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ON A CERTAIN CLASS OF CONCAVE MEROMORPHIC HARMONIC FUNCTIONS DEFINED BY INVERSE OF INTEGRAL OPERATOR

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Abstract. In this paper we introduce new class of meromorphic harmonic concave functions defined by an integral operator and establish some of the properties of this class.

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1. Introduction

Let A denotes the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

that map the unit disc conformally onto a domain whose complement with respect to a convex that satisfies the normalization $f(1) = \infty$,the opening angle of $f(U)$ at infinity is less than or equal to $\alpha\pi$.

The families of these functions is referred to as a concave univalent function denoted as $C_o(\alpha)$ if it satisfies the condition $P_f > 0$, where

$$P_f = \frac{2}{\alpha - 1}\left[\frac{\alpha + 1}{2}\frac{1 + z}{1 - z} - 1 - z\frac{f''(z)}{f'(z)}\right]$$

In [11], the concept of meromorphic concave function was introduced, that a conformal mapping of meromorphic functions on the unit disc is referred to as a concave function if its image is the complement of a compact convex function. We define the class of meromorphic functions to be of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k. \tag{2}$$

A function of the form (2) is referred to as a concave meromorphic if it satisfied the condition

$$1 + Re\left[\frac{zf^{''}(z)}{f^{'}(z)}\right] < 0. \tag{3}$$

The concept of harmonic univalent function was first introduced by Clunie and Sheil-Small in [5]. The class of function is applied in the study of minimal surfaces and other areas of sciences. We say that a continuous function $f = u+iv$ is complex harmonic function in a domain $\mathbb{U} \subset \mathbb{C}$, if both u and v are real harmonic in $\mathbb{U}$. Let $f = h+\overline{g}$, where h and g are analytic in $\mathbb{U}$. A necessary and sufficient condition for f to be locally univalent and preserving in $\mathbb{U}$ is that $|h^{'}(z)| > |g^{'}(z)|$ in $\mathbb{U}$.

In [6], it was shown that a complex valued, harmonic, sense preserving, univalent mapping f must admit the representation

$$f(z) = h(z) + \overline{g(z)} + Alog|z| \tag{4}$$

where $h(z)$ and $g(z)$ are defined by

$$h(z) = \alpha z + \sum_{k=1}^{\infty} a_k z^{-n}, g(z) = \beta\overline{z} + \sum_{k=1}^{\infty} b_n z^{-n} \tag{5}$$

for $0 \leq |\beta| < |\alpha|$. In [7], For $z \in \mathbb{U}$. S_H was define to be the class of functions

$$f(z) = h(z) + \overline{g(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}, \tag{6}$$

which are harmonic in the unit disc $\mathbb{U}$, $h(z)$ and $g(z)$ are analytic in $\mathbb{U}$, respectively.

A function $f \in S_H$ is said to be in the subclass S_H^* of meromorphic harmonic starlike functions in $\mathbb{U}$, if its satisfied the condition

$$Re\left[-\frac{zh^{'}(z) - \overline{zg'(z)}}{h(z) + g(z)}\right] > 0, (z \in \mathbb{U} \setminus 0).$$

Also A function $f \in S_H$ is said to be in the subclass SC_H of meromorphic harmonic convex functions in $(U \setminus 0)$, if its satisfied the condition

$$Re\left[-\frac{zh^{''}(z)+h^{'}(z)\overline{-zg^{''}(z)+g^{'}(z)}}{h^{'}(z)-g^{'}(z)}\right] > 0, (z \in \mathbb{U}\setminus 0).$$

These two classes ware studied in [8, 9, 10].

In [12], the integral operator was introduced as the following:

$$\mathcal{L}_{\sigma,\gamma}f(z) = \int_0^z \frac{(\gamma+1)^2 t^{\sigma-1}}{z^{\gamma}\Gamma(\sigma)}\left(log\frac{z}{t}\right)^{\sigma-1} f(t)dt,$$

and expressed as

$$\mathcal{L}_{\sigma,\gamma}f(z) = z + \sum_{k=2}^{\infty}\left(\frac{\gamma+1}{\gamma+k}\right) a_k z^k \tag{7}$$

In [13], the inverse of the integral operator was considered as

$$\mathcal{J}_{\sigma,\gamma}f(z) = \int_0^z \frac{(\gamma+1)^2 t^{\sigma-1}}{z^{\gamma}\Gamma(\sigma)}\left(log\frac{z}{t}\right)^{-(\sigma-1)} f(t)dt \tag{8}$$

an expressed as

$$\mathcal{L}_{\sigma,\gamma}f(z) = z + \sum_{k=2}^{\infty}\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} a_k z^k \tag{9}$$

so that

$$\mathcal{L}_{\sigma,\gamma}(\mathcal{J}_{\sigma,\gamma}f(z)) = f(z). \tag{10}$$

If $\gamma = 0$, $n = \sigma$ we have $D^n f(z)$, known as the Salagean operator.

In this work , we studied a new class of meromorphic concave functions defined by inverse of an integral operator denoted $SH^{\sigma}_{\gamma}C_0$ and define the class as follows:

Definition 1 Let $SH^{\sigma}_{\gamma}C_0$ denote the class of meromorphic harmonic concave function define by inverse of Integral Operator on the function of the form (2)

$$\mathcal{L}_{\sigma,\gamma}f(z) = \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} a_k z^k + \sum_{k=1}^{\infty}\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} \overline{b_k z^k}, (\sigma > 0, \gamma > 1) \tag{11}$$

such that

$$Re\left[1 + \frac{z(\mathcal{L}_{\sigma,\gamma}f(z))^{'}}{\mathcal{L}_{\sigma,\gamma}f(z)}\right] < 0.$$

2. Main Results

2.1. Coefficient Inequalities for the class $SH^{\sigma}_{\gamma}C_0$

Theorem 1. *Let $\mathcal{L}_{\sigma,\gamma}f = h + g$ be of the form* (11) *, if*

$$\sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} (|a_n| + |b_n|) \leq 1 \tag{12}$$

then f is harmonic univalent, sense preserving in U

Proof. For $0 < |z_1| \leq |z_2| < 1$, we have

$$|\mathcal{L}_{\sigma,\gamma}f(z_1) - \mathcal{L}_{\sigma,\gamma}f(z_2)| = |\frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} a_k z_1^k + \sum_{k=1}^{\infty}\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}\overline{b_k z_1^k}$$

$$-\frac{1}{z} - \sum_{k=1}^{\infty}\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} a_k z_2^k - \sum_{k=1}^{\infty}\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}\overline{b_k z_2^k}|$$

$$\geq \frac{1}{|z_1|} - \frac{1}{|z_2|} - \sum_{k=1}^{\infty}\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}|a_k||z_1^k - z_2^k| - \sum_{k=1}^{\infty}\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}|b_k||z_1^k - z_2^k|$$

$$> \frac{|z_1 - z_2|}{|z_1 z_2|} - |z_1 - z_2|\sum_{k=1}^{\infty} k \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}(|a_k| + |b_k|)$$

$$> \frac{|z_1 - z_2|}{|z_1 z_2|}\left[1 - |z_2|^2 \sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}(|a_k| + |b_k|)\right]$$

$$> \frac{|z_1 - z_2|}{|z_1 z_2|}\left[1 - \sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}(|a_k| + |b_k|)\right]$$

The last expression is non negative by $\sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}(|a_k| + |b_k|) < 1$, $\mathcal{L}_{\sigma,\gamma}f(z)$ is univalent in U.

Now we want to show that f is sense preserving in U , we need to show that $|h^{'}(z)| \geq |g^{'}(z)|$ in U,

$$|h^{'}(z)| \geq \frac{1}{|z|^2} - \sum_{k=1}^{\infty} k \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}|a_k||z|^{k-1}$$

$$= \frac{1}{r^2} - \sum_{k=1}^{\infty} k \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}|a_k| r^{k-1}$$

$$> 1-\sum_{k=1}^{\infty} k\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}|a_k|$$

$$\geq 1-\sum_{k=1}^{\infty} k^2\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}|a_k|$$

$$\geq \sum_{k=1}^{\infty} k^2\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}|b_k|$$

$$> \sum_{k=1}^{\infty} k\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}|b_k|r^{k-1} = \sum_{k=1}^{\infty} k\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}|b_k||z|^{k-1} \geq |g^{'}(z)|.$$

Thus this completes the proof of the theorem.

Theorem 2. *Let* $\mathcal{L}_{\sigma,\gamma}f = h+g$ *be of the form* (11) , *then* $f \in SH_{\gamma}^{\sigma}C_0$ *if the condition holds*

$$\sum_{k=1}^{\infty} k^2\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}(|a_n|+|b_n|) \leq 1 \tag{13}$$

Proof. with the condition that $Rew < 0 \leftrightarrow |\frac{w+1}{w-1}| < 1$, it suffices to show that $|\frac{w+1}{w-1}| < 1$.
Let

$$w = Re\left[1+\frac{z(\mathcal{L}_{\sigma,\gamma}f(z))^{'}}{\mathcal{L}_{\sigma,\gamma}f(z)}\right]$$

such that $w = \frac{zg^{'}(z)}{g(z)}$, where $g(z) = z(\mathcal{L}_{\sigma,\gamma}f(z))^{'}$ we have that

$$\left|\frac{w+1}{w-1}\right| = \left|\frac{\sum_{k=1}^{\infty}(k^2+k)(\gamma+k/\gamma+1)a_kz^k - \sum_{k=1}^{\infty}(k^2+k)(\gamma+k/\gamma+1)b_kz^k}{\frac{2}{z}+\sum_{k=1}^{\infty}(k^2+k)(\gamma+k/\gamma+1)a_kz^k - \sum_{k=1}^{\infty}(k^2+k)(\gamma+k/\gamma+1)b_kz^k}\right|$$

$$< \frac{\sum_{k=1}^{\infty}(k^2+k)(\gamma+k/\gamma+1)|a_k| - \sum_{k=1}^{\infty}(k^2+k)(\gamma+k/\gamma+1)|b_k|}{2-\sum_{k=1}^{\infty}(k^2-k)(\gamma+k/\gamma+1)|a_k| - \sum_{k=1}^{\infty}(k^2-k)(\gamma+k/\gamma+1)|b_k|}. \tag{14}$$

The last expression is bounded above by 1 if

$$\sum_{k=1}^{\infty}(k^2+k)(\gamma+k/\gamma+1)a_k + \sum_{k=1}^{\infty}(k^2+k)(\gamma+k/\gamma+1)b_k$$

$$\leq 2-\sum_{k=1}^{\infty}(k^2-k)(\gamma+k/\gamma+1)a_k - \sum_{k=1}^{\infty}(k^2-k)(\gamma+k/\gamma+1)b_k$$

which is equivalent to our condition by

$$\sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} (|a_n| + |b_n|) \leq 1.$$

Conversely, assume $f \in SH_{\gamma}^{\sigma} C_0$, then we have

$$\left| \frac{1 + \frac{z\mathcal{L}_{\sigma,\gamma}h(z))'' - z\mathcal{L}_{\sigma,\gamma}g(z))''}{\mathcal{L}_{\sigma,\gamma}h(z))' - \mathcal{L}_{\sigma,\gamma}g(z))'} + 1}{1 + \frac{z\mathcal{L}_{\sigma,\gamma}h(z))'' - z\mathcal{L}_{\sigma,\gamma}g(z))''}{\mathcal{L}_{\sigma,\gamma}h(z))' - \mathcal{L}_{\sigma,\gamma}g(z))'} - 1} \right| < 1$$

$$= \left| \frac{\sum_{k=1}^{\infty}(k^2+k)(\gamma+k/\gamma+1)a_k - \sum_{k=1}^{\infty}(k^2+k)(\gamma+k/\gamma+1)\overline{b_k}}{\frac{2}{z^2} - \sum_{k=1}^{\infty}(k^2-k)(\gamma+k/\gamma+1)a_k - \sum_{k=1}^{\infty}(k^2-k)(\gamma+k/\gamma+1)\overline{b_k}} \right| < 1.$$

By letting $|z| \to 1$, we obtain (12).

Theorem 3. *Let $f = h + \overline{g}$ of the form* (11)*, then a necessary and sufficient condition for $\mathcal{L}_{\sigma,\gamma}f(z)$ to be in $SH_{\gamma}^{\sigma}C_0$ is that*

$$\sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} (|a_k| + |b_k|) \leq 1.$$

Proof. From Theorem 2, we assume that

$$\sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} (|a_k| + |b_k|) > 1.$$

Since $\mathcal{L}_{\sigma,\gamma}f(z) \in \mathcal{L}_{\sigma,\gamma}S_H C_0$, then $1 + Rez(\mathcal{L}_{\sigma,\gamma}f(z))''/(\mathcal{L}_{\sigma,\gamma}f(z))'$ is equivalent to

$$Re\frac{zg'(z)}{g(z)} = Re\frac{z\left(\frac{1}{z^2} + \sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} a_k z^{k-1} + \sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} \overline{b_k z^{k-1}}\right)}{\frac{1}{z} + \sum_{k=1}^{\infty} k \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} a_k z^k + \sum_{k=1}^{\infty} k \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} \overline{b_k z^k}}$$

$$= Re\frac{\left(\frac{1}{z} + \sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} a_k z^k + \sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} \overline{b_k z^k}\right)}{\frac{1}{z} + \sum_{k=1}^{\infty} k \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} a_k z^k + \sum_{k=1}^{\infty} k \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} \overline{b_k z^k}} \leq 0$$

for $|z| = r > 1$, the above expression reduce to

$$Re\left(\frac{1 + \sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} (|a_k| + |b_k|) r^k}{1 + \sum_{k=1}^{\infty} k \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} (|a_k| + |b_k|) r^k}\right) = \left(\frac{A(r)}{B(r)}\right) \leq 0$$

from our assumption that $\sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} (|a_k|+|b_k|) > 1$, then $A(r)$ and $B(r)$ are positive for r sufficiently close to 1. Thus there exists a $z_0 = r_0 > 1$ for which the quotient is positive. This contradicts the required condition that $\frac{A(r)}{B(r)} \leq 0$, so the proof is complete.

2.2. Distortion and Extreme point

Theorem 4. *If $\mathcal{L}_{\sigma,\gamma} f_k = h_k + \overline{g_k}$ be of the form* (11) *and $0 < |z| = r < 1$, then*

$$|\mathcal{L}_{\sigma,\gamma} f_k(z)| \leq \frac{1+r^2}{r}$$

and

$$|\mathcal{L}_{\sigma,\gamma} f_k(z)| \leq \frac{1-r^2}{r}.$$

Proof. Taking the absolute of f_k, we have that

$$|f_k| = \left|\frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} a_k z^k + \sum_{k=1}^{\infty} \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} \overline{b_k z^k}\right|$$

$$\geq \frac{1}{r} - \sum_{k=1}^{\infty} \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} (|a_k|+|b_k|) r^k$$

$$\geq \frac{1}{r} - \sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} (|a_k|+|b_k|) r$$

by applying $\sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} (|a_k|+|b_k|) \leq 1$

$$|f_k| \geq \frac{1}{r} - r = \frac{1-r^2}{r}.$$

Also

$$|f_k| = \left|\frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} a_k z^k + \sum_{k=1}^{\infty} \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} \overline{b_k z^k}\right|$$

$$\leq \frac{1}{r} + \sum_{k=1}^{\infty} \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} (|a_k|+|b_k|) r^k$$

$$\leq \frac{1}{r} + \sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} (|a_k|+|b_k|) r$$

by applying $\sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} (|a_k| + |b_k|) \leq 1$

$$|f_k| \leq \frac{1}{r} + r = \frac{1+r^2}{r}.$$

Theorem 5. *Let* $\mathcal{L}_{\sigma,\gamma} f = h + \overline{g}$ *be of the form* (11). *Set* $h_{n,0} = g_{n,0} = \frac{1}{z}$ *for* $k = 1, 2, 3, \cdots$ *set*

$$h_{n,k}(z) = \frac{1}{z} + \frac{1}{k^2} z^k, g_{n,k}(z) = \frac{1}{z} + \frac{1}{k^2} \overline{z^k}$$

then $\mathcal{L}_{\sigma,\gamma} f(z)$ *to be in* $SH_{\gamma}^{\sigma} C_0$ *if and only if* f_k *can be expressed as*

$$f_{n,k} = \sum_{k=0}^{\infty} (\Psi_k h_{n,k}(z) + \Phi_k g_{n,k}(z))$$

where $\Psi_k \geq 0, \Phi_k \geq 0$ *and* $\sum_{k=0}^{\infty} (\Psi_k + \Phi_k) = 1$.

Proof. For function $f = h + g$ to be of the form (11), we have that

$$f_{n,k}(z) = \sum_{k=1}^{\infty} (\Psi_k h_{n,k}(z) + \Phi_k g_{n,k}(z))$$

$$= \Psi_0 h_{n,0} + \Phi_0 g_{n,0} + \sum_{k=1}^{\infty} (\Psi_k h_{n,k}(z) + \Phi_k g_{n,k}(z))$$

$$= \Psi_0 h_{n,0} + \Phi_0 g_{n,0} + \sum_{k=1}^{\infty} \Psi_k \left(\frac{1}{z} + \frac{1}{k^2} z^k\right) + \sum_{k=1}^{\infty} \Phi_k \left(\frac{1}{z} + \frac{1}{k^2} \overline{z^k}\right)$$

$$\sum_{k=0}^{\infty} (\Psi_k + \Phi_k) \frac{1}{z} + \sum_{k=1}^{\infty} \frac{1}{k^2} (\Psi_k z^k + \Phi_k \overline{z^k}).$$

Now by Theorem 1,

$$\sum_{k=1}^{\infty} (\Psi_k \frac{1}{k^2} k^2 + \Phi_k \frac{1}{k^2} k^2) = \sum_{k=1}^{\infty} \Psi_k + \Phi_k = 1 - \Psi_0 - \Phi_0 \leq 1$$

we have $\mathcal{L}_{\sigma,\gamma} f(z)$ to be in $SH_{\gamma}^{\sigma} C_0$. The converse is similar to the above proof

2.3. Convolution Properties

For harmonic functions

$$\mathcal{L}_{\sigma,\gamma}f_n(z)=\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}a_kz^k+\sum_{k=1}^{\infty}\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}\overline{b_kz^k} \tag{15}$$

and

$$\mathcal{L}_{\sigma,\gamma}F_n(z)=\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}A_kz^k+\sum_{k=1}^{\infty}\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}\overline{B_kz^k}. \tag{16}$$

The convolution of $\mathcal{L}_{\sigma,\gamma}f_n(z)$ and $\mathcal{L}_{\sigma,\gamma}F_n(z)$ is given by $(\mathcal{L}_{\sigma,\gamma}f_n*\mathcal{L}_{\sigma,\gamma}F_n)(z)=\mathcal{L}_{\sigma,\gamma}f_n(z)*\mathcal{L}_{\sigma,\gamma}F_n(z)$

$$=\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}|a_k||A_k|z^k+\sum_{k=1}^{\infty}\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}|b_k||B_k|\overline{z^k}. \tag{17}$$

The geometric convolution of f_k and F_k is given by

$$(f(z)*F_k)(z)=f_k(z)\bullet F_k(z)=\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}\sqrt{|a_kA_k|}z^k+\sum_{k=1}^{\infty}\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}\sqrt{|b_kB_k|}\overline{z^k}. \tag{18}$$

The integral convolution of f_k and F_k is given by

$$(f_k\circ F_k)(z)=f_k(z)\circ F_k(z)=\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}\frac{|a_kA_k|}{k}z^k+\sum_{k=1}^{\infty}\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}\frac{|b_kB_k|}{k}\overline{z^k}. \tag{19}$$

Theorem 6. *Let $\mathcal{L}_{\sigma,\gamma}f_k(z)\in SH^{\sigma}_{\gamma}C_0$ and $\mathcal{L}_{\sigma,\gamma}F_k(z)\in SH^{\sigma}_{\gamma}C_0$. Then the convolution $\mathcal{L}_{\sigma,\gamma}f_k(z)*\mathcal{L}_{\sigma,\gamma}F_k(z)\in SH^{\sigma}_{\gamma}C_0$.*

Proof. From (17), (18), then the convolution given by (19). We need to show that the coefficients of $\mathcal{L}_{\sigma,\gamma}f_k(z)*\mathcal{L}_{\sigma,\gamma}F_k(z)$ satisfy the condition of theorem (2.1). We obtain that

$$\sum_{k=1}^{\infty}k^2\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}(|a_k||A_k|)+\sum_{k=1}^{\infty}k^2\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}(|b_k||B_k|)$$

$$\le\sum_{k=1}^{\infty}k^2\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}|a_k|+\sum_{k=1}^{\infty}k^2\left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma}|b_k|\le 1.$$

Therefore $\mathcal{L}_{\sigma,\gamma}f_k(z)*\mathcal{L}_{\sigma,\gamma}F_k(z)\in\mathcal{L}_{\sigma,\gamma}S_HC_0$, where $|A_k|\le 1$, $|B_k|\le 1$. This completes the proof.

Theorem 7. *Given f_k and F_k of the form* (17) *and* (18) *belong to the class $SH_{\gamma}^{\sigma}C_0$, then the geometric condition $(f(z) \bullet F_k)(z) \in SH_{\gamma}^{\sigma}C_0$.*

Proof. From (19), and by Cauchy-Schwartz's inequality, it follows that

$$\sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} (\sqrt{|A_k a_k|} + \sqrt{|B_k b_k|}) \leq 1.$$

Theorem 8. *Given f_k and F_k of form* (17) *and* (18) *belong to the class $SH_{\gamma}^{\sigma}C_0$, then the integral convolution $(f(z) \circ F_k)(z) \in SH_{\gamma}^{\sigma}C_0$.*

Proof. Let $|A_k| \leq 1$ and $|B_k| \leq 1$, then

$$\sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} (\frac{|A_k a_k|}{k} + \frac{|B_k b_k|}{k})$$

$$\leq \sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} (\frac{|a_k|}{k} + \frac{|b_k|}{k})$$

$$\leq \sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} (\frac{|a_k|}{k} + \frac{|b_k|}{k}) \leq 1.$$

The proof is complete.

2.4. Convex Combinations

Theorem 9. *The class $SH_{\gamma}^{\sigma}C_0$ is closed under convex combination.*

Proof. Let $i = 1, 2, \cdots$, then

$$\mathcal{L}_{\sigma,\gamma} f_i(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} a_{ik} z^k + \sum_{k=1}^{\infty} \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} \overline{i b_k z^k}$$

where $a_{ik} > 0$, $b_{ik} > 0$, by theorem (2)

$$\sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+}\right)^{\sigma} (|a_{ik}| + |b_{ik}|) \leq 1.$$

For $\sum_{k=1}^{\infty} t_i = 1$, $0 \leq t \leq 1$, the convex combinations of $\mathcal{L}_{\sigma,\gamma} f(z)$ is written as

$$\sum_{k=1}^{\infty} t_i \mathcal{L}_{\sigma,\gamma} f_i(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} (t_i a_{ik}) z^k + \sum_{k=1}^{\infty} \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} \overline{t_i b_{ik}) z^k}.$$

Then by

$$\sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} (|a_{ik}|+|b_{ik}|) \leq 1$$

$$\sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} \left[|\sum_{k=1}^{\infty}(t_i a_{ik}| + |\sum_{k=1}^{\infty} t_i b_{ik}|) \right]$$

$$= \sum_{k=1}^{\infty} t_i \left[\sum_{k=1}^{\infty} k^2 \left(\frac{\gamma+k}{\gamma+1}\right)^{\sigma} (|a_{ik}|+|b_{ik}|) \right] \leq \sum_{k=1}^{\infty} t_i = 1$$

. Then $\sum_{k=1}^{\infty} t_i \mathcal{L}_{\sigma,\gamma} f_i(z) \in SH_{\gamma}^{\sigma} C_0$.

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