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ON A STUDY OF BINOMIAL FORM TO THE NEW (S,T)-JACOBSTHAL SEQUENCE

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Abstract. Many (s,t)-type of sequences has been introduced earlier such as (s,t)-Fibonacci sequence, (s,t)-Lucas sequence, (s,t)-Jacobsthal sequence, (s,t)-Jacobsthal-Lucas sequence etc . However in this article, we give a new type of (s,t)-Jacobsthal sequence $\langle U_n(s,t)\rangle_{n\in\mathbb{N}}$

$$U_n = iU_{n-1} + 2U_{n-2},\ n \geq 2 \text{ and } U_0 = s - 2t,\ U_1 = i(s-t)$$

where $i = \sqrt{-1}$ and $s,t \in \mathbb{Z}^+$. Next we define a binomial form $\langle X_n(s,t)\rangle_{n\in\mathbb{N}}$ to the new (s,t)-Jacobsthal sequence and then some fundamental properties for the binomial form $\langle X_n(s,t)\rangle_{n\in\mathbb{N}}$ are obtained. Furthermore a new kind of matrix sequence $\langle Z_n(s,t)\rangle_{n\in\mathbb{N}}$ will be presented for the binomial form $\langle X_n(s,t)\rangle_{n\in\mathbb{N}}$.

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1. Introduction

In the mathematical world, the Fibonacci sequence (see [1]) have great importance and plays an important role almost in the every arena of science. Some sequences such as Jacobsthal and Jacobsthal-Lucas sequences etc have similar structure to the Fibonacci sequence and in another words, we can say that these sequences are the extensions or generalizations of Fibonacci sequence.

In 1961 Horadam [2] introduced the first ever generalization of Fibonacci sequence and denoted it by $\langle H_n\rangle$.

$$H_n = H_{n-1} + H_{n-2}, \quad n \geq 3 \text{ and } H_0 = p,\ H_1 = p + q \tag{1.1}$$

where p and q are arbitrary integers. Then in 1965 Horadam introduced another generalized Fibonacci sequence $\langle W_n \rangle$ called as Horadam sequence (see [3]). The sequence $\langle W_n \rangle$ is generated by the following recurrence relation

$$W_n = pW_{n-1} - qW_{n-2}, \quad n \geq 2 \text{ and } W_0 = a, \; W_1 = b \tag{1.2}$$

where a, b, p and q are real constants. After that a lot of work has been done to study the generalizations of Fibonacci sequences by several methods.

In formal terms, a complex sequence is a function whose domain is the positive integers and co-domain is a set of the complex numbers. Gaussian numbers were first investigated in 1832 by the German mathematician Karl Friedrich Gauss. A Gaussian number is a complex number $Z = a + ib$, where a and b are any integers and $i = \sqrt{-1}$. In 1963 Horadam [4] considered the generalized Fibonacci sequence (1.1) and then delineated generalized complex Fibonacci sequence $\langle D_n \rangle$.

$$\begin{aligned} D_n &= H_n + iH_{n+1}, \quad n \geq 1 \\ &= \big(p_H - q_H + iq_H\big)F_n + \big(q_H + ip_H\big)F_{n+1}, \quad n \geq 1 \\ &= D_{n-1} + D_{n-2}, \quad n \geq 3 \end{aligned} \tag{1.3}$$

with $D_1 = \big(1+i\big)p_H + q_H$ and $D_2 = p_H + q_H + i\big(2p_H + q_H\big)$.
As a special case of equation (1.3) the author defined a complex Fibonacci sequence $\langle C_n \rangle$ such that

$$\begin{aligned} C_n &= F_n + iF_{n+1}, \quad n \geq 1 \\ &= C_{n-1} + C_{n-2}, \quad n \geq 3 \end{aligned} \tag{1.4}$$

with $C_1 = 1 + i$ and $C_2 = 1 + 2i$.
Jordan [5] in 1965 presented a Gaussian Fibonacci sequence $\langle GF_n \rangle$ and established some results between Gaussian Fibonacci sequence and classical Fibonacci sequence.

$$GF_n = GF_{n-1} + GF_{n-2}, \quad n \geq 2 \text{ and } GF_0 = i, \; GF_1 = 1 \tag{1.5}$$

Later on Berzsenyi [6], Harman [7] and Pethe [8] used different approaches of extensions of Fibonacci numbers on the complex plane.

Now we discuss some other literature where the authors studied the generalizations of Jacobsthal numbers and Jacobsthal-Lucas numbers. Asci ans Gurel [9] introduced and studied Gaussian Jacobsthal $\langle GJ_n \rangle$ and Gaussian Jacobsthal-Lucas $\langle Gj_n \rangle$ numbers.

$$GJ_{n+1} = GJ_n + 2GJ_{n-1}, \quad n \geq 1 \;\; \textit{and} \;\; GJ_0 = \frac{i}{2}, \; GJ_1 = 1 \tag{1.6}$$

$$Gj_{n+1} = Gj_n + 2Gj_{n-1},\ \ n \geq 1 \ \ and \ \ Gj_0 = 2 - \frac{i}{2},\ Gj_1 = 1 + 2i. \tag{1.7}$$

Asci ans Gurel [10] defined the polynomials of Gaussian Jacobsthal and Gaussian Jacobsthal-Lucas numbers. Catarino et al. [11] studied the new generalizations of Jacobsthal and Jacobsthal-Lucas sequences. Uygun [12] also defined new generalizations for Jacobsthal and Jacobsthal-Lucas sequences called $p(x)$-Jacobsthal polynomial sequences $\langle J_{p,n}(x)\rangle$ and $p(x)$-Jacobsthal-Lucas polynomial sequences $\langle C_{p,n}(x)\rangle$ by

$$J_{p,n}(x) = p(x)\,J_{p,n-1}(x) + 2J_{p,n-2}(x),\ \ n \geq 2 \ \ and \ \ J_{p,0}(x) = 0,\ J_{p,1}(x) = 1 \tag{1.8}$$

$$C_{p,n}(x) = p(x)\,C_{p,n-1}(x) + 2C_{p,n-2}(x),\ \ n \geq 2 \ \ and \ \ C_{p,0}(x) = 2,\ C_{p,1}(x) = p(x) \tag{1.9}$$

where $p(x)$ is a polynomial with real coefficients.

In [13] a sequence $\langle b_n\rangle_{n\in\mathbb{Z}_0}$ is the binomial transform to the sequence $\langle a_n\rangle_{n\in\mathbb{Z}_0}$ if

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \tag{1.10}$$

Chen [13] obtained various identities related to binomial transform. In [14] and [15] the authors discussed the binomial transforms to the Dold and Fibonacci-Like sequences respectively.

From past several years many authors investigated the generalizations of Fibonacci, Lucas, Jacobsthal sequences etc by adding parameters s and t to the recurrence relations of these sequences then named the resulted sequences as (s,t)-type sequences. In addition to this they also defined the matrix sequences for (s,t)-type sequences and called the matrix sequences as (s,t)-type matrix sequences. A matrix sequence is the sequence in which the terms of the sequences are in the form of matrices and the elements of these matrices are the terms of general sequences. In 2008 Civciv and and Turkmen [16] presented (s,t)-Fibonacci sequence $\langle F_n(s,t)\rangle$ and (s,t)-Fibonacci matrix sequence $\langle \mathcal{F}_n(s,t)\rangle$.

$$F_{n+1}(s,t) = sF_n(s,t) + tF_{n-1}(s,t),\ \ n \geq 1 \ \text{ and } \ F_0(s,t) = 0,\ F_1(s,t) = 1 \tag{1.11}$$

$$\mathcal{F}_{n+1}(s,t) = s\mathcal{F}_n(s,t) + t\mathcal{F}_{n-1}(s,t),\quad n \geq 1 \tag{1.12}$$

with $\mathcal{F}_0(s,t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathcal{F}_1(s,t) = \begin{bmatrix} s & 1 \\ t & 0 \end{bmatrix}$

whereas $\mathcal{F}_n(s,t) = \begin{bmatrix} F_{n+1}(s,t) & F_n(s,t) \\ tF_n(s,t) & tF_{n-1}(s,t) \end{bmatrix}$ for $s > 0,\ t \neq 0,\ s^2 + 4t > 0$.

In 2011 Yazlik et al. [17] introduced the generalizations of (s,t)-Fibonacci sequence and (s,t)-Fibonacci matrix sequence by defining the sequences $\langle G_n(s,t)\rangle_{n\in\mathbb{N}}$ called the generalized (s,t)-Fibonacci sequence and $\langle \mathfrak{R}_n(s,t)\rangle_{n\in\mathbb{N}}$ called the generalized (s,t)-Fibonacci matrix sequence. The sequences $\langle G_n(s,t)\rangle_{n\in\mathbb{N}}$ and $\langle \mathfrak{R}_n(s,t)\rangle_{n\in\mathbb{N}}$ are recurrently defined by

$$G_{n+1}(s,t) = sG_n(s,t) + tG_{n-1}(s,t)\,,\ \ n \geq 1 \ \text{ and } \ G_0(s,t) = a,\ G_1(s,t) = bs \tag{1.13}$$

and

$$\mathfrak{R}_{n+1}(s,t) = s\mathfrak{R}_n(s,t) + t\mathfrak{R}_{n-1}(s,t)\,,\ \ \text{for } n \geq 1 \tag{1.14}$$

with $\mathfrak{R}_0(s,t) = \begin{bmatrix} bs & a \\ at & (b-a)s \end{bmatrix}$, $\mathfrak{R}_1(s,t) = \begin{bmatrix} bs^2+at & bs \\ bst & at \end{bmatrix}$ whereas $\mathfrak{R}_n(s,t) = \begin{bmatrix} G_{n+1}(s,t) & G_n(s,t) \\ tG_n(s,t) & tG_{n-1}(s,t) \end{bmatrix}$ for $s>0,\ t\neq 0,\ s^2+4t>0$ and $a,b\in\mathbb{R}$.
Again in 2015 Ipek et al. [18] delineated the another generalized (s,t)-Fibonacci sequence $\langle G_n(s,t)\rangle_{n\in\mathbb{N}}$ and its matrix sequence $\langle \mathfrak{R}_n(s,t)\rangle_{n\in\mathbb{N}}$ by

$$G_{n+1}(s,t) = sG_n(s,t) + tG_{n-1}(s,t)\,,\ \ n \geq 1 \ \text{ and } \ G_0(s,t) = a_0,\ G_1(s,t) = a_1 \tag{1.15}$$

and

$$\mathfrak{R}_{n+1}(s,t) = s\mathfrak{R}_n(s,t) + t\mathfrak{R}_{n-1}(s,t)\,,\ \ \ n \geq 1 \tag{1.16}$$

with $\mathfrak{R}_0(s,t) = \begin{bmatrix} a_1 & a_0 \\ ta_0 & a_1 - sa_0 \end{bmatrix}$, $\mathfrak{R}_1(s,t) = \begin{bmatrix} sa_1+ta_0 & a_1 \\ ta_1 & ta_0 \end{bmatrix}$ whereas $\mathfrak{R}_n(s,t) = \begin{bmatrix} Gn+1(s,t) & G_n(s,t) \\ tG_n(s,t) & tG_{n-1}(s,t) \end{bmatrix}$ for $s>0,\ t\neq 0,\ s^2+4t>0$ and $a_0,a_1\in\mathbb{R}$.
Yazlik et al. [19] applied binomial transforms to the (s,t)-Fibonacci matrix sequence (1.12) and generalized (s,t)-Fibonacci matrix sequence (1.14). Uygun [20] presented (s,t)-Jacobsthal sequence $\langle \hat{j}_n(s,t)\rangle$ and (s,t)-Jacobsthal-Lucas sequence $\langle \hat{c}_n(s,t)\rangle$ such that

$$\hat{j}_n(s,t) = s\hat{j}_{n-1}(s,t) + 2t\hat{j}_{n-2}(s,t)\,,\ \ n \geq 2 \ \text{ and } \ \hat{j}_0(s,t) = 0,\ \hat{j}_1(s,t) = 1 \tag{1.17}$$

$$\hat{c}_n(s,t) = s\hat{c}_{n-1}(s,t) + 2t\hat{c}_{n-2}(s,t)\,,\ \ n \geq 2 \ \text{ and } \ \hat{c}_0(s,t) = 2,\ \hat{c}_1(s,t) = s \tag{1.18}$$

where $s > 0$, $t \neq 0$ and $s^2 + 8t > 0$.
Then in [21] the authors defined (s,t)-Jacobsthal matrix sequence $\langle J_n(s,t)\rangle_{n\in\mathbb{N}}$ and (s,t)-Jacobsthal-Lucas matrix sequence $\langle C_n(s,t)\rangle_{n\in\mathbb{N}}$ as

$$J_{n+1}(s,t) = sJ_n(s,t) + 2tJ_{n-1}(s,t), \quad n \geq 1 \tag{1.19}$$

with $J_0(s,t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $J_1(s,t) = \begin{bmatrix} s & 2 \\ t & 0 \end{bmatrix}$ whereas $J_n(s,t) = \begin{bmatrix} \hat{j}_{n+1}(s,t) & \hat{j}_n(s,t) \\ t\hat{j}_n(s,t) & t\hat{j}_{n-1}(s,t) \end{bmatrix}$
and

$$C_{n+1}(s,t) = sC_n(s,t) + 2tC_{n-1}(s,t), \quad n \geq 1 \tag{1.20}$$

with $C_0(s,t) = \begin{bmatrix} s & 4 \\ 2t & -s \end{bmatrix}$, $C_1(s,t) = \begin{bmatrix} s^2+4t & 2s \\ st & 4t \end{bmatrix}$
whereas and $C_n(s,t) = \begin{bmatrix} \hat{c}(s,t) & \hat{c}_n(s,t) \\ t\hat{c}_n(s,t) & t\hat{c}_{n-1}(s,t) \end{bmatrix}$ for $t \neq 0$, $s^2 + 8t \neq 0$.
Uygun [22] gave some summation identities for the (s,t)-Jacobsthal and (s,t)-Jacobsthal-Lucas matrix sequences.
In 2016 Uygun and Uslu [23] studied the generalizations of (s,t)-Jacobsthal and (s,t)-Jacobsthla-Lucas sequences as well as generalizations of their matrix ones. Thus in [23] the authors presented (s,t)-generalized Jacobsthal sequence $\langle G_n(s,t)\rangle_{n\in\mathbb{N}}$ and (s,t)-generalized Jacobsthal matrix sequence $\langle \mathfrak{R}_n(s,t)\rangle_{n\in\mathbb{N}}$ by the following equations

$$G_{n+1}(s,t) = sG_n(s,t) + 2tG_{n-1}(s,t), \; n \geq 1 \text{ and } G_0(s,t) = a, \; G_1(s,t) = bs \tag{1.21}$$

and

$$\mathfrak{R}_{n+1}(s,t) = s\mathfrak{R}_n(s,t) + 2t\mathfrak{R}_{n-1}(s,t), \; n \geq 1 \tag{1.22}$$

with $\mathfrak{R}_0(s,t) = \begin{bmatrix} bs & 2a \\ at & (b-a)s \end{bmatrix}$, $\mathfrak{R}_1(s,t) = \begin{bmatrix} bs^2+2at & 2bs \\ bst & 2at \end{bmatrix}$ whereas $\mathfrak{R}_n(s,t) = \begin{bmatrix} G_{n+1}(s,t) & G_n(s,t) \\ tG_n(s,t) & tG_{n-1}(s,t) \end{bmatrix}$ for $s > 0$, $t \neq 0$, $s^2 + 4t > 0$ and $a, b \in \mathbb{R}$.
In the rest of the paper, we use symbols $\langle U_n\rangle$, $\langle X_n\rangle$ and $\langle Z_n\rangle$ instead of $\langle U_n(s,t)\rangle$, $\langle X_n(s,t)\rangle$ and $\langle Z_n(s,t)\rangle$.

2. (s,t)-Jacobsthal sequence

Definition 1. *For $s,t \in \mathbb{Z}^+$ and $i\left(=\sqrt{-1}\right)$, the (s,t)-Jacobsthal sequence $\langle U_n\rangle_{n\in\mathbb{N}}$ is recurrently defined by*

$$U_n = iU_{n-1} + 2U_{n-2}, \quad n \geq 2 \tag{2.1}$$

with seeds $U_0 = s-2t$ and $U_1 = i\,(s-t)$

The recurrence relation (2.1) have the characteristic equation $u^2 - iu - 2 = 0$ and suppose that θ and ϑ are the roots of this characteristic equation.

$$\theta = \frac{\sqrt{7}+i}{2} \quad \text{and} \quad \vartheta = \frac{-\left(\sqrt{7}-i\right)}{2} \tag{2.2}$$

3. Binomial form to the (s,t)-Jacobsthal sequence

In this section first and foremost we give a binomial form $\langle X_n\rangle$ of (s,t)- Jacobsthal sequence $\langle U_n\rangle$ and after that a recurrence relation and Binet's formula for $\langle X_n\rangle$ are presented.

Definition 2. *For $n \in \mathbb{Z}_0$, the binomial form to the (s,t)-Jacobsthal sequence $\langle U_n\rangle$ is defined by*

$$X_n = \sum_{l=0}^{n} \binom{n}{l} U_l \tag{3.1}$$

Lemma 1. *For $n \in \mathbb{Z}_0$, the following property holds for $\langle X_n\rangle$*

$$X_{n+1} = \sum_{l=0}^{n} \binom{n}{l} \left(U_l + U_{l+1}\right) \tag{3.2}$$

Proof. Its proof can be easily obtained by using the relation $\binom{n+1}{l} = \binom{n}{l} + \binom{n}{l-1}$

Theorem 2. **(Recurrence relation for $\langle X_n\rangle$)** *If $s,t \in \mathbb{Z}^+$ and $i\left(=\sqrt{-1}\right)$, the recurrence relation of the binomial form $\langle X_n\rangle$ is given by*

$$X_{n+1} = (2+i)\,X_n + (1-i)\,X_{n-1}, \quad n \geq 1 \tag{3.3}$$

with $X_0 = s-2t$ and $X_1 = s\,(1+i) - t\,(2+i)$

Proof. Since

$$\begin{aligned}
X_{n+1} &= \sum_{l=0}^{n}\binom{n}{l}\big(U_l+U_{l+1}\big)\\
&= U_0+U_1+\sum_{l=1}^{n}\binom{n}{l}\big(U_l+U_{l+1}\big)\\
&= U_0+U_1+\sum_{l=1}^{n}\binom{n}{l}\big(U_l+iU_l+2U_{l-1}\big) \qquad \text{By the Eqn.(2.1)}\\
&= U_0+U_1+\sum_{l=1}^{n}\binom{n}{l}\Big[(1+i)\,U_l+2U_{l-1}\Big]\\
&= (1+i)\sum_{l=1}^{n}\binom{n}{l}U_l+2\sum_{l=1}^{n}\binom{n}{l}U_{l-1}+U_0+U_1\\
&= (1+i)\sum_{l=1}^{n}\binom{n}{l}U_l+(1+i)\,U_0+2\sum_{l=1}^{n}\binom{n}{l}U_{l-1}-(1+i)\,U_0+U_0\\
&\quad +U_1\\
&= (1+i)\sum_{l=0}^{n}\binom{n}{l}U_l+2\sum_{l=1}^{n}\binom{n}{l}U_{l-1}-iU_0+U_1\\
&= (1+i)\,X_n+2\sum_{l=1}^{n}\binom{n}{l}U_{l-1}-iU_0+U_1 \qquad \text{By the Eqn.(3.1)}
\end{aligned} \tag{3.4}$$

By replacing n by $n-1$, we get

$$\begin{aligned}
X_n &= (1+i)\,X_{n-1}+2\sum_{l=1}^{n-1}\binom{n-1}{l}U_{l-1}-iU_0+U_1\\
&= iX_{n-1}+\sum_{l=0}^{n-1}\binom{n-1}{l}U_l+2\sum_{l=1}^{n-1}\binom{n-1}{l}U_{l-1}-iU_0+U_1\\
&= iX_{n-1}+\sum_{l=1}^{n}\binom{n-1}{l-1}U_{l-1}+2\Bigg[\binom{n-1}{1}U_0+\binom{n-1}{2}U_1+\binom{n-1}{3}\\
&\qquad U_2+\cdots+\binom{n-1}{n-1}U_{n-2}+\binom{n-1}{n}U_{n-1}\Bigg]-iU_0+U_1
\end{aligned}$$

After using the fact $\binom{n-1}{n}=0$, we have

$$X_n = iX_{n-1} + \sum_{l=1}^{n}\binom{n-1}{l-1}U_{l-1} + 2\sum_{l=1}^{n}\binom{n-1}{l}U_{l-1} - iU_0 + U_1$$

$$\begin{aligned}
X_n &= iX_{n-1} + \sum_{l=1}^{n}\left[\binom{n-1}{l-1} + 2\binom{n-1}{l}\right]U_{l-1} - iU_0 + U_1 \\
&= iX_{n-1} + \sum_{l=1}^{n}\left[\binom{n-1}{l-1} + 2\binom{n-1}{l} + 2\binom{n-1}{l-1} - 2\binom{n-1}{l-1}\right]U_{l-1} \\
&\quad - iU_0 + U_1 \\
&= iX_{n-1} + \sum_{l=1}^{n}\left[(1-2)\binom{n-1}{l-1} + 2\binom{n}{l}\right]U_{l-1} - iU_0 + U_1 \\
&= iX_{n-1} - \sum_{l=1}^{n}\binom{n-1}{l-1}U_{l-1} + 2\sum_{l=1}^{n}\binom{n}{l}U_{l-1} - iU_0 + U_1 \\
&= iX_{n-1} - \sum_{l=0}^{n-1}\binom{n-1}{l}U_{l} + 2\sum_{l=1}^{n}\binom{n}{l}U_{l-1} - iU_0 + U_1 \\
&= iX_{n-1} - X_{n-1} + 2\sum_{l=1}^{n}\binom{n}{l}U_{l-1} - iU_0 + U_1 \qquad \text{By the Eqn.(3.1)} \\
&= (i-1)\,X_{n-1} + 2\sum_{l=1}^{n}\binom{n}{l}U_{l-1} - iU_0 + U_1
\end{aligned}$$

Thus

$$X_n - (i-1)\,X_{n-1} = 2\sum_{l=1}^{n}\binom{n}{l}U_{l-1} - iU_0 + U_1$$

Hence from the equation (3.4), we get

$$\begin{aligned}
X_{n+1} &= (1+i)\,X_n + X_n - (i-1)\,X_{n-1} \\
&= (2+i)\,X_n + (1-i)\,X_{n-1}
\end{aligned}$$

as required.

One can obtain the characteristic equation of $\langle X_n\rangle$ in the form $v^2-(2+i)\,v-(1-i)=0$. Let γ and δ be its two roots such that

$$\gamma = \theta + 1 \ \text{ and } \ \delta = \vartheta + 1 \tag{3.5}$$

Some noticeable points about γ and δ are

$$\gamma + \delta = 2 + i, \;\; \gamma\delta = i - 1 = -(1 - i) \;\; \text{and} \;\; \gamma - \delta = \sqrt{7} \tag{3.6}$$

Lemma 3. *For a square matrix* $X = \begin{bmatrix} 2+i & 1-i \\ 1 & 0 \end{bmatrix}$ *and* $n \in \mathbb{Z}_0$*, the following results hold*

$$\begin{bmatrix} X_{n+1} \\ X_n \end{bmatrix} = X^n \begin{bmatrix} X_1 \\ X_0 \end{bmatrix} \tag{3.7}$$

$$X^n = (\gamma - \delta)^{-1} \begin{bmatrix} \gamma^{n+1} - \delta^{n+1} & -\delta\gamma^{n+1} + \gamma\delta^{n+1} \\ \gamma^n - \delta^n & -\delta\gamma^n + \gamma\delta^n \end{bmatrix} \tag{3.8}$$

Proof. The equation (3.7) can be easily proved by the induction method. Certainly X is a square matrix and let v be the eigen value of X. Then by Cayley Hamilton theorem the characteristic equation of X is given by the equation

$$\left| X - uI \right| = 0$$

$$\begin{vmatrix} 2+i & 1-i \\ 1 & 0 \end{vmatrix} = 0$$

$$v^2 - (2+i)\,v - (1-i) = 0$$

Let γ and δ be the characteristic roots as well as eigen values of matrix X. Now same as in [24] the eigen vectors corresponding to γ and δ are $\begin{bmatrix} \gamma \\ 1 \end{bmatrix}$ and $\begin{bmatrix} \delta \\ 1 \end{bmatrix}$ respectively. Let $V_1 = \begin{bmatrix} \gamma & \delta \\ 1 & 1 \end{bmatrix}$ be the matrix of eigen vectors and $V_2 = \begin{bmatrix} \gamma & 0 \\ 0 & \delta \end{bmatrix}$ is the diagonal matrix. Then by the process of diagonalization of matrices, we get

$$\begin{aligned} X^n &= V_1 V_2^n V_1^{-1} \\ &= (\gamma - \delta)^{-1} \begin{bmatrix} \gamma & \delta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \gamma^n & 0 \\ 0 & \delta^n \end{bmatrix} \begin{bmatrix} 1 & -\delta \\ -1 & \gamma \end{bmatrix} \\ &= (\gamma - \delta)^{-1} \begin{bmatrix} \gamma^{n+1} - \delta^{n+1} & -\delta\gamma^{n+1} + \gamma\delta^{n+1} \\ \gamma^n - \delta^n & -\delta\gamma^n + \gamma\delta^n \end{bmatrix} \end{aligned}$$

Hence the proof of the equation (3.8).

Theorem 4. (Binet's formula for the binomial form $\langle X_n \rangle$) *For a square matrix* $X = \begin{bmatrix} 2+i & 1-i \\ 1 & 0 \end{bmatrix}$ *and* $n \in \mathbb{Z}_0$, *we have*

$$X_n = A\gamma^n + B\delta^n, \quad A = \frac{X_1 - \delta X_0}{\gamma - \delta} \text{ and } B = \frac{\gamma X_0 - X_1}{\gamma - \delta} \tag{3.9}$$

$$= s\left(\frac{\gamma^{n+1} - \delta^{n+1}}{\gamma - \delta} - \frac{\gamma^n - \delta^n}{\gamma - \delta}\right) - t\left(\gamma^n + \delta^n\right) \tag{3.10}$$

Proof. Since $X = \begin{bmatrix} 2+i & 1-i \\ 1 & 0 \end{bmatrix}$ and $n \in \mathbb{Z}_0$ then clearly from the equation (3.8), we have

$$X^n = (\gamma - \delta)^{-1} \begin{bmatrix} \gamma^{n+1} - \delta^{n+1} & -\delta\gamma^{n+1} + \gamma\delta^{n+1} \\ \gamma^n - \delta^n & -\delta\gamma^n + \gamma\delta^n \end{bmatrix}$$

By using the equation (3.7), we have

$$\begin{bmatrix} X_{n+1} \\ X_n \end{bmatrix} = (\gamma - \delta)^{-1} \begin{bmatrix} \gamma^{n+1} - \delta^{n+1} & -\delta\gamma^{n+1} + \gamma\delta^{n+1} \\ \gamma^n - \delta^n & -\delta\gamma^n + \gamma\delta^n \end{bmatrix} \begin{bmatrix} X_1 \\ X0 \end{bmatrix}$$

$$= (\gamma - \delta)^{-1} \begin{bmatrix} X_1\gamma^{n+1} - X_1\delta^{n+1} - X_0\delta\gamma^{n+1} + X_0\gamma\delta^{n+1} \\ X_1\gamma^n - X_1\delta^n - X_0\delta\gamma^n + X_0\gamma\delta^n \end{bmatrix}$$

Thus

$$\begin{aligned} X_n &= \frac{X_1\gamma^n - X_1\delta^n - X_0\delta\gamma^n + X_0\gamma\delta^n}{\gamma - \delta} \\ &= \frac{1}{\gamma - \delta}\Big[\big(X_1 - \delta X_0\big)\gamma^n + \big(\gamma X_0 - X_1\big)\delta^n\Big] \\ &= A\gamma^n + B\delta^n \end{aligned}$$

Where

$$\begin{aligned} A &= (\gamma - \delta)^{-1}\big(X_1 - \delta X_0\big)\gamma^n \\ &= (\gamma - \delta)^{-1}\Big[s(1+i) - t(2+i) - \delta(s - 2t)\Big]\gamma^n \\ &= (\gamma - \delta)^{-1}\big(is + s - it - 2t - \delta s + 2\delta t\big)\gamma^n \\ &= (\gamma - \delta)^{-1}\Big(is\gamma^n + s\gamma^n - s\delta\gamma^n - it\gamma^n - 2t\gamma^n + 2\delta t\gamma^n\Big) \end{aligned}$$

$$= (\gamma-\delta)^{-1}\left[is\gamma^n + s\gamma^n - s\left(2+i-\gamma\right)\gamma^n - it\gamma^n - 2t\gamma^n + 2t\left(2+i-\gamma\right)\gamma^n\right]$$

By the Eqn. (3.6)

$$= (\gamma-\delta)^{-1}\Big(is\gamma^n + s\gamma^n - 2s\gamma^n - is\gamma^n + s\gamma^{n+1} - it\gamma^n - 2t\gamma^n + 4t\gamma^n + i2t\gamma^n - 2t\gamma^{n+1}\Big)$$

$$= (\gamma-\delta)^{-1}\Big(-s\gamma^n + s\gamma^{n+1} + it\gamma^n + 2t\gamma^n - 2t\gamma^{n+1}\Big)$$

$$= (\gamma-\delta)^{-1}\Big[s\gamma^{n+1} - s\gamma^n + t\gamma^n\big(2+i-2\gamma\big)\Big]$$

$$= (\gamma-\delta)^{-1}\Big[s\gamma^{n+1} - s\gamma^n + t\gamma^n\big(\gamma+\delta-2\gamma\big)\Big] \qquad \text{By the Eqn. (3.6)}$$

$$= (\gamma-\delta)^{-1}\Big[s\gamma^{n+1} - s\gamma^n - t\gamma^n\left(\gamma-\delta\right)\Big]$$

Similarly

$$B = (\gamma-\delta)^{-1}\Big[-s\delta^{n+1} + s\delta^n - t\delta^n\left(\gamma-\delta\right)\Big]$$

Therefore, we have

$$\begin{aligned} X_n &= \frac{1}{\gamma-\delta}\Big[s\gamma^{n+1} - s\gamma^n - t\gamma^n\left(\gamma-\delta\right) - s\delta^{n+1} + s\delta^n - t\delta^n\left(\gamma-\delta\right)\Big] \\ &= \frac{1}{\gamma-\delta}\Big[s\gamma^{n+1} - s\delta^{n+1} - s\gamma^n + s\delta^n - t\gamma^n\left(\gamma-\delta\right) - t\delta^n\left(\gamma-\delta\right)\Big] \qquad (3.11) \\ &= s\left(\frac{\gamma^{n+1}-\delta^{n+1}}{\gamma-\delta} - \frac{\gamma^n-\delta^n}{\gamma-\delta}\right) - t\Big(\gamma^n+\delta^n\Big) \qquad (3.12) \end{aligned}$$

Hence the result.

For the sake of convenience we express the large sized equation (3.12) into two sequences $\langle M_n\rangle$ and $\langle N_n\rangle$. Then, we have

$$X_n = s\left(M_{n+1} - M_n\right) - tN_n, \;\; n \in \mathbb{Z}_0 \qquad (3.13)$$

Clearly

$$M_n = \frac{\gamma^n-\delta^n}{\gamma-\delta} \;\text{ and }\; N_n = \gamma^n + \delta^n, \;\; n \in \mathbb{Z}_0 \qquad (3.14)$$

The following corollaries are useful while proving the subsequent results.

Corollary 5. (Binet's formula of the (s,t)-Jacobsthal sequence $\langle U_n\rangle$) *Let $n \geq 0$, we have*

$$U_n = s\frac{\theta^{n+1}-\vartheta^{n+1}}{\theta-\vartheta} - t\left(\theta^n + \vartheta^n\right)$$

Proof. From the equation (3.11), we have

$$\begin{aligned} X_n &= \frac{1}{\gamma-\delta}\Big[s\gamma^{n+1} - s\delta^{n+1} - s\gamma^n + s\delta^n - t\gamma^n(\gamma-\delta) - t\delta^n(\gamma-\delta)\Big] \\ &= s\frac{\gamma^n(\gamma-1) - \delta^n(\delta-1)}{\gamma-\delta} - t(\gamma^n+\delta^n) \\ &= s\frac{(\theta+1)^n\theta - (\vartheta+1)^n\vartheta}{\theta-\vartheta} - t\Big[(\theta+1)^n + (\vartheta+1)^n\Big] \qquad \text{By the Eqn. (2.2)} \\ &= s\frac{\sum_{l=0}^{n}\binom{n}{l}\theta^l\theta - \sum_{l=0}^{n}\binom{n}{l}\vartheta^l\vartheta}{\theta-\vartheta} - t\left[\sum_{l=0}^{n}\binom{n}{l}\theta^l + \sum_{l=0}^{n}\binom{n}{l}\vartheta^l\right] \\ &= \sum_{l=0}^{n}\binom{n}{l}\left[s\frac{\theta^{l+1}-\vartheta^{l+1}}{\theta-\vartheta} - t\big(\theta^l+\vartheta^l\big)\right] \end{aligned} \tag{3.15}$$

If we compare the equations (3.1) and (3.15), we get

$$U_l = s\frac{\theta^{l+1}-\vartheta^{l+1}}{\theta-\vartheta} - t\big(\theta^l+\vartheta^l\big) \text{ or } U_n = s\frac{\theta^{n+1}-\vartheta^{n+1}}{\theta-\vartheta} - t\big(\theta^n+\vartheta^n\big)$$

Hence the result.

Corollary 6. *For $m, n \geq 1$, the sequences $\langle X_n\rangle$, $\langle M_n\rangle$ and $\langle N_n\rangle$ satisfy the following properties*

$$M_{m+n-1} = \frac{X_0X_{m+n} - X_1X_{m+n-1}}{X_0X_2 - X_1^2} \tag{3.16}$$

$$M_{m+n-1} = \frac{N_0N_{m+n} - N_1N_{m+n-1}}{(\gamma-\delta)^2} \tag{3.17}$$

$$M_{m+n-1} = \frac{M_m\gamma^n - M_m\delta^n - M_{m-1}\delta\gamma^n + M_{m-1}\gamma\delta^n}{\gamma-\delta} \tag{3.18}$$

$$N_{m+n} = M_{m+n+1} + (1-i)\,M_{m+n-1} \tag{3.19}$$

Proof. The proof of all the equations can be given by using the equations (3.9) and (3.14).

Corollary 7. *If $m, q \in \mathbb{Z}_0$ and $n \in \mathbb{N}$, we get*

$$N_mX_{mn+q} - \big(i-1\big)^m X_{m(n-1)+q} = X_{m(n+1)+q} \tag{3.20}$$

$$N_mM_{mn+q} - \big(i-1\big)^m M_{m(n-1)+q} = M_{m(n+1)+q} \tag{3.21}$$

Proof. The proof is clearly seen by the equations (3.9) and (3.14).

4. Some generalized results

In this section we obtain the various generalized results for the sequences $\langle X_n \rangle$ and $\langle M_n \rangle$ whose terms of the form $mn+q,\ m,n,q \geq 0$.

Theorem 8. (Generalized sum for $\langle X_n \rangle$) *For $m,n,q \in \mathbb{Z}_0$, we have*

$$\sum_{j=1}^{n} X_{mj+q} = \frac{X_{m(n+1)+q} - \big(i-1\big)^m X_{mn+q} - X_{m+q} + \big(i-1\big)^m X_q}{N_m - \big(i-1\big)^m - 1} \tag{4.1}$$

Proof. Let

$$S = \sum_{j=1}^{n} X_{mj+q}$$

Multiplying both sides by $\Big[N_m - \big(i-1\big)^m - 1\Big]$, we have

$$S\Big[N_m - \big(i-1\big)^m - 1\Big] = N_m \sum_{j=1}^{n} X_{mj+q} - \big(i-1\big)^m \sum_{j=1}^{n} X_{mj+q} - \sum_{j=1}^{n} X_{mj+q}$$

Let

$$S\Big[N_m - \big(i-1\big)^m - 1\Big] = S_1 + S_2 + S_3$$

Here

$$S_1 = N_m \sum_{j=1}^{n} X_{mj+q}$$

Now

$$\begin{aligned} S_2 &= -\big(i-1\big)^m \sum_{j=1}^{n} X_{mj+q} \\ &= -\big(i-1\big)^m \Big(X_{m+q} + X_{2m+q} + \cdots + X_{mn+q}\Big) \end{aligned}$$

Add and subtract X_q on R. H. S, we get

$$\begin{aligned} S_2 &= -\big(i-1\big)^m \Big(X_{mn+q} - X_q + X_q + X_{m+q} + X_{2m+q} + \cdots + X_{m(n-1)+q}\Big) \\ &= -\big(i-1\big)^m X_{mn+q} + \big(i-1\big)^m X_q - \big(i-1\big)^m \sum_{j=1}^{n} X_{m(j-1)+q} \end{aligned}$$

and

$$S_3 = -\sum_{j=1}^{n} X_{mj+q}$$

Add and subtract $X_{m(n+1)+q}$ on R. H. S, we have

$$\begin{aligned} S_3 &= -\Big(-X_{m(n+1)+q} + X_{m+q} + X_{2m+q} + \cdots + X_{mn+q} + X_{m(n+1)+q}\Big) \\ &= X_{m(n+1)+q} - X_{m+q} - \sum_{j=1}^{n} X_{m(j+1)+q} \end{aligned}$$

Thus, we have

$$\begin{aligned} & S\Big[N_m - \big(i-1\big)^m - 1\Big] \\ &= X_{m(n+1)+q} - \big(i-1\big)^m X_{mn+q} - X_{m+q} + \big(i-1\big)^m X_q + \sum_{j=1}^{n}\Big(N_m X_{mj+q} \\ &\quad - \big(i-1\big)^m X_{m(j-1)+q} - X_{m(j+1)+q}\Big) \\ &= X_{m(n+1)+q} - \big(i-1\big)^m X_{mn+q} - X_{m+q} + \big(i-1\big)^m X_q + \sum_{j=1}^{n}\Big(X_{m(j+1)+q} \\ &\quad - X_{m(j+1)+q}\Big) \qquad \text{By the Eqn. (3.20)} \\ &= X_{m(n+1)+q} - \big(i-1\big)^m X_{mn+q} - X_{m+q} + \big(i-1\big)^m X_q \end{aligned}$$

Hence

$$\sum_{j=1}^{n} X_{mj+q} = \frac{X_{m(n+1)+q} - \big(i-1\big)^m X_{mn+q} - X_{m+q} + \big(i-1\big)^m X_q}{N_m - \big(i-1\big)^m - 1}$$

This completes the proof of the theorem.

Theorem 9. *For $m, n, q \in \mathbb{Z}_0$, the following property holds for $\langle M_n \rangle$*

$$\sum_{j=1}^{n} M_{mj+q} = \frac{M_{m(n+1)+q} - \big(i-1\big)^m M_{mn+q} - M_{m+q} + \big(i-1\big)^m M_q}{N_m - \big(i-1\big)^m - 1} \tag{4.2}$$

Theorem 10. *If $x \in \mathbb{R}$ and $0 \leq m \leq q$, we get*

$$\sum_{n=0}^{\infty} M_{mn+q} = \frac{M_q + \big(i-1\big)^{-q} M_{m+q} x}{\big(i-1\big)^m x^2 - N_m x + 1} \tag{4.3}$$

Proof. Let

$$T=\sum_{n=0}^{\infty} M_{mn+q}x^n$$

Multiplying both sides by $\Big[\big(i-1\big)^m x^2 - N_m x + 1\Big]$, we obtain

$$\begin{aligned}
&T\Big[\big(i-1\big)^m x^2 - N_m x + 1\Big]\\
&=\big(i-1\big)^m \sum_{n=0}^{\infty} M_{mn+q}x^{n+2} - N_m \sum_{n=0}^{\infty} M_{mn+q}x^{n+1} + \sum_{n=0}^{\infty} M_{mn+q}x^n\\
&=\big(i-1\big)^m \sum_{n=0}^{\infty} M_{mn+q}x^{n+2} - N_m M_q x - N_m \sum_{n=0}^{\infty} M_{m\left(n+1\right)+q}x^{n+2} + M_q + M_{m+q}x\\
&\qquad+\sum_{n=0}^{\infty} M_{m\left(n+2\right)+q}x^{n+2}\\
&=M_q + \big(M_{m+q} - N_m M_q\big)x + \sum_{n=0}^{\infty}\Big[M_{m\left(n+2\right)+q} - N_m M_{m\left(n+1\right)+q}\\
&+\big(i-1\big)^m M_{mn+q}\Big]x^{n+2} \qquad\qquad \text{By the Eqn. (3.21)}\\
&=M_q - \big(i-1\big)^m M_{q-m}x + \sum_{n=0}^{\infty}\Big(M_{m\left(n+2\right)+q} - M_{m\left(n+2\right)+q}\Big)x^{n+2}
\end{aligned}$$

Since $-\left(i-1\right)^m M_{q-m} = \left(i-1\right)^{-q} M_{m+q}$, we get

$$T\Big[\big(i-1\big)^m x^2 - N_m x + 1\Big] = M_q + \left(i-1\right)^{-q} M_{m+q}x$$

Therefore, we have

$$\sum_{n=0}^{\infty} M_{mn+q}x^n = \frac{M_q + \left(i-1\right)^{-q} M_{m+q}x}{\big(i-1\big)^m x^2 - N_m x + 1}$$

Hence the proof.

5. Matrix sequence of the binomial form $\langle X_n \rangle$

In this section we define a kind type of matrix sequence $\langle Z_n \rangle$ to binomial form $\langle X_n \rangle$. In addition to this we investigate some results for the matrix sequence $\langle Z_n \rangle$.

Definition 3. *For $i\left(=\sqrt{-1}\right)$, the matrix sequence $\langle Z_n\rangle_{n\in\mathbb{N}}$ is defined by the following equation*

$$Z_{n+1}=(2+i)\,Z_n+(1-i)\,Z_{n-1},\ \ n\geq 1 \tag{5.1}$$

with $Z_0=\begin{bmatrix}M_2 & (1-i)\,M_1\\ M_1 & (1-i)\,M_0\end{bmatrix}$ *and* $Z_1=\begin{bmatrix}M_3 & (1-i)\,M_2\\ M_2 & (1-i)\,M_1\end{bmatrix}$

Lemma 11. *For a square matrix* $X=\begin{bmatrix}2+i & 1-i\\ 1 & 0\end{bmatrix}$ *and $n\geq 0$, we have*

$$\begin{bmatrix}Z_{n+1}\\ Z_n\end{bmatrix}=X^n\begin{bmatrix}Z_1\\ Z_0\end{bmatrix} \tag{5.2}$$

Theorem 12. *For a square matrix* $X=\begin{bmatrix}2+i & 1-i\\ 1 & 0\end{bmatrix}$ *and $n\geq 0$, the n^{th} term of matrix sequence $\langle Z_n\rangle$ is given as*

$$Z_n=\begin{bmatrix}M_{n+2} & \left(1-i\right)M_{n+1}\\ M_{n+1} & \left(1-i\right)M_n\end{bmatrix} \tag{5.3}$$

$$=\left(X_0X_2-X_1^2\right)^{-1}\begin{bmatrix}X_0X_{n+3}-X_1X_{n+2} & \left(1-i\right)\left(X_0X_{n+2}-X_1X_{n+1}\right)\\ X_0X_{n+2}-X_1X_{n+1} & \left(1-i\right)\left(X_0X_{n+1}-X_1X_n\right)\end{bmatrix} \tag{5.4}$$

$$=(\gamma-\delta)^{-2}\begin{bmatrix}N_0N_{n+3}-N_1N_{n+2} & \left(1-i\right)\left(N_0N_{n+2}-N_1N_{n+1}\right)\\ N_0N_{n+2}-N_1N_{n+1} & \left(1-i\right)\left(N_0N_{n+1}-N_1N_n\right)\end{bmatrix} \tag{5.5}$$

Proof. Since $X=\begin{bmatrix}2+i & 1-i\\ 1 & 0\end{bmatrix}$ then by the equation (3.8), we have

$$X^n=(\gamma-\delta)^{-1}\begin{bmatrix}\gamma^{n+1}-\delta^{n+1} & -\delta\gamma^{n+1}+\gamma\delta^{n+1}\\ \gamma^n-\delta^n & -\delta\gamma^n+\gamma\delta^n\end{bmatrix}$$

Since $\begin{bmatrix}Z_{n+1}\\ Z_n\end{bmatrix}=X^n\begin{bmatrix}Z_1\\ Z_0\end{bmatrix}$, we have

$$\begin{bmatrix}Z_{n+1}\\ Z_n\end{bmatrix}=(\gamma-\delta)^{-1}\begin{bmatrix}\gamma^{n+1}-\delta^{n+1} & -\delta\gamma^{n+1}+\gamma\delta^{n+1}\\ \gamma^n-\delta^n & -\delta\gamma^n+\gamma\delta^n\end{bmatrix}\begin{bmatrix}Z_1\\ Z_0\end{bmatrix}$$

$$= (\gamma-\delta)^{-1}\begin{bmatrix} Z_1\gamma^{n+1} - Z_1\delta^{n+1} - Z_0\delta\gamma^{n+1} + Z_0\gamma\delta^{n+1} \\ Z_1\gamma^{n} - Z_1\delta^{n} - Z_0\delta\gamma^{n} + Z_0\gamma\delta^{n} \end{bmatrix}$$

Therefore

$$\begin{aligned}
Z_n &= \frac{Z_1\gamma^n - Z_1\delta^n - Z_0\delta\gamma^n + Z_0\gamma\delta^n}{\gamma-\delta} \\
&= \frac{1}{\gamma-\delta}\left\{\begin{bmatrix} M_3 & (1-i)\,M_2 \\ M_2 & (1-i)\,M_1 \end{bmatrix}\gamma^n - \begin{bmatrix} M_3 & (1-i)\,M_2 \\ M_2 & (1-i)\,M_1 \end{bmatrix}\delta^n - \begin{bmatrix} M_2 & (1-i)\,M_1 \\ M_1 & (1-i)\,M_0 \end{bmatrix}\delta\gamma^n \right. \\
&\qquad \left. + \begin{bmatrix} M_2 & (1-i)\,M_1 \\ M_1 & (1-i)\,M_0 \end{bmatrix}\gamma\delta^n \right\} \\
&= \frac{1}{\gamma-\delta}\begin{bmatrix} M_3\gamma^n - M_3\delta^n - M_2\gamma\delta^n + M_2\gamma\delta^n & (1-i)\big(M_2\gamma^n - M_2\delta^n \\ & -M_1\gamma\delta^n + M_1\gamma\delta^n\big) \\ M_2\gamma^n - M_2\delta^n - M_1\gamma\delta^n + M_1\gamma\delta^n & (1-i)\big(M_1\gamma^n - M_1\delta^n \\ & -M_0\gamma\delta^n + M_0\gamma\delta^n\big) \end{bmatrix} \\
&= \begin{bmatrix} M_{n+2} & (1-i)\,M_{n+1} \\ M_{n+1} & (1-i)\,M_n \end{bmatrix} \qquad \text{By the Eqn. (3.18)} \\
&= \left(X_0X_2 - X_1^2\right)^{-1}\begin{bmatrix} X_0X_{n+3} - X_1X_{n+2} & (1-i)\big(X_0X_{n+2} - X_1X_{n+1}\big) \\ X_0X_{n+2} - X_1X_{n+1} & (1-i)\big(X_0X_{n+1} - X_1X_n\big) \end{bmatrix} \\
&\qquad \text{By the Eqn. (3.16)} \\
&= (\gamma-\delta)^{-2}\begin{bmatrix} N_0N_{n+3} - N_1N_{n+2} & (1-i)\big(N_0N_{n+2} - N_1N_{n+1}\big) \\ N_0N_{n+2} - N_1N_{n+1} & (1-i)\big(N_0N_{n+1} - N_1N_n\big) \end{bmatrix} \\
&\qquad \text{By the Eqn. (3.17)}
\end{aligned}$$

Hence the result.

Corollary 13. *If $n \geq 0$, the following relation is true for the sequences $\langle X_n\rangle$, $\langle M_n\rangle$ and $\langle N_n\rangle$*

$$N_{n+1} = \frac{X_0\Big[X_{n+3} + \big(1-i\big)X_{n+1}\Big] - X_1\Big[X_{n+2} + \big(1-i\big)X_n\Big]}{X_0X_2 - X_1^2} \tag{5.6}$$

Proof. If we compare corresponding terms of matrices from the equations (5.3) and (5.4). Therefore we have

$$M_{n+2} = \frac{X_0X_{n+3} - X_1X_{n+2}}{X_0X_2 - X_1^2}$$

$$\left(1-i\right)M_n = \left(1-i\right)\frac{X_0X_{n+1}-X_1X_n}{X_0X_2-X_1^2}$$

If we add these equations together, we have

$$\begin{aligned} M_{n+2}+\left(1-i\right)M_n &= \frac{X_0\left[X_{n+3}+\left(1-i\right)X_{n+1}\right]-X_1\left[X_{n+2}+\left(1-i\right)X_n\right]}{X_0X_2-X_1^2} \\ N_{n+1} &= \frac{X_0\left[X_{n+3}+\left(1-i\right)X_{n+1}\right]-X_1\left[X_{n+2}+\left(1-i\right)X_n\right]}{X_0X_2-X_1^2} \end{aligned}$$

By the Eqn. (3.19)

This proves the equation (5.6).

Theorem 14. (Sum of the first n terms of matrix sequence $\langle Z_n\rangle$)

$$\sum_{j=1}^{n} Z_n = 2^{-1}\begin{bmatrix} M_{n+3}-\left(1-i\right)M_{n+2}-\left(7+2i\right) & \left(1-i\right)M_{n+2}-2iM_{n+1}-3\left(1-i\right) \\ M_{n+2}-\left(1-i\right)M_{n+1}-3 & \left(1-i\right)M_{n+1}-2iM_n-\left(1-i\right) \end{bmatrix} \tag{5.7}$$

Proof. From the equations (4.2) and (5.3), we have

$$\begin{aligned} \sum_{j=1}^{n} Z_n &= \sum_{j=1}^{n}\begin{bmatrix} M_{n+2} & \left(1-i\right)M_{n+1} \\ M_{n+1} & \left(1-i\right)M_n \end{bmatrix} \\ &= \begin{bmatrix} \frac{M_{n+3}+\left(1-i\right)M_{n+2}-\left(7+2i\right)}{2} & \left(1-i\right)\frac{M_{n+2}+\left(1-i\right)M_{n+1}-3}{2} \\ \frac{M_{n+2}+\left(1-i\right)M_{n+1}-3}{2} & \left(1-i\right)\frac{M_{n+1}+\left(1-i\right)M_n-1}{2} \end{bmatrix} \\ &= 2^{-1}\begin{bmatrix} M_{n+3}-\left(1-i\right)M_{n+2}-\left(7+2i\right) & \left(1-i\right)M_{n+2}-2iM_{n+1}-3\left(1-i\right) \\ M_{n+2}-\left(1-i\right)M_{n+1}-3 & \left(1-i\right)M_{n+1}-2iM_n-\left(1-i\right) \end{bmatrix} \end{aligned}$$

Hence the proof.

Theorem 15. (Generating function of the binomial matrix sequence $\langle Z_n\rangle$) *Let $x \in \mathbb{R}$, we get*

$$\sum_{j=1}^{n} Z_n x^n = \begin{bmatrix} \frac{2+i+(i-1)^{-2}(4+3i)x}{(i-1)x^2-(2+i)x+1} & \frac{1-i-(2+i)x}{(i-1)x^2-(2+i)x+1} \\ \frac{1+(i-1)^{-1}(2+i)x}{(i-1)x^2-(2+i)x+1} & \frac{(1-i)}{(i-1)x^2-(2+i)x+1} \end{bmatrix} \tag{5.8}$$

Proof. The proof of this theorem is established by using the equations (5.3) and (4.3).

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