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ON SOME NEW ESTIMATES RELATED TO POISSON-TYPE INTEGRAL IN PRODUCT DOMAINS

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Abstract. In this note we define new Poisson-type integral in the unit ball and unit polyball and we extend a known classical maximal theorem related with it. Related results for such type integrals will be given.

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1. Introduction

In this short note we extend the notion of classical Poisson integral in a simple natural way in the unit ball and extend some well known estimates related with it, in particular an extension of a known maximal theorem will be provided in the unit ball. Such type results probably can be proved in context of more general pseudoconvex domains with smooth boundary.

Poisson type integrals on product domains considered recently also in papers [1] and [3].

In this short note, also, in particular, we provide an extension of a known result concerning estimate from below of Poisson kernel (see [7]).

Let $B_n = \{z \in \mathbb{C}^n : |z| < 1\}$ be the unit ball and let $S_n = \{z \in \mathbb{C}^n : |z| = 1\}$ be the unit sphere.

Let $\xi \in S_n$, $r > 0$, $Q_r(\xi) = \{z \in B_n,\ d(z, \xi) < r\}$, $d(z, w) = |1- < z, w > |^{\frac{1}{2}}$, $z, w \in \overline{B}_n$.

We recall $Q_r(\xi)$ is Carleson tube at ξ. Let $M_\alpha(f)(\xi) = \sup_{z \in D_\alpha(\xi)} |f(z)|$, where

$$D_\alpha(\xi) = \{z \in B_n : |1- < z, \xi > | < \frac{\alpha}{2}(1 - |z|^2),\ \alpha > 1\}$$

where $\xi \in S_n$.

Let

$$P[\mu](z) = \int_{S_n} \frac{(1-|z|)^n d\mu(\xi)}{|1-\overline{\xi}z|^{2n}}, \; z \in B_n,$$

$$(M\mu)(\xi) = \sup_{\delta>0} \frac{1}{\sigma(Q(\xi,\delta))} \int_{Q(\xi,\delta)} d\mu(\tilde{\xi}),$$

where $Q(\xi,\delta) = \{\eta \in S_n : d(\xi,\eta) < \delta\}$, $\delta > 0$, $\xi \in S_n$, $z \in B_n$.

We denote, as usual, by c_1, c_2, etc various positive constants in estimates in this paper which depend on various parameters.

For a function f we denote by $P[f]$, Mf similar new expressions using standard changes (see [2], [7]).

Theorem A. (a maximal theorem in the ball, see [2], [7]) Let $\alpha > 1$, then

$$I_\mu(\xi) = (M_\alpha P[\mu])(\xi) \le \bar{c} M\mu(\xi), \; \xi \in S_n,$$

and $\|I_f(\xi)\|_{L^p(S_n)} \le \tilde{c}\|f\|_{L^p}$, $p > 1$ for a μ positive complex finite Borel measure on B_n.

This maximal theorem result has many applications (see, for example, [2], [7]).

2. A maximal theorem in the unit ball related with the Poisson type integral

Let further

$$P_{\overrightarrow{\alpha}}(\overrightarrow{z}, \xi) = \frac{(1-|z|)^n}{\prod_{j=1}^m |1-z_j\xi|^{\alpha_j}},$$

where $\xi \in S_n$, $z_j \in B_n$, $|z_j| = |z|$, $\alpha_j > 0$, $j = 1, \ldots, m$.

Theorem 1. *Let*

$$P_{\vec{\alpha}}[\mu](\vec{z}) = \int_{S_n} \frac{(1-|z|)^n d\mu(\xi)}{\prod_{j=1}^m |1-z_j\xi|^{\alpha_j}}, \; z_j \in B, \; j = 1, \ldots, m, \; \sum_{j=1}^m \alpha_j = 2n.$$

Then we have that

$$K_\xi(\mu) = \sup_{z_j \in D_\tau(\xi), |z_j| = r} P_{\vec{\alpha}}[\mu](\vec{z}) \le c_1 M\mu(\xi), \; r \in (0,1),$$

where μ is a positive Borel measure, and

$$\|K_\xi(f)\|_{L^p(S_n, d\sigma(\xi))} \le c_2 \|f\|_{L^p(S_n, d\sigma(\xi))}, \; p > 1.$$

Proof. Let $\vec{z} \in D_\alpha(\xi)$. Let $z = (|z_1|\varphi_1, \dots, |z_m|\varphi_m)$, $|z_j| = |z_i|$, $1 \le i \le j \le m$, $r = |z_j|$, $t = 8\alpha(1-r)$. Let $V_0 = \{\eta \in S_n : |1- <\eta, \xi>| < t\}$, $1 \le k \le N$, $2^N t > 2$, $V_k = \{\eta \in S_n : 2^{k-1}t \le |1- <\eta,\xi>| < 2^k t\}$.

Note that we have obviously

$$\int_{S_n} P_{\vec{\alpha}}(\vec{z},\eta)d\mu(\eta) = \int_{V_0} P_{\vec{\alpha}}(\vec{z},\eta)d\mu(\eta) + \sum_{k=1}^{N}\int_{V_k} P_{\vec{\alpha}}(\vec{z},\eta)d\mu(\eta).$$

It is enough to show that

$$\int_{S_n} P_{\vec{\alpha}}(\vec{z},\eta)d\mu(\eta) \le c(\alpha,n)(M\mu(\xi)),\ \xi \in S_n,\ z_j \in D_\alpha(\xi),\ j=1,\dots,m,$$

for some constant $c(\alpha,n)$. We have by definition that $V_k \subset Q(\xi,\sqrt{2^k t})$, $1 \le k \le N$ and hence

$$\mu(V_k) \le \mu(Q(\xi,\sqrt{2^k t})) \le ((M\mu)(\xi))\sigma(Q(\xi,\sqrt{2^k t})) \le c(2^k t)^n M\mu(\xi),\ 0 \le k \le N.$$

$$P_{\vec{\alpha}}(\vec{z},\eta) \le 2^n(1-r)^{-n}.$$

We have hence, since $\mu(V_0) \le ct^n M\mu(\xi)$

$$\int_{V_0} P_{\vec{\alpha}}(\vec{z},\eta)d\mu(\eta) \le c\left(\frac{t}{1-r}\right)^n (M\mu(\xi)) \le C_{\alpha,n}(M\mu)(\xi).$$

Then if $\eta \in S_n$ we have

$$|1- <\tilde{z},\xi>| \le \tilde{c}_\alpha(1-|\tilde{z}|^2) \le \tilde{c}'_\alpha|1- <\tilde{z},\eta>|$$
$$d(\tilde{z},\xi) < \sqrt{\alpha'}d(\tilde{z},\eta).$$

Hence we have from definition of d

$$d(\xi,\eta) \le d(\xi,\tilde{z}) + d(\tilde{z},\eta) \le \tilde{c}''_\alpha d(\tilde{z},\eta),\ \tilde{z} = z_j,\ j=1,\dots,m.$$

Hence

$$\int_{V_k}(P_{\vec{\alpha}}(\overrightarrow{z},\eta))d(\mu(\eta)) \le (\frac{c(\alpha,n)}{2^{kn}})(M\mu)(\xi),$$

where $1 \le k \le N$, the rest is clear.

The last estimate follows from the fact that

$|1- <\xi,\eta>| \le \tilde{c}(\alpha,n)|1- <z_j,\eta>|$ for every z_j, $j=1,\dots,m$, $1 \le k \le N$, $\eta \in V_k$. And hence we have that

$$P_{\vec{\alpha}}(\vec{z},\eta) \le \frac{\tilde{c}'(\alpha,n)t^n}{|1- <\xi,\eta>|^{2n}} \le \frac{\tilde{c}''(\alpha,n)}{4^{kn}t^n}.$$

The second assertion of theorem follows from the first part of our theorem and the well known maximal theorem, (see [2], [7]).

Theorem is proved.

Let further, as usual, $P(z,\xi) = \dfrac{(1-|z|^2)^n}{|1-<z,\xi>|^{2n}}$, $z \in B_n$, $\xi \in S_n$ be the Poisson kernel on B_n unit ball (see [2], [6], [7]). Let $P[f](z) = \int_{S_n} P(z,\xi) f(\xi) d\sigma(\xi)$, $f \in L^1(S_n, d\sigma)$, where σ is a normalized measure on S_n.

It is natural to study the following extension of $P[f]$, $\tilde{P}[f](\vec{z})$, $z_j \in B_n$, $j = 1,\ldots,m$, where

$$\tilde{P}[f](\vec{z}) = \int_{S_n} \tilde{P}(\vec{z},\xi) f(\xi) d\sigma(\xi),$$

where

$$\tilde{P}(\vec{z},\xi) = \tilde{P}_{\vec{\alpha},\vec{\beta}}(\vec{z},\xi) = \frac{\prod_{j=1}^m (1-|z_j|^2)^{\alpha_j}}{\prod_{j=1}^m |1-<z_j,\xi>|^{\beta_j}},$$

$\alpha_j, \beta_j > 0,\ j = 1,\ldots,m,\ \sum_{j=1}^m \alpha_j = n,\ \sum_{j=1}^m \beta_j = 2n$ and even more generally on product domains.

$$G(z_1^1,\ldots,z_m^1,\ldots,z_1^k,\ldots,z_m^k,\xi_1,\ldots,\xi_k) = \prod_{i=1}^k \widetilde{P}^i(\overrightarrow{z},\xi),\ \widetilde{P}^i = \widetilde{P}_{\overrightarrow{\alpha_j}^i,\overrightarrow{\beta_j}^i},$$

$z_i^j, \xi_j \in B_n,\ i,j = 1,\ldots,k$, and try to expand classical known assertions.

Very similar procedure of extension of Bergman projection was provided and used intensively in connection with trace problem (see [4], [5]).

We found the following results for Poisson $\tilde{P}$ kernel as modification of known proofs for P kernel (see [7] for $m=1$ case).

Theorem 2. *Let μ, μ_j be positive Borel measures on B_n, and let $p_i \in (0,\infty)$, $i = 1,\ldots,m$. Then we have that*

1)

$$\sup_{z_j \in B_n, j=1\ldots m} \int_{B_n} \tilde{P}(w,\vec{z}) d\mu(w) \geq c\, \frac{\mu(Q_r(\xi))}{r^{2n}},\ \xi \in S_n,\ r \in (0,1),$$

and even generally we have for $r_j \in (0,1)$, $j = 1,\ldots,m$.

2)

$$\sup_{w_j \in B_n, j=1\ldots m} \left(\int_{B_n} \cdots \left(\int_{B_n} |G(\vec{z},\vec{w})|^{p_1} d\mu_1(z_1) \right)^{\frac{p_2}{p_1}} d\mu_2(z_2) \cdots d\mu_m(z_m) \right)^{\frac{1}{p_m}} \geq$$

$$\geq \tilde{C}_1 \frac{\mu_1(Q_{r_1}(\xi_1)) \dots \mu_m(Q_{r_m}(\xi_m))}{r_1^{\alpha_0^1} \dots r_m^{\alpha_0^m}}, \ 0 < p_j < \infty, \ j = 1 \dots m,$$

for some α_0^j, $\alpha_0^j > 0$, $j = 1, \dots, m$, *where* μ, μ_j, $j = 1, \dots, m$ *are positive Borel measures on* B_n *for some constants* $\tilde{c}_1$, c *where* $\xi_j \in S_n$, $j = 1, \dots, m$.

Proofs are heavily based on arguments of proofs of $m = 1$ case and they will be given elsewhere.

Remark 1. *These results of Theorem 2 extend some known assertions from [2] and [7].*

References

[1] I. Chyzhykov, O. Zolota, *Growth of the Poisson - Stieltjes integral in the polydisk*, Zh. Mat. Fiz. Anal. Geom., 7(2), (2011), 141-157.

[2] W. Rudin, *Function Theory in the Unit Ball of* C^n, Springer, New York, 2008.

[3] R. Shamoyan, *A note on Poisson type integrals in pseudoconvex and convex domains of finite type and some related results*, Acta Universitatis Apulensis, 40, (2017), 19-27.

[4] R. Shamoyan, O. Mihić, *In search of traces of some holomorphic functions in polyballs*, Revista Notas de Matemtica 4, (2008), 1-23.

[5] R. Shamoyan, O. Mihić *On traces of holomorphic functions on the unit polyball*, Appl. Anal. Discrete Math. 3, (2009), 198-211.

[6] S. V. Shvedenko, *Hardy classes and related spaces of analytic functions in the unit disc, polydisc and ball*, in Russian, Itogi nauki i tekhniki, VINITI, 23, (1985), 3-124.

[7] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Graduate Texts in Mathematics 226, Springer, New York, 2005.

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