\geq 1$. If $\operatorname{Re}p(z)\not>0$, $z\in U$, then there is a point $z_0\in U$, and there are two real numbers $x,y\in\mathbb{R}$ such that*
(i) $p(z_0)=ix$
(ii) $z_0p'(z_0)=y\leq-\frac{n(x^2+1)}{2}$,
(iii) $\operatorname{Re}z_0^2p''(z_0)+z_0p'(z_0)\leq 0$.

3. Main Results

Theorem 3. *We have $S^{**} \subset S^{*}$.*

Proof. Let f be a function from the class S^{**}.
We will prove that $p(z) = \dfrac{zf'(z)}{f(z)} > 0, \quad z \in U$.
It is easily seen that
$$1 + \frac{zf''(z)}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)},$$
and consequently the following equivalence holds
$$\left|1 + \frac{zf''(z)}{f'(z)}\right| < \sqrt{\frac{5}{4}},\ z \in U \Leftrightarrow \left|p(z) + \frac{zp'(z)}{p(z)}\right| < \sqrt{\frac{5}{4}},\ z \in U. \tag{2}$$

If the condition $\operatorname{Re} p(z) = \operatorname{Re} \dfrac{zf'(z)}{f(z)} > 0, \quad z \in U$, does not hold, then according to the Miller-Mocanu lemma (Lemma 2) there is a point $z_0 \in U$, and there are two real numbers $x, y \in \mathbb{R}$ such that
$$p(z_0) = ix,$$
and
$$z_0 p'(z_0) = y \le -\frac{1+x^2}{2}.$$
These equalities imply
$$\left|p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)}\right| = \left|ix - i\frac{y}{x}\right| = \left|x - \frac{y}{x}\right| \ge \left|x + \frac{1+x^2}{2x}\right| \ge \sqrt{3} > \sqrt{\frac{5}{4}}.$$

This inequality contradicts (2) and consequently $\operatorname{Re} p(z) = \operatorname{Re} \dfrac{zf'(z)}{f(z)} > 0, \quad z \in U$ holds.

Remark 1. *The class S^{**} is not empty. It is easily seen that if $f(z) = z - \dfrac{z^2}{100}$, then*
$$\left|1 + \frac{zf''(z)}{f'(z)}\right| = \left|\frac{100-4z}{100-2z}\right| \le \frac{104}{98} < \sqrt{\frac{5}{4}}\ z \in U,$$
and consequently $f \in S^{**}$.

Theorem 4. *If $f \in S^{**}$, then $\left|\arg \dfrac{zf'(z)}{f(z)}\right| < \dfrac{\pi}{4},\ z \in U.$*

Proof. The inequality $\left|\arg\dfrac{zf'(z)}{f(z)}\right| < \dfrac{\pi}{4},\ z \in U$ is equivalent to

$$p(z) = \frac{zf'(z)}{f(z)} \prec \sqrt{\frac{1+z}{1-z}} = q(z), \quad z \in U. \tag{3}$$

We will prove the subordination (3) using again the Miller-Mocanu lemma.
If the subordination (3) does not hold, then according to Lemma 1 there are two points $z_0 \in U$ and $\zeta_0 = e^{i\theta} \in \partial U$, and a real number $m \in [1, \infty)$, such that

$$p(z_0) = q(\zeta_0) = q(e^{i\theta}) = \sqrt{\frac{1+e^{i\theta}}{1-e^{i\theta}}},$$

$$\frac{z_0 p'(z_0)}{p(z_0)} = m\frac{\zeta_0 q'(\zeta_0)}{q(\zeta_0)} = m\frac{e^{i\theta}}{1-e^{2i\theta}}.$$

According to (2) the function f belongs to the class S^{**} if and only if

$$\left|p(z) + \frac{zp'(z)}{p(z)}\right| < \sqrt{\frac{5}{4}},\ z \in U. \tag{4}$$

On the other hand we have

$$\begin{aligned}\left|p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)}\right| &= \left|q(\zeta_0) + m\frac{\zeta_0 q'(\zeta_0)}{q(\zeta_0)}\right| = \left|\sqrt{\frac{1+e^{i\theta}}{1-e^{i\theta}}} + m\frac{e^{i\theta}}{1-e^{2i\theta}}\right| \\ &= \left|\sqrt{i\cot\frac{\theta}{2}} + m\frac{i}{2\sin\theta}\right|.\end{aligned}$$

Denoting $x = \sqrt{\left|\cot\frac{\theta}{2}\right|}$, it follows that $x \in (0, \infty)$, and

$$\begin{aligned}\left|p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)}\right| &= \left|\Big(\pm\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\Big)x + im\frac{x^4+1}{4x^2}\right| \\ = \sqrt{\frac{x^2}{2} + \Big(\frac{x}{\sqrt{2}} + m\frac{x^4+1}{4x^2}\Big)^2} &\geq \sqrt{\frac{x^2}{2} + \Big(\frac{x}{\sqrt{2}} + \frac{x^4+1}{4x^2}\Big)^2} \\ &= \sqrt{x^2 + \frac{x^4+1}{2\sqrt{2}x} + \Big(\frac{x^4+1}{4x^2}\Big)^2}.\end{aligned} \tag{5}$$

We will prove in the followings that

$$x^2 + \frac{x^4+1}{2\sqrt{2}x} + \Big(\frac{x^4+1}{4x^2}\Big)^2 > \frac{5}{4}, \quad x \in (0, \infty). \tag{6}$$

Indeed we have

$$x^2 + \frac{x^4+1}{2\sqrt{2}x} + \Big(\frac{x^4+1}{4x^2}\Big)^2 > x^2 + \frac{x^4+1}{2\sqrt{2}x} \geq 1 + \frac{1}{\sqrt{2}} > \frac{5}{4}, \quad x \in [1,\infty).$$

Since the mapping $\chi : (0, \frac{1}{2}] \to \mathbb{R},\ \chi(x) = \dfrac{x^4+1}{2\sqrt{2}x} + \Big(\dfrac{x^4+1}{4x^2}\Big)^2$ is strictly decreasing, it follows that

$$x^2 + \frac{x^4+1}{2\sqrt{2}x} + \Big(\frac{x^4+1}{4x^2}\Big)^2 \geq \chi(x) \geq \chi\Big(\frac{1}{2}\Big) > \frac{5}{4}, \quad x \in \Big(0, \frac{1}{2}\Big],$$

finally if $x \in \Big[\dfrac{1}{2}, 1\Big]$ then $x^2 \geq \dfrac{1}{4}$, $\Big(\dfrac{x^4+1}{4x^2}\Big)^2 > \dfrac{1}{4}$ and $\dfrac{x^4+1}{2\sqrt{2}x} > \dfrac{2\sqrt[4]{3}}{3\sqrt{2}} > 0.62$. We introduce the notation $t = x^2 \Rightarrow x^2 + \Big(\dfrac{x^4+1}{4x^2}\Big)^2 = t + \Big(\dfrac{t^2+1}{4t}\Big)^2$. We will prove that $x^2 + \Big(\dfrac{x^4+1}{4x^2}\Big)^2 > \dfrac{3}{4}$, $t \in [\frac{1}{4}, 1]$. This inequality is equivalent to

$$u(t) = t^4 + 16t^3 - 10t^2 + 1 > 0,\ t \in [\frac{1}{4}, 1.]$$

We have $u'(t) = 4t^3 + 48t^2 - 20t = 4t(t^2 + 12t - 5)$ and $u(t) \geq \sqrt{41} - 6 > 0$, $t \in [\frac{1}{4}, 1]$. Thus the proof of (6) is finished. Now (5) and (6) imply

$$\Big|p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)}\Big| > \sqrt{\frac{5}{4}},$$

and this contradicts (4). This contradiction implies the subordination (3).

Theorem 5. *If* $f \in S^{**}$, *then* $\Big|\arg f'(z)\Big| < \dfrac{\pi}{4},\ z \in U.$

Proof. The inequality $\Big|\arg f'(z)\Big| < \dfrac{\pi}{4},\ z \in U$ is equivalent to

$$f'(z) \prec \sqrt{\frac{1+z}{1-z}} = q(z), \quad z \in U. \tag{7}$$

If the subordination (7) does not hold, then according to Lemma 1 there are two points $z_0 \in U$ and $\zeta_0 = e^{i\theta} \in \partial U$, and a real number $m \in [1, \infty)$, such that

$$f'(z_0) = q(\zeta_0) = q(e^{i\theta}) = \sqrt{\frac{1+e^{i\theta}}{1-e^{i\theta}}},$$

$$\frac{z_0 f''(z_0)}{f'(z_0)} = m\frac{\zeta_0 q'(\zeta_0)}{q(\zeta_0)} = m\frac{e^{i\theta}}{1-e^{2i\theta}} = \frac{im}{2\sin\theta}.$$

Thus we get

$$\Big|1+\frac{z_0 f''(z_0)}{f'(z_0)}\Big| = \sqrt{1+\Big(\frac{m}{2\sin\theta}\Big)^2} \geq \sqrt{1+\Big(\frac{1}{2}\Big)^2} = \sqrt{\frac{5}{4}}.$$

This inequality contradicts $f \in S^{**}$. The contradiction implies that the subordination (7) holds, and the proof is done.

Now we are able to prove the result proposed in the Introduction for composition of functions.

Theorem 6. *If $f, g \in S^{**}$, and $r_0 = \sup\{r \in (0,1] \big| g\big(U(r)\big) \subset U\}$, then $f \circ g$ is starlike in $U(r_0)$.*

Proof. We have

$$\frac{z(f\circ g)'(z)}{(f\circ g)(z)} = \frac{zf'(g(z))}{f(g(z))}g'(z). \tag{8}$$

If $f, g \in S^{**}$, then Theorem 4 and Theorem 5 imply the inequalities

$$\Big|\arg\frac{zf'(z)}{f(z)}\Big| < \frac{\pi}{4}, z \in U \text{ and } \Big|\arg g'(z)\Big| < \frac{\pi}{4},\ z \in U.$$

The equality (8) implies that

$$\arg\frac{z(f\circ g)'(z)}{(f\circ g)(z)} = \arg\frac{zf'(g(z))}{f(g(z))} + \arg g'(z).$$

Thus we get

$$\Big|\arg\frac{z(f\circ g)'(z)}{(f\circ g)(z)}\Big| \leq \Big|\arg\frac{zf'(g(z))}{f(g(z))}\Big| + \Big|\arg g'(z)\Big| \leq \frac{\pi}{2},\quad z \in U(r_0).$$

This inequality means that

$$\mathrm{Re}\,\frac{z(f\circ g)'(z)}{(f\circ g)(z)} > 0,\ z \in U(r_0),$$

and consequently $f \circ g$ is starlike in $U(r_0)$.

4. Final Conclusions

We have proved that the class S^{**} is a subclass of the class S^{*}, the class of starlike functions. Also we have proved that the class S^{**} has the property that every two functions from this class can be composed, and the composition of such two starlike functions is also starlike.

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Olga Engel
Department of Mathematics,
Babeş - Bolyai University,
Cluj - Napoca, Romania
email: *engel_olga@hotmail.com*