ON BERGMAN TYPE OPERATORS IN NEW HARDY AND HERZ TYPE ANALYTIC SPACES IN BOUNDED STRICTLY PSEUDOCONVEX AND TUBULAR DOMAINS

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Abstract. We provide some new estimates on Traces in new mixed norm Hardy and Herz type spaces and related new results on Bergman type integral operators in Hardy type spaces and Herz type spaces in the unit ball, in tubular domains over symmetric cones and in bounded strictly pseudoconvex domains with smooth boundary. We in particular generalize in various directions a well-known one dimensional result concerning Traces of Hardy spaces obtained previously in the unit disk and polydisk by various authors.

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1. Introduction

This paper is continuation of our previous paper [31] and a large series of papers of author on Traces of analytic function spaces of several variables in various types of domains in C^n. We heavily enlarge the list of previously obtained assertions related with traces of Hardy type and Herz type spaces .The approaches we use here to obtain new results are mainly based on methods and ideas of [31].In particular we define among other things two new Hardy type and Herz type mixed norm spaces in tubular domains over symmetric cones and in bounded strictly pseudoconvex domains with smooth boundary and provide some new estimates of sizes of their traces generalizing at the same time previously known assertions obtained earlier by various authors.Similar results were obtained by us in less general form recently in [31].

Further we consider various new scales of Hardy type and Herz type spaces in the unit ball and products of unit balls.In particular some new extentions of so-

called Lizorkin-Triebel type spaces via Luzin cone in the unit ball will be provided and some results concerning their traces will be discussed and given .

This will enlarge the list of similar type results provided recently in [31]. We also generalize a well-known one-dimensional result concerning Traces of Hardy spaces obtained previously in the unit disk by various authors. This trace problem in Hardy spaces(diagonal map problem) in the unit disk and in the unit polydisk was considered by many authors during last several decades. In the polydisk some estimates related with this problem can be seen, for example, in [15], [16]. In the polydisk, but for particular values of parameters this problem was considered in [7] and also in some papers from list of references of [15] and [16]. A rather long history related with these type estimates of traces of Hardy spaces can be read in [6] and in [16] also. In this paper we extend a known crucial estimate related with this problem to more general and complicated cases of tubular domains over symmetric cones and to pseudoconvex domains with smooth boundary putting a natural condition on Bergman kernel of these domains. This paper can be considered also as continuation of a long series of papers of first author on Traces of function spaces (see, for example, [18], [19], [20], [21] and various references there).

In recent decades the same definition can be given for any domain in C^n and any function spaces on them, many papers appeared where various Hardy and other analytic spaces were studied from various points of views in higher dimension in various domains in C^n. We refer for example to a series of papers of Krantz and coauthors (see [23], [24], [25] in particular) and also to [8], [4], [5], [23] for results in this direction. For some new interesting results on analytic spaces in tubular domains over symmetric cones we refer the reader to [2], [11], [27], [28] and various references there also.We will heavily use nice techniques which was developed in these papers related with the so-called r- lattices.

We start hovewer with a result in the unit ball. Then we define new mixed norm Hardy type classes in tubular domains and pseudoconvex domains (see [15], [16] for much simpler case of the unit polydisk). Then we provide a complete proof of our assertion in the polyball and then provide assertions in bounded pseudoconvex domains with smooth boundary and in tubular domains over symmetric cones. In all cases proofs actually are the same.

Note that first such type sharp trace results in more complicated bounded pseudoconvex domains were provided recently in [17], [18], [19] and in unbounded tubular domains over symmetric cones in [21]. Hovewer these papers recently covered only various analytic Bergman type and Herz type spaces cases.

The situation with Hardy spaces is more complicated even in simplest case of product domains that is the case of the unit polydisk (see [15], [16] and references there). We note trace results have many applications (see [17] for example). Note

in more general than unit disk and unit polydisk cases the case of polyball such type results were proved earlier in [18], [19], [20] in BMOA type and other analytic function spaces.

To provide our assertion we will need some basic notations and lemmas in the case of the unit ball and the unit polyball, in tubular domains over symmetric cones and in bounded strongly pseudoconvex domains with smooth boundary (see, for example, [18], [19], [20] and references there).

Let B be the unit ball in $\mathbb{C}^n$, let $H(B)$ be the space of all analytic functions in B. Moreover, let $d\nu$ and ν_α(or dv and dv_α) denote the Lebesgue measure on B, normalizes such that $\nu(B) = 1$, and, for any $\alpha \in \mathbb{R}$, let $d\nu_\alpha(z) = c_\alpha(1-|z|^2)^\alpha d\nu(z)$ for $z \in B$. Here, if $\alpha \leq -1$, $c_\alpha = 1$ and if $\alpha > -1$, $c_\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}$ is the normalizing constant so that v_α has unit total mass. The Bergman metric on B is $\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}$, where φ_z is the Mobius transformation of B that interchanges 0 and z. Let $D(a,r) = \{z \in B : \beta(z,a) < r\}$ denote the Bergman metric ball centered at $a \in B$ with radius $r > 0$. Let further $\vec{p}$ denote $(p_1,\ldots,p_n)$, we use below also $\vec{z}$ instead of $(z_1,\ldots,z_n)$,the same notation below will be used for $\vec{x}$ or $\vec{y}$. Let further

$$(T_\alpha f)(z_1,\ldots,z_m) = c_\alpha \int_B \frac{f(w)(1-|w|^2)^\alpha dv(w)}{\prod_{j=1}^m (1 - \langle z_j, w \rangle)^{\frac{n+1+\alpha}{m}}}$$

, $\alpha > -1$, $z_j \in B$, $j = 1,\ldots,m$. The question is to study the action of this operator in Hardy, Bergman spaces. For $m = 1$ we have $(T_\alpha f)(z) = f(z)$ for every $f \in H(B)$ and each $z,z \in B$, and for $\alpha > -1$,so there is nothing to show, because of Bergman integral representation formula. This Bergman type operator or it"s various analogues were used in various estimates related with trace problem in Hardy, Bergman and BMOA classes in tubular and pseudoconvex domains (see [18], [19], [20], [21], [29]).

Let $B^m = B \times \cdots \times B$, $m > 1$.Let $H(B^m)$ be the space of all analytic functions in B^m.

Definition 1. Let X and Y be Banach analytic function spaces on the ball and the polyball so that $X \subset H(B)$ and $Y \subset (H(B^m))$. Then X is called the trace of Y if, for every function f, $f \in Y, f(z,...,z)$ is in X and the reverse is also true, for every function $g, g \in X$, there exists a function f in Y such that $f(z,...,z) = g(z)$ for all $z \in B$. In this case we write $Trace Y = X$ everwhere below. The same definition of traces can be stated for any bounded (or unbounded) domain in C^n and analytic function spaces on them. In particular we consider in this paper bounded pseudoconvex domains and tubular domains over symmetric cones.

We define Bergman space A_α^p in the ball as usual

$$A_\alpha^p(B) = \{f \in H(B) : \int_B |f(z)|^p(1-|z|)^\alpha dv(z) < \infty\}, 0 < p < \infty, \alpha > -1$$

This is Banach space for $p > 1$ or for $p = 1$ and complete metric space for other values of p. This space had been heavily studied by many authors (see [22]).

The proofs of the following properties of the Bergman balls can be found in [22] (see Lemmas 1.24, 2.20, 2.24 and 2.27 in [22]).

Lemma A.

a) *There exists a positive integer number $N \geq 1$ such that, for any $0 < r \leq 1$, we can find a sequence $\{v_k\}_{k=1}^{\infty}$ in B to be r-lattice in the Bergman metric of B. This means that $B = \cup_{k=1}^{\infty} D(v_k, r)$, $D(v_l, \frac{r}{4}) \cap D(v_k, \frac{r}{4}) = 0$ if $k \neq l$ and each $z \in B$ belongs to at most N of the sets $D(v_k, 2r)$.*

b) *For any $r > 0$, there is a constant $C > 0$ so that $\frac{1}{C} \leq \left| \frac{1-\langle z,w \rangle}{1-\langle z,v \rangle} \right| \leq C$ for all $z \in B$ and all w, v with $\beta(w, v) < r$.*

c) *For any $\alpha > -1$ and $r > 0$, $\int_{D(z,r)}(1 - |w|^2)^{\alpha}dv(w)$ is comparable with $(1 - |z|^2)^{n+1+\alpha}$ for all $z \in B$.*

d) *Suppose $r > 0$, $p > 0$ and $\alpha > -1$. Then there is a constant $C > 0$ such that $|f(z)|^p \leq \frac{c}{(1-|z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f(z)|^p dv_{\alpha}(w)$, for all $f \in H(B)$ and all $z \in B$.*

Let

$$H^{p_1,\ldots,p_m}(B^m) = \left\{ g \in H(B^m) : \sup_{r_m < 1} \int_S \left(\ldots \sup_{r_2 < 1} \int_S \left(\sup_{r_1 < 1} \int_S |g(r_1\zeta_1,\ldots,r_m\zeta_m)|^{p_1} d\sigma(\zeta_1) \right)^{\frac{p_2}{p_1}} \right. \right.$$

$$\left. \left. d\sigma(\zeta_2)\ldots \right)^{\frac{p_m}{p_{m-1}}} d\sigma(\zeta_m) < \infty \right\};$$

$0 < p_i < \infty$, $i = 1,\ldots,m$. Let

$$\tilde{H}^{p_1,\ldots,p_m}(B^m) = \left\{ g \in H(B^m) : \sup_{r_j < 1} \int_S \left(\ldots \int_S \left(\int_S |g(r_1\zeta_1,\ldots,r_m\zeta_m)|^{p_1} d\sigma(\zeta_1) \right)^{\frac{p_2}{p_1}} \right. \right.$$

$$\left. \left. d\sigma(\zeta_2)\ldots \right)^{\frac{p_m}{p_{m-1}}} d\sigma(\zeta_m) < \infty \right\};$$

$0 < p_i < \infty$, $i = 1,\ldots,m$, $j = 1,..,m$ where σ is a Lebegues measure on $S = \{z : |z| = 1\}$, $S = \partial B$. These are a new mixed norm analytic Hardy class on products of unit balls (polyballs). Previously many authors studied such type spaces for $m = 1$ case only.

Similarly based on definition of one -domain case we can define mixed norm Hardy classes on products of tubular and bounded strongly pseudoconvex domains as subspaces of $H(\Omega^m)$ and $H(T_{\Omega}^m)$, $m \in \mathbb{N}$ of spaces of all analytic functions on

products of pseudoconvex domains $\Omega^m = \Omega \times \cdots \times \Omega$ or on products of tubular domains $T_\Omega^m = T_\Omega \times \cdots \times T_\Omega$. We denote them by $H^{p_1,\ldots,p_m}(\Omega^m)$ and $H^{p_1,\ldots,p_m}(T_\Omega^m)$, for all positive values of these parameters.

The well-known definitions of classical Hardy spaces in tubular and in pseudoconvex domains can be seen in [2], [9], [10], [26], [27]. In case of bounded strongly pseudoconvex domains with smooth boundary the sphere S in our defintion must be simply replaced by special $\partial\Omega_\varepsilon$ domains closely related with so-called r defining function of pseudoconvex D or Ω domains. To be more precise it is a set of those points of this domain for which defining r function is constant and is equal to ε, (see for one domain analytic Hardy classes in pseudoconvex domains, for example [9], [23], [24], [25]).

In the last part of our paper we consider some new mixed norm analytic Herz type spaces in product domains and we provide some new estimates for traces of such type spaces.On one hand this part is continuation of our previous research in this direction which started in [18 19] and on the other hand it is based on arguments of the proof of theorem 1 of previous section.Herz type analytic spaces in simpler domains like unit disk and unit polydisk studied in recent years by various authors (see for example for this [18, 19]and various references there). We in this section among other things also discuss some extentions of our results on traces in Herz type spaces in the polyball (under some condition on Bergman kernel) to tubular domains over symmetric cones and to bounded strongly pseudocnvex domains with smooth boundary.Some other new analytic function spaces will also be considered and discussed in this section.

2. On Bergman type operators and traces in Hardy type spaces in product domains

In this section we provide main results of this paper on Traces of some new mixed norm Hardy type spaces in product domains in higher dimensions. We generalize some previously known assertions provided earlier by various authors in the unit polydisk to more complicated domains.

We provide first a full and a rather short proof of our main result in simple polyball case based on lemmas we have just formulated.Then we add some vital comments concerning this proof and formulate complete analogues of this result in tubular and pseudoconvex domains under an additional natural condition on Bergman kernel of these domains.

Theorem 1. *Let $f \in H(B); 0 < p_j < 1, j = 1, \ldots, m$. If*

$$\int_B |f(z)|^{p_m}(1 - |z|)^{np_m(\sum_{i=1}^{m-1} \frac{1}{p_i} - 1) - 1} dv(z) < \infty$$

and $\alpha > \frac{2n}{p_m} - n - 1$ then we have $T_\alpha f \in H^{p_1,\dots,p_m}(B^m)$, and $T_\alpha f \in \tilde{H}^{p_1,\dots,p_m}(B^m)$.

So we have $Trace(H^{\vec{p}}) \supset A_\tau^{p_m}(B), \tau = np_m(\sum_{i=1}^{m-1} \frac{1}{p_i} - 1) - 1$ and also $Trace(\tilde{H}^{\vec{p}}) \supset A_\tau^{p_m}(B), \tau = np_m(\sum_{i=1}^{m-1} \frac{1}{p_i} - 1) - 1$.

The statement in theorem concerning traces follows directly from first assertion and on results on integral representations in Bergman spaces in one- domain namely in the unit ball (not in product domains) and we refer for this to [4,6,22] and various references there.

Remark 1. *This result is well-known when $n = 1$, $p_i = p$, $i = 1,\dots,m$. (see [6], [15], [16] for example and references there in the case of unit polydisk). For these values of parameters it provides a sharp embedding for traces of classical H^p Hardy spaces in the unit polydisk (see, for example, [6], [15], [16]).*

The Proof of theorem 1.

The proof of theorem 1 is fully based on properties of r-lattices of the unit ball (see lemma A). We consider the case $m = 2$, i.e. if $\int_B |f(w)|^{p_2}(1 - |z|)^{\frac{np_2}{p_1} - 1} dv(z) < \infty$, $p_2, p_1 < 1$ we prove that

$$\sup_{r_1,r_2<1} \int_S \left(\int_S |T_\alpha f(z_1, z_2)|^{p_1} d\sigma(\zeta_1) \right)^{\frac{p_2}{p_1}} d\sigma(\zeta_2) < \infty; \; z_j = r_j\zeta_j; \; j = 1,2.$$

The general case is the same. We use properties of r-lattices listed above (see Lemma A). We have the following estimates ($\gamma = \frac{n+1+\alpha}{2}$):

$$J = \int_S \left(\int_S |T_\alpha f(z_1, z_2)|^{p_1} d\sigma(\zeta_1) \right)^{\frac{p_2}{p_1}} d\sigma(\zeta_2) \leq$$

$$\leq c \sum_{k=1}^\infty (\sup_{z \in D(a_k, r)} |f(z)|^{p_2} \int_S \left(\int_S \left(\int_{D(a_k,r)} \frac{(1 - |w|)^\alpha dv(w)}{\prod_{j=1}^2 (|1 - \langle z_j, \zeta_j \rangle|^\gamma)} \right)^{p_1} d\sigma(\zeta_1) \right)^{\frac{p_2}{p_1}} d\sigma(\zeta_2) =$$

$$= c \sum_{k=1}^\infty (F_k G_k); \; F_k = \sup_{z \in D(a_k, r)} |f(z)|^{p_2}.$$

Note

$$G_k \leq (1 - |a_k|)^{p_2(n+1+\alpha)} \left(\sup_{r_2<1} \int_S \frac{d\sigma(\zeta_1)}{|1 - \langle r_1\zeta_1, a_k \rangle|^{p_1\gamma}} \right)^{\frac{p_2}{p_1}} \left(\sup_{r_2<1} \int_S \frac{d\sigma(\zeta_1)}{|1 - \langle r_2\zeta_2, a_k \rangle|^{p_2\gamma}} \right).$$

Hence we have that the following chain of estimates is valid based on Lemma A (note this chain of estimates is valid also in tubular and pseudoconevx domains based on properties of r-lattices of these domains, we refer to [8] and [30] for this)

$$J \leq c \sum_{k=1}^\infty \sup_{z \in D(a_k, r)} |f(z)|^{p_2} (1 - |a_k|)^{n + \frac{np_2}{p_1}} \leq$$

$$\leq c \sum_{k=1}^{\infty} \int_{D(a_k, 2r)} |f(z)|^{p_2} (1-|z|)^{n\frac{p_2}{p_1}-1} dv(z) \leq c_N \int_B |f(z)|^{p_2} (1-|z|)^{n\frac{p_2}{p_1}-1} dv(z) < \infty.$$

This completes the proof of theorem in case of the unit ball.

We provide some discussion on this proof. A carefull analysis of this proof shows similar results should be valid also in other domains. Indeed our arguments are valid, for example, in tubular domains and strongly pseudoconvex domains based on r-lattices of those domains (see [2], [30]). First for the proof in tubular, pseudoconvex domains we need a version of Lemma A in these domains). Complete analogues of first, third and forth assertions of lemma A in tubular domains and pseudoconvex domains can be seen in pseudoconvex domains in [8], [10] , in tubular domains in [2], [11], [26], [27]. The second assertion of lemma A is also valid in these domains, but only in milder form [8], [30]. This hovever, is vital for our proof and will stand in our formulations below as an additional condition on Bergman kernel.

Note further the last chain of estimates in our proof is valid also in tubular and pseudoconvex domains based on same properties of r-lattices of these domains (we refer to [8] and [30] for this standard arguments).

For analogues of estimates of G_k (see again the proof above in the ball) we refer to theorem 2 of [12] for pseudoconvex domains and to basic properties of defining r function of the pseudoconvex domain and for tubular domains to a lemma from [11], and [28] concerning integrability of Δ function. To be more precise, for example, in tubular domains it is the following well-known basic integrability property for the determinant Δ function (see for this assertion [2] or [27]).

Lemma B.

1) The integral

$$J_\alpha(y) = \int_{\mathbb{R}^n} \left| \Delta^{-\alpha} \left(\frac{x+iy}{i} \right) \right| dx$$

converges if and only if $\alpha > 2\frac{n}{r} - 1$. In that case

$$J_\alpha(y) = \widetilde{C}_\alpha \Delta^{-\alpha+n/r}(y),$$

$\alpha \in R$, $y \in \Omega$.

In addition we must use the fact that Δ function is "monotone" on Ω cone (see for example [2] or [27] for these basic properties).

We denote by $d\tilde{v}$ and dv as usual Lebegues measures on tube T_Ω and pseudoconvex Ω domains.

The Bergman kernels in tubular domains and pseudoconvex domains will be denoted as usual by $K_t(z,w)$ and $B_\nu(z,w)$ (see, for example, [2], [8], [10], [27]). The

Bergman B_ν kernel for tubular domains can be written via so-called Δ function in the following manner.

$$B_\nu(z, w) = C_\nu \Delta^{-(\nu + \frac{n}{r})}((z - \overline{w})/i),$$

$z \in T_\Omega$, $w \in T_\Omega$ (see [1], [2] for more details).

We formulate analogues of our first theorem in tubular and bounded strictly pseudoconvex domains. We omit calculations leaving them to interested readers.

We first define the complete analogue of Bergman type T_α integral operator in pseudoconvex and tubular domains using Bergman kernels in the following way. In the tubular domain over symmetric cones we define them as the following Bergman-type integral operators.

$$T_\alpha(f)(\vec{z}) = \int_{T_\Omega} \frac{f(w) \Delta^\alpha (Im\ w) dv(w)}{\prod_{j=1}^m \Delta^{\frac{\alpha + 2\frac{n}{r}}{m}}(\frac{z_j - \overline{w}}{i})};\ f \in L^1(T_\Omega),$$

$\alpha > -1$, $z_j \in T_\Omega$, $j = 1, \ldots, m$.

In bounded pseudoconvex domains with smooth boundary Ω we define them as

$$T_\alpha f(\vec{z}) = \int_\Omega f(w) \prod_{j=1}^m K_\tau(z_j, w) \delta^\alpha(w) dv(w);\ \tau = \frac{\alpha + n + 1}{m},$$

$f \in L^1(\Omega)$, $\alpha > -1$, $z_j \in \Omega$, $j = 1, \ldots, m$, where $\delta(w)$ is a distance from w to the boundary of Ω. Such type operators were studied before in papers [18], [19], [20] written in cooperation with author. They play a crucial role in various Trace theorems related with Herz and Bergman spaces in such type domains (see [18], [19], [20], and also in [21]). We define new mixed norm analytic Hardy spaces in products of tubular domains. Let further for all positive values of parameters

$$H^{p_1, \ldots, p_m}(T_\Omega^m) = \left\{ f \in H(T_\Omega^m) : \sup_{y_j \in \Omega} \left[\int_{R^n} \int_{R^n} \left(\int_{R^n} |f(\vec{x} + i\vec{y})|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} \ldots dx_m \right]^{\frac{1}{p_m}} < \infty \right\};$$

$$\tilde{H}^{p_1, \ldots, p_m}(T_\Omega^m) = \left\{ f \in H(T_\Omega^m) : \sup_{y_m \in \Omega} \left[\int_{R^n} \sup_{y_3 \in \Omega} \int_{R^n} \sup_{y_2 \in \Omega} \left(\int_{R^n} \sup_{y_1 \in \Omega} |f(\vec{x} + i\vec{y})|^{p_1} dx_1 \right)^{\frac{p_2}{p_1}} \ldots \right.\right.$$

$$\left.\left. \ldots dx_m \right]^{\frac{1}{p_m}} < \infty \right\};$$

$0 < p_i < \infty$; $i = 1, \ldots, m$. We assume that for $B_t(z, w)$ function (the Bergman kernel of T_Ω domain) the following condition is valid. $|B_t(z, w)| \asymp |B_t(z, w_k)|$, $w \in$

$B_{T_\Omega}(w_k, R)$; $z \in T_\Omega$, where t is positive and $B_{T_\Omega}(w, R)$ is a Bergman ball in tubular domain (see, for example, [2],[26], [27], for definitions of these objects and more discussions). We refer to [30] for very similar, but milder version of this condition on Bergman kernel.

Theorem 2. *Let* $f \in H(T_\Omega)$ *and let* $0 < p_i < 1$, $\tau = \frac{n}{r} p_m \sum_{i=1}^{m-1} \frac{1}{p_i} - 1$, $i = 1, \ldots, m$, *if*

$$\int_{T_\Omega} |f(z)|^{p_m} \Delta(Im\ z)^{\frac{n}{r} p_m \sum_{i=1}^{m-1} \frac{1}{p_i} - 1} d\tilde{v}(z) < \infty$$

and if $\alpha > \alpha_0$ *for large enough* α_0, *then* $T_\alpha f \in H^{p_1,\ldots,p_m}(T_\Omega^m)$, *so* $Trace(H^{\vec{p}}(T_\Omega^m)) \supset A_\tau^{p_m}(T_\Omega)$ *and let* $f \in H(T_\Omega)$ *and let* $0 < p_i < 1$, $\tau = \frac{n}{r} p_m \sum_{i=1}^{m-1} \frac{1}{p_i} - 1$, $i = 1, \ldots, m$, *if*

$$\int_{T_\Omega} |f(z)|^{p_m} \Delta(Im\ z)^{\frac{n}{r} p_m \sum_{i=1}^{m-1} \frac{1}{p_i} - 1} d\tilde{v}(z) < \infty$$

and if $\alpha > \alpha_0$ *for large enough* α_0, *then* $T_\alpha f \in \tilde{H}^{p_1,\ldots,p_m}(T_\Omega^m)$, *so* $Trace(\tilde{H}^{\vec{p}}(T_\Omega^m)) \supset A_\tau^{p_m}(T_\Omega)$.

The statement in theorem concerning traces follows directly from first assertion of theorem and from results on integral representations in Bergman spaces in pseudoconvex domains (not in product domains). We refer for this for example to [5,8-9] and various references there.

Let further,

$$\tilde{H}^{p_1,\ldots,p_m}(\Omega^m) = \left\{ f \in H(\Omega^m) : \sup_{\varepsilon > 0} \left[\int_{\Omega_\varepsilon} \sup_{\varepsilon > 0} \int_{\Omega_\varepsilon} \sup_{\varepsilon > 0} \left(\int_{\Omega_\varepsilon} |f(\vec{w})|^{p_1} dw_1 \right)^{\frac{p_2}{p_1}} \ldots dw_m \right]^{\frac{1}{p_m}} < \right.$$

$$\left. < \infty \right\}$$

We can similarly define also $H^{p_1,\ldots,p_m}$ spaces as we did above in tube adding one sup for all integrals.

We assume that for the Bergman K_t kernel of bounded strongly pseudoconvex Ω domain with smooth boundary the following assertion is valid. $|K_t(z, w)| \asymp |K_t(z, w_k)|$, $w \in B_\Omega(w_k, R)$; $z \in \Omega$, where $B_\Omega(w, R)$ is a Kobayashi ball. (we refer for more details on these objects, for example, to [8], [10]). In these papers similar, but milder condition on Bergman kernel can be also found.

Theorem 3. *Let* $0 < p_i < 1$, $i = 1, \ldots, m$, $\tau = n p_m (\sum_{i=1}^{m-1} \frac{1}{p_i} - 1)$, *if*

$$\int_\Omega |f(z)|^{p_m} \delta^{n p_m (\sum_{i=1}^{m-1} \frac{1}{p_i} - 1)}(z) dv(z) < \infty$$

and if $\alpha > \alpha_0$ for large enough α_0 then $T_\alpha f \in H^{p_1,\ldots,p_m}(\Omega^m)$, so $Trace(H^{\vec{p}}) \supset A^{p_m}_\tau(\Omega)$ and if $\alpha > \alpha_0$ for large enough α_0 then $T_\alpha f \in \tilde{H}^{p_1,\ldots,p_m}(\Omega^m)$, so $Trace(\tilde{H}^{\vec{p}}) \supset A^{p_m}_\tau(\Omega)$.

The statement in this theorem concerning traces follows directly from first assertion of this theorem and on results on integral representations in Bergman spaces in tubular domains (not in product domains) and we refer the reader for this representation to [2,26,27,30] and various references there.

We add some similar type results where Lusin cone $\Gamma_\alpha(\xi)$ is involved.

Let further $\Gamma_\alpha(\xi) = \{z \in B : |1 - \bar{\xi}z| < \alpha(1 - |z|)\}$. For $p_i \in (0, \infty), i = 1, \ldots, m$ we will define mixed norm new Hardy classes in $B^m = B \times \cdots \times B$ with the following finite quasinorms

$$\left(\int \sup_{z_m \in \Gamma_\alpha(\xi_m)} \int_S \cdots \left(\int_S \sup_{z_1 \in \Gamma_\alpha(\xi_1)} |f(z)|^{p_1} d\xi_1 \right)^{\frac{p_2}{p_1}} d\xi_m \right)^{\frac{1}{p_m}} \quad (1)$$

$$\left(\int_S \cdots \left(\int_S \left(\sup_{z_m \in \Gamma_\alpha(\xi_m)} \cdots \sup_{z_1 \in \Gamma_\alpha(\xi_1)} |f(z_1, \ldots, z_m)|^{p_1} \right) d\xi_1 \right)^{\frac{p_2}{p_1}} d\xi_m \right)^{\frac{1}{p_m}} \quad (2)$$

$$\left(\left(\int_S \cdots \left(\int_S \sup_{0 < r_j < 1, j=1,..m} |f(r_1\xi_1, \ldots, r_m\xi_m)|^{p_1} d\xi_1 \right)^{\frac{p_2}{p_1}} \right) d\xi_m \right)^{\frac{1}{p_m}} \quad (3)$$

It is possible to show that all of them are the closed quazinormed subspaces $H(B \times \cdots \times B)$ of a class of all analytic functions in $B^m, m \geq 1, m \in N$. Similar proof work for these classes. We omit easy details.

These are a new mixed norm analytic Hardy class on products of unit balls (polyball). For $m = 1$ we have well-known classical Hardy spaces in the unit ball.

(Some weighted version can be defined similarly and similar results can be even proved also)

Let Y be one of these classes in products of balls B^m.

Theorem 4. Let $f \in H(B)$; $0 < p_j < 1$,$j = 1, \ldots, m$. If

$$\int_B |f(z)|^{p_m}(1 - |z|)^{np_m(\sum_{i=1}^{m-1} \frac{1}{p_i} - 1) - 1} dv(z) < \infty$$

and $\alpha > \frac{2n}{p_m} - n - 1$ then we have $T_\alpha f \in Y(B^m)$ and we have $Trace(Y) \supset (A_\tau^{p_m})$;

$$\tau = (np_m)\left(\sum_{j=1}^{m-1}(\frac{1}{p_j} - 1) - 1\right)$$

.

The proof of this theorem is very similar to the proof of our first theorem. The proof hinges on known estimates like (see [3,4]).

$$\sup_{z \in \Gamma(\xi)} \frac{1}{|1 - wz|^\alpha} \le \frac{c}{|1 - w\xi|^\alpha}; \xi \in S, w \in B, \alpha > 0.$$

Let further

$$\int_{S^m}\left(\int_{\Gamma(\xi_m)}\cdots\left(\int_{\Gamma(\xi_2)}\left(\int_{\Gamma(\xi_1)}|f(\vec{z})|^{p_1}(1 - |z_1|)^{\alpha_1}d|z_1|\right)^{\frac{p_2}{p_1}}(1 - |z_m|)^{\alpha_m}d|z_m|\right)\right)^{\frac{q}{p_m}}$$

$$d\sigma(\xi_1)\cdots d\sigma(\xi_m)$$

$$\int_{S^m}\left(\int_0^1\left(\cdots\int_0^1\left(\int_0^1|f(\vec{z})|^{p_1}(1 - |z_1|)^{\alpha_1}d|z_1|\right)^{\frac{p_2}{p_1}}(1 - |z_m|)^{\alpha_m}d|z_m|\right)\right)^{\frac{q}{p_m}}d\sigma(\xi_1)\cdots d\sigma(\xi_m),$$

Where $p_i \in (0, \infty), \alpha_j > -1, j = 1, ..., m$.

Note for $m = 1$ last two classes coincide with $F_\alpha^{q,p}$ spaces (see [3]) and $F_0^{p,2} = H^p, 0 < p < \infty$ (see [3]) in the unit ball. Using the same proof we can calculate sizes of traces of these spases also. We leave these details to interested readers since the proof is very similar to the proof of theorem 1.

The base of our proof is the estimate (see for example[3,4])

$$\int_{\Gamma_\alpha(\xi)} \frac{(1 - |z|)^\alpha}{|1 - wz|^\beta}dv(z) \le \frac{c}{|1 - \xi w|^{\beta - (\alpha+2)}},$$

$\xi \in S, w \in B, \alpha > -1, \beta > \alpha + 2$ and similar estimate for integral by $(0,1)$.

With this we proved various extensions of the embedding $Y \subset TraceH^p$ obtained in the unit disk by various authors before (see ,for example, [6, 7]).

Indeed following carefully arguments of the proof of our first theorem we have the following chain of estimates

$$\int_0^1 \left(\int_S \left(\int_S |T_\alpha(f)(z_1,z_2)|^{p_1} d\sigma(\zeta_1) \right)^{\frac{p_2}{p_1}} \times d\sigma(\zeta_2) \right)^{\frac{p_3}{p_2}} \le (p_{j+1} \le p_j)$$

$$\le c_1 \sum_{k=1}^{\infty} \sup_{z \in D(a_k,r)} (|f(z)|^{p_3}) \int_0^1 \left((1-|a_k|)^{p_3(n+1+\alpha)} \right) \times (1-|a_k|)^{(-p_1\gamma+n)\frac{p_3}{p_1}} \times$$

$$\left(\gamma = \left(\frac{n+1+\alpha}{2} \right) \right) \times (1-|a_k|)^{(-p_2\gamma+n)\frac{p_3}{p_2}} \le c_2 \int_B |f(z)|^{p_3} \times (1-|z|)^{np_3(\frac{1}{p_2}+\frac{1}{p_1})-(n)} dv(z)$$

$$\int_S \int_S \left(\int_0^1 \left(\int_0^1 |(T_\alpha(f))(z_1,z_2)|^{p_1} d|z_1| \right)^{\frac{p_2}{p_1}} d|z_2| \right)^{\frac{p_3}{p_1}} \le (p_{j+1} \le p_j)$$

$$\le c_3 \sum_{k=1}^{\infty} (1-|a_k|)^{p_3(n+1+\alpha)+n} \times (1-|a_k|)^{(-p_1\gamma+1)(\frac{p_3}{p_1})+n}$$

$$\times (1-|a_k|)^{(-p_2\gamma+1)(\frac{p_3}{p_2})} \left(\max_{z \in D(a_k,r)} |f(z)|^{p_3} \right)$$

$$\le c_4 \int_B |f(z)|^{p_3} (1-|z|)^{\tau} dv(z)$$

$$\int_S \int_S \left(\int_{\Gamma_{\alpha_1}(\zeta_2)} \left(\int_{\Gamma_{\alpha_2}(\zeta_1)} |T_\alpha(f)(z_1,z_2)|^{p_1} dm_2(z_1) \right)^{\frac{p_2}{p_1}} dm_2(z_2) \right)^{\frac{p_3}{p_1}} \le (p_{j+1} \le p_j)$$

$$\le c_5 \sum_{k=1}^{\infty} (\sup_{z \in D(a_k,r)} (|f(z)|^{p_3}) (1-|a_k|)^{p_3(n+1+\alpha)} \times (1-|a_k|)^{(-p_1\gamma+n+1)\frac{p_3}{p_1}+n} \times$$

$$(1-|a_k|)^{(-p_2\gamma+n+1)\frac{p_3}{p_2}+n} \le c_6 \int_B |f(z)|^{p_3} \times (1-|z|)^{\tau_1} dv(z)$$

where $\tau = p_3(\frac{1}{p_1} + \frac{1}{p_2}) + n - 1$; $\tau_1 = (n+1)p_3(\frac{1}{p_1} + \frac{1}{p_2}) + n - 1$

The general case of m variable spaces can be considered similarly, for example we have in this case instead of τ $(np_m) \left(\sum_{j=1}^{m-1} \frac{1}{p_j} \right) - (n+1)$ and instead of τ_1 we have $(p_m) \left(\sum_{j=1}^{m-1} (\frac{1}{p_j}) \right) + n - 1$

The same type of arguments based only on proof of theorem 1 will be used for us in our next section,where we provide sizes of traces of some new analytic Herz type spaces in product domains.

3. On Bergman type operators in some new mixed norm Herz type function spaces in product domains and some related function spaces

In this section we discuss some issues related with Traces of some new mixed norm analytic Herz -type spaces in product domains. We in this section also look at Hardy-Sobolev type spaces and weighted Hardy spaces with (Muckenhoupt weights) and discuss issues related with traces of these classes. Next we look also at trace of some new Morrey-Campanato type function classes and pose some interesting problems for these classes of functions also.

Let $B_D(z, r)$ on $B_{T_\Omega}(z, r)$ be Bergman or Kobayashi ball in bounded strongly pseudoconvex or in tube domain which we studied in this paper. We define analytic Herz type spaces in these domains as follows

$$M_{\vec{\alpha}}^{\vec{p}, q}(\Omega^m) = \left\{ f \in H(\Omega^m) : \int_D \left[\int_{B_{D(z,r)}} \cdots \left(\int_{B_{D(z,r)}} |f(\vec{w})|^{p_1} dv_{\alpha_1}(w_1) \right)^{\frac{p_2}{p_1}} dv_{\alpha_m}(w_m) \right]^{\frac{q}{p_m}} \right.$$

$$\left. dv(z) < \infty \right\}$$

$0 < p_j < \infty$, $\alpha_j > -1$, $j = 1, .., m$, $q \in (0, \infty)$.

$$\tilde{M}_{\vec{\alpha}}^{\vec{p}, q}(T_\Omega^m) = \left\{ f \in H(T_\Omega^m) : \int_{T_\Omega} \left[\int_{B_{T_\Omega(z,r)}} \cdots \left(\int_{B_{T_\Omega(z,r)}} |f(\vec{w})|^{p_1} dv_{\alpha_1}(w_1) \right)^{\frac{p_2}{p_1}} dv_{\alpha_m}(w_m) \right]^{\frac{q}{p_m}} \right.$$

$$\left. dv(z) < \infty \right\}$$

$0 < p_j < \infty$, $\alpha_j > -1$, $j = 1, .., m$, $q \in (0, \infty)$.

Very similar results by very similar methods can be obtained for these spaces under condition on kernel we discussed. We will present these results in a separate paper.

Namely we have

$$(Trace)(M_{\vec{\alpha}}^{\vec{p}, q}) \supset (A_\tau^s)(B);$$

$$(Trace)(\tilde{M}_{\vec{\alpha}}^{\vec{p}, q}) \supset (A_\tau^s)(B)$$

for some $0 < p_j < \infty$, $q, s \in (0, \infty)$, $\alpha_j > -1$, $j = 1, .., m$, τ.

Finally we remark some analogues of these assertions with similar proofs may be true in other domains, for example, in minimal homogeneous domains, in bounded symmetric domains, but again under some additional condition on Bergman kernel. We note for these domains we also have similar properties of r-lattices and the analogue of Lemma A (see for example [13], [14] and various references there).

The weighted Hardy-Sobolev space $H_s^p(\omega)$, $0 < s$, $p < +\infty$, consists of those functions f holomorphic in the unit ball $\mathbb{B}$ such that if $f(z) = \sum\limits_{k>0} f_k(z)$ is its so-called homogeneous polynomial expansion in the unit ball, and $(D)^s f(z) = \sum\limits_{k>0} (1 + k)^s f_k(z)$, we have that

$$\|f\|_{H_s^p}(\omega) = \sup_{0<r<1} \|(D)^s f_r\|_{L^p(\omega)} < +\infty$$

where $f_r(\zeta) = f(r\zeta)$, $r \in (0, 1)$. and D^s is an ordinary operator of fractional differentiation in B. Readers may find these objects in [3],[4],[9]. Similary to theorem 1 some results concerning traces of (H_s^p) spaces can be probably provided also. We leave this to interested readers. Note theorem 1 provides results only for $s = 0$ case for these spaces.

We will consider ω weights in A_p function classes in $\mathbb{S}^n$, $1 < p < +\infty$, that is, ω weights in $\mathbb{S}^n$ such that there exists a constant $C > 0$ such that for any nonisotropic ball $\tilde{B} \subset \mathbb{S}^n$, $\tilde{B} = B(\zeta, r) = \{\eta \in \mathbb{S}^n; |1 - \zeta\overline{\eta}| < r\}$,

$$\left(\frac{1}{|\tilde{B}|} \int\limits_{\tilde{B}} \omega d\sigma\right)\left(\frac{1}{|\tilde{B}|} \int\limits_{\tilde{B}} \omega^{\frac{-1}{p-1}} d\sigma\right)^{p-1} \le C,$$

where σ is the Lebesgue measure on $\mathbb{S}^n$ and $|\tilde{B}|$ the Lebesgue measure of $\tilde{B}$. We will use as usual the notation $\zeta\overline{\eta}$ to indicate the complex inner product in $\mathbb{C}^n$ given by $\zeta\overline{\eta} = \sum\limits_{i-1}^{n} \zeta_i\overline{\eta_i}$, if $\zeta = (\zeta_1, ..., \zeta_n)$, $\eta = (\eta_1, ..., \eta_n)$.

If $0 < s < n$, and function f in $H_s^p(\omega)$ can be expressed as

$$f(z) = C_s(g)(z) := \int\limits_{\mathbb{S}^n} \frac{g(\zeta)}{(1-z\overline{\zeta})^{n-s}} d\sigma(\zeta),$$

where $d\sigma$ is the normalized Lebesgue measure on the unit sphere $\mathbb{S}^n$ and $g \in L^p(\omega)$. (see[3,4,9]) For $1 < p < \infty$, and $\theta \in A_p$, the weighted Hardy space $H^p(\theta)$ consists of holomorphic functions f on $\mathbb{B}$ such that

$$\|f\|_{H^p(\theta)} = (\sup_r \int\limits_{\mathbb{S}} |f(r\zeta)|^p \theta(\zeta) d\sigma(\zeta))^1/p < \infty. \tag{M}$$

For $1 < p < \infty$ and $-1 < s \le n/p$, we also define the Morrey-Campanato space $M^{p,s}$ on $\mathbb{S}$ given by

$$M^{p,s} = \{f \in L^p(\mathbb{S} : \|f\|_{M^{p,s}}) < \infty\},$$

where

$$\|f\|_{p,s} = \|f\|_p + \sup_{\tilde{B}_{\zeta,\varepsilon}} (\varepsilon^{sp-n} \int_{\tilde{B}_{\zeta,\varepsilon}} |f(\eta) - f(\zeta)|^p d\sigma(\eta))^1/p, \|f\|_p \qquad (M_1)$$

denotes the usual $L^p(\mathbb{S})$ quazinorms of f, and $\tilde{B}_\zeta, \varepsilon = \{\eta \in \mathbb{S}; |1 - \bar{\zeta}\eta| < \varepsilon\}$.

It is clear that for $s = n/p$ the space $M^{p,n/p}$ coincides with $L^p(\mathbb{S})$ and that for $s = 0$, $M^{p,0}$ coincides with the non isotropic BMO space. It is also well-known that for $0 < p < 1$ the space $M^{p,s}$ coincides with the non isotropic Lipschitz space Λ_s.

Let $HM^{p,s} = M^{p,s} \cap H^p$ be the corresponding holomorphic Morrey space.

We may now based on this define $(HM^{p,s})$ new analytic function spaces in product domains and pose a problem of calculating of traces of these interesting function classes also.

First define H^p Hardy spaces in product of balls, so that we for $n = 1$ we have usual H^p class in the unit polydisk $0 < p \le \infty$ with the following quazinorm

$$\|f\|_{H^p(B^m)} = \left(\sup_{r_j < 1, j = 1 \cdots n} \right) \left(\int_S \cdots \int_S |f(r_1\zeta_1, \cdots, r_m\zeta_m)|^p d\sigma(\vec{\zeta}) < \infty \right)^{\frac{1}{p}};$$

$0 < p \le \infty$; and also define

$$M^{p,s}(S^m) = \left\{ f \in L^p(S^m) : \|f\|_{M^{p,s}(S^n)} < \infty \right\},$$

where we define on S^m

$$\|f\|_{M^{p,s}} = \|f\|_{L^p(S^m)} + \left(\sup_{\tilde{I}_{\zeta,\varepsilon}} \right) \left((\varepsilon^{sp-n}) \int_{(\tilde{I}_{\zeta,\varepsilon})} |f(\vec{\eta}) - f(\vec{\zeta})|^p d\sigma(\vec{\eta}) \right)^{\frac{1}{p}}$$

$$(\tilde{I}_{\zeta,\varepsilon}) = I_{\zeta_1,\varepsilon} \times \cdots I_{\zeta_m,\varepsilon}; \zeta_j \in S; j = 1, \cdots, m; \varepsilon > 0,$$

$$\vec{\eta} = (\eta_1, \cdots, \eta_m); \vec{\zeta} = (\zeta_1, \cdots, \zeta_m);$$

$$(-1) < s \le \frac{n}{p}; 1 < p < \infty;$$

The question is to find sizes of Traces of these new analytic spaces on product domains in higher dimension. New appoaches should be developed here probably or $H^{\frac{n}{s}} \subset (HM^{p,s})$ embedding should be used ,but in case of product domains (see[3],[4])

Proposition C . Let $1 < p < +\infty$, and $\theta \in A_p$, and let λ be the constant in the doubling condition of θ, $\theta(I_{\zeta,2r}) \lesssim C2^{\lambda}\theta(I_{\zeta,2r})$, and $N_p > \lambda$. Let $f_z(\omega) = \dfrac{1}{(1-\omega\bar{z})^N}$, for $z \in \mathbb{B}$. Then we have

$$\|f_z\|^p_{H^p(\theta)} \lesssim \frac{\theta(I_z)}{(1-|z|^2)^{N_p}} \qquad (K)$$

,

Let $\theta_i \in A^{p_i}(S)$; $1 < p_i < \infty$, $i = 1,..,n$ be Muckenhoupt weights on S. We define new weighted Hardy class in the polyball as follows

$$H^{\vec{p}}_{\vec{\theta}}(B^n) = \left\{ f \in H(B^n) : \sup_{r_j < 1, j \geq 1} \left(\int_S \cdots \left(\int_s |f(r_1\zeta_1,..,r_n\zeta_n)|^{p_1}\theta_1(\zeta_1)d\sigma(\zeta_1) \right)^{\frac{p_2}{p_1}} \theta_n(\zeta_n)d\sigma(\zeta_n) \right)^{\frac{1}{p_n}} \right.$$

$$\left. < +\infty \right\}$$

and

$$\tilde{H}^{\vec{p}}_{\vec{\theta}}(B^n) = \left\{ f \in H(B^n) \left(\sup_{r_n < 1} \int_S \cdots \left(\sup_{r_1 < 1} \int_S |f(r_1\zeta_1,\ldots,r_n\zeta_n)|^{p_1}\theta_1(\zeta_1)d\sigma(\zeta_1) \right)^{\frac{p_2}{p_1}} \right.\right.$$

$$\left.\left. \ldots \theta_n(\zeta_n)d\sigma(\zeta_n) \right)^{\frac{1}{p_n}} < +\infty \right\}$$

The natural question to find sizes of Traces of these spaces also.Below we provide some answers to these questions.

We give some estimates of traces of these Hardy type spaces with Muckenhoupt weights below

following Proposition C and following the proof of theorem 1 . .

We have following Proposition C.

$$\|T_\alpha f\|_{H^{p_1,p_2}_{\theta,\theta}} \leq c \sum_{k \geq 0} \left(\max_{z \in D(a_k,r)} |f(z)|^{p_2} \right) (\theta(I_{a_k}))^{\frac{p_2}{p_1}+1}$$

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Let $B(a_k, r)$ be a Kobayashi or Bergman ball in bounded pseudoconvex domain or tubular domain over symmetric cones. Let $\{a_k\}_{k\geq 1}$ be the r-lattice in such domain (tube or pseudoconvex). We define some new spaces in these domains as subspaces of $H(D)$ so that the following quazinorm is finite.

$$\sum_{k_1 \geq 0} \cdots \sum_{k_n \geq 0} \left(\int_{D(a_{\frac{k}{n}}, r)} \cdots \left(\int_{D(a_{k_1}, r)} |f(z_1, .., z_n)|^{p_1} \times (1 - |z_1|)^{\alpha_1} dv(z_1) \right)^{\frac{p_2}{p_1}} \times$$

$$\times (1 - |z_n|)^{\alpha_n} dv(z_n) \right)^{\frac{1}{p_n}}$$

,

where $p_i \in (0, \infty)$; $\alpha_j \in (-1, \infty)$, $j = 1, .., n$
or quazinorms like

$$\sum_{k \geq 0} \left(\int_{D(a_k, r)} \cdots \left(\int_{D(a_k, r)} |f(z_1, .., z_n)|^{p_1} \times (1 - |z_1|)^{\alpha_1} dv(z_1) \right)^{\frac{p_2}{p_1}} \times$$

$$\times (1 - |z_n|)^{\alpha_n} dv(z_n) \right)^{\frac{1}{p_n}}$$

,

where $p_i \in (0, \infty)$; $\alpha_j \in (-1, \infty)$, $j = 1, .., n$.

Similar questions concerning sizes of traces of these analytic function spaces can also be posed, we provide some answers below. (note defining these type spaces in tube and pseudoconvex domains we formally simply replace standard $(1 - |z|)^\alpha$ weights in the unit ball as usual by appropriate weights in bounded pseudoconevex and tubular domains over symmetric cones.(see above for this))

Let first our domain be the unit ball.

Let X be one of these spaces. We present some calculations and then explain how to pass them to tubular domains over symmetric cones or bounded strongly pseudoconvex domains with smooth boundary in C^n. It is easy to note from the proof of the first theorem that we arrive at estimates of the following integrals for example for some Hardy Herz-type spaces we arrive at the following type estimates. (see the proof of theorem 1)

$$\sum_{k=1}^{\infty} \left(\sup_{z \in D(a_k, r)} \right) |f(z)|^{p_2} \times$$

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$$\times \left\| \left(\int_{D(a_k,r)} \frac{(1 - |w|)^{\alpha} dv(w)}{\prod_{j=1}^{2} |1 - (z_j, \zeta_j)|^{\gamma}} \right) \right\|_X$$

And the only problem now to estimate the last expression for various type of analytic spaces (Hardy, Herz, and so on) for large enough α, $\alpha > \alpha_0$.

In the unit ball

$$\left(\int_{B(z,r)} \frac{(1 - |w|)^{\alpha} dv(w)}{|1 - \bar{w} w_1|^{\beta}} \right) \leq \frac{c}{|1 - \bar{z} w_1|^{\beta - (\alpha + n + 1)}}; \qquad (K_1)$$

$$\left(\int_{D(a_k,r)} \frac{(1 - |w|)^{\alpha} dv(w)}{|1 - \bar{w} w_1|^{\beta}} \right) \leq \frac{c_1}{|1 - a_k w_1|^{\beta - (\alpha + n + 1)}} \qquad (K_2)$$

$\alpha > -1$; $\beta > (\alpha + n + 1)$,; $a_k, z, w_1 \in B$

For example for Herz type spaces with quazinorms

$$\|f\|_{G^{\vec{p}}}^{p_m} = \int_B \left(\int_{B(z,r)} \cdots \left(\int_{B(z,r)} |f(w_1, .., w_n)|^{p_1} dv(w_1) \right)^{\frac{p_2}{p_1}} \cdots dv(w_m) \right)^{\frac{p_m}{p_m - 1}}$$

$0 < p_i < \infty$, $i = 1, .., m$.

we have the following theorem based on these ideas.

Theorem 5. *Let $f \in H(B)$, $0 < p_j \leq 1$, $i = 1, 2.., m$, $p_{j+1} \leq p_j$, $j = 1, ..., m$. If*

$$\int_B |f(z)|^{p_m} \times (1 - |z|)^{((n+1)p_m)\left(\frac{1}{p_1} + \cdots + \frac{1}{p_{m-1}}\right)} dv(z) < \infty$$

Then for $\alpha > \alpha_0$, where α_0 is large enough we have that

$$(T_{\alpha} f) \in G^{p_1 \ldots p_m} (B \times \ldots B).$$

So we have the following inclusion

$$\left(Trace \left(G^{\vec{p}} \right) \right) \supset (A_{\tau}^{p_m});$$

where $\tau = ((n + 1) p_m) \times \left(\sum_{i=1}^{m-1} \left(\frac{1}{p_i} \right) \right)$.

The same results is valid for weighted $\left(G_{\vec{\alpha}}^{\vec{p}} \right) (B \times \cdots \times B)$ Herz type spaces.

The proof is based on remarks above and Forelli-Rudin estimate in the ball and is similar to the proof of theorem 1.

We have to use estimates K_1, K_2. We omit details leaving them to reader.

For $\alpha > n$, $0 < p_i < \infty$, $i = 1,..,m$, define spaces $\left(N_\alpha^{\vec{p}} \right)(B \times \ldots B)$ with quazinorms

$$\|f\|_{N_\alpha^{\vec{p}}} = \sum_{k \geq 0}(1 - |a_k|)^\alpha \times \int_{D(a_k,r)} \left(\int_{D(a_k,r)} \cdots \left(\int_{D(a_k,r)} |f(w_1,..,w_n)|^{p_1} dv(w_1) \right)^{\frac{p_2}{p_1}} \right.$$

$$\left. \ldots dv(w_m) \right)^{\frac{p_m}{p_{m-1}}}$$

Then using similar arguments as on proof of theorem 1 and estimates

$$\left(\sum_{k \geq 0} \frac{((1 - |a_k|)^\alpha)}{|1 - \bar{w}a_k|^\tau} \right) \leq \sum_{k \geq 0} \int_{D(a_k,r)} \frac{(1 - |v|)^{\alpha - (n+1)} d\tilde{v}(v)}{|1 - w\bar{v}|^\tau} \leq$$

$$\leq \frac{c}{(1 - |w|)^r}$$

$w \in B$, $r = \tau - \alpha$, $\alpha > n$, $\tau > \alpha$.

we have the following theorem for Herz spaces.

Theorem 6. *Let $f \in H(B)$, $0 < p_{j+1} \leq p_j \leq 1$; $p_j \leq 1$, $j = 1,..,m$. If*

$$\int_B |f(z)|^{p_m} \times (1 - |z|)^s dv(z) < \infty,$$

$$s = (n + 1)(\sum_{j=1}^{m-1} \frac{1}{p_j}) - (n + 1) + \beta;$$

then for $\alpha > \alpha_0$, (α_0) is large enough we have

$$(T_\alpha) f \in \left(N_\beta^{\vec{p}} \right)(B^m).$$

and

$$\left(Trace N_\beta^{\vec{p}} \right)(B^m) \supset (A_s^{p_m})(B);$$

The proof of this theorem use estimates (K_1), $(K_2,)$ also. We omit some simple details here leaving them to readers.

We may also define other $\tilde{N}_\alpha^{\vec{p}}$; $\tilde{G}_\alpha^{\vec{p}}$ analytic spaces similarly via n-sums and by integralion on $B \times \cdots B$, using the following expression.

$$\left(\int_{B(z_m,r)} \left(\int_{B(z_1,r)} \cdots \right)^{\frac{p_2}{p_1}} \right)^{\frac{1}{p_m}}; \left(\int_{D(a_{k_m},r)} \left(\int_{D(a_{k_1},r)} \cdots \right)^{\frac{p_2}{p_1}} \right)^{\frac{1}{p_m}}$$

and provide sizes of their traces also.

Theorems 5 and 6 under certain additional condition on kernel can be extended to tubular and to pseudoconvex domains with smooth boundary also. We mean conditions like (K_1) and (K_2) for Bergman kernel in tubular and in bounded strongly pseudoconvex domains inC^n. We leave details to interested readers.

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