Acta Universitatis Apulensis No. 64/2020
ISSN: 1582-5329 pp. 127-136
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2020.64.09

APPLICATION OF OPERATORS $H(\alpha,\beta)$ AND $\bar{H}(\alpha,\beta)$ TO HYPERGEOMETRIC GAUSSIAN FUNCTIONS IN THREE VARIABLES

Anvar Hasanov and Jihad A. Younis

Abstract. By using the reciprocally inverse operators of the Burchnall and Chaundy type introduced by Choi and Hasanov in 2011, we aim at deriving certain decomposition formulas for some Gauss's triple hypergeometric functions. We also present some transformation formulas by using our decomposition formulas.

2010 *Mathematics Subject Classification*: Primary 33C20; Secondary 44A45.

Keywords: Inverse pairs of symbolic operators, decomposition formulas, Gaussian hypergeometric functions in three variables, transformation formulas, Appell functions, Horn functions.

1. Introduction

The use of many mathematical operations goes beyond the class of elementary functions. Calculation of integrals, summation of series, solution of algebraic, transcendental, difference and differential equations and their systems require expanding the class of functions studied. The development of the concept of a function, going in parallel with the development of the concepts of number and space, led to the emergence of new hypergeometric functions of many complex variables.

The theory of hypergeometric functions is one of the main branches of mathematical physics [9], [17], [18]. Over the past three centuries, the need to solve problems in hydrodynamics, control theory, classical and quantum mechanics, as well as numerous problems in probability theory and mathematical statistics has stimulated the development of the theory of special functions of one and several variables. Mathematical models of physical processes contain, as a rule, ordinary differential equations, partial differential equations or systems of such equations. However, only a few of the equations encountered in practice can be solved in the class of elementary functions. New functions were often defined as solutions to differential equations or their systems and were called hypergeometric functions. This is how the Bessel functions, Hermite functions and the Gaussian hypergeometric function arose. In the monograph [20, pp. 75-77], the following second-order Gaussian hypergeometric functions in three variables were introduced:

$$F_{6a}\left(a_1,a_2,a_3,a_4,a_5;c_1,c_2;x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_m(a_3)_n(a_4)_p(a_5)_p}{(c_1)_m(c_2)_{n+p}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{1}$$

$$F_{8c}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{m+n}(a_3)_p(b)_{p-m}}{(c_1)_n(c_2)_p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{2}$$

$$F_{10a}\left(a_1,a_2,a_3,a_4;c_1,c_2,c_3;x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{n+p}(a_3)_m(a_4)_p}{(c_1)_m(c_2)_n(c_2)_p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{3}$$

$$F_{10b}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{n+p}(a_2)_m(a_3)_p(b)_{m+n-p}}{(c_1)_m(c_2)_n} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{4}$$

$$F_{10d}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n}(a_2)_{n+p}(a_3)_p(b)_{m-n}}{(c_1)_m(c_2)_p} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{5}$$

$$F_{10l}\left(a_1,a_2,b_1,b_2;c;x,y,z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{n+p}(a_2)_m(b_1)_{m+n-p}(b_2)_{p-n}}{(c)_m} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}, \tag{6}$$

where $(a)_m$ denotes the Pochhammer symbol given by

$$(a)_m = a(a+1)...(a+m-1) \ \ (m \in \mathbb{N} := \{1,2,...\}) \text{ and } (a)_0 = 1.$$

Over seven decades ago, Burchnall and Chaundy (see [3], [4]) and Chaundy [5] systematically presented a number of expansion and decomposition formulas for some double hypergeometric functions in series of simpler hypergeometric functions. Their method is based upon the following inverse pairs of symbolic operators:

$$\nabla(h) = \frac{\Gamma(h)\Gamma(\delta_1+\delta_2+h)}{\Gamma(\delta_1+h)\Gamma(\delta_2+h)} = \sum_{k=0}^{\infty} \frac{(-\delta_1)_k(-\delta_2)_k}{(h)_k k!} \tag{7}$$

and

$$\Delta(h) = \frac{\Gamma(\delta_1+h)\Gamma(\delta_2+h)}{\Gamma(h)\Gamma(\delta_1+\delta_2+h)} = \sum_{k=0}^{\infty} \frac{(-\delta_1)_k(-\delta_2)_k}{(1-h-\delta_1-\delta_2)_k k!}, \tag{8}$$

where

$$\delta_1 := x\frac{\partial}{\partial x} \text{ and } \delta_2 := y\frac{\partial}{\partial y}.$$

For various multivariable hypergeometric functions including the Lauricella multivariable functions $F_A^{(n)}, F_B^{(n)}, F_C^{(n)}$ and $F_D^{(n)}$, Hasanov and Srivastava [10], [11] presented a number of decomposition formulas in terms of such simpler hypergeometric functions as the Gauss and Appell functions. Choi and Hasanov [6] showed how some rather elementary techniques based upon certain inverse pairs of symbolic operators

$$\begin{aligned} H_{x_1,\ldots,x_r}(\alpha,\beta) &= \frac{\Gamma(\beta)\Gamma(\alpha+\delta_1+\cdots+\delta_r)}{\Gamma(\alpha)\Gamma(\beta+\delta_1+\cdots+\delta_r)} \\ &= \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(\beta-\alpha)_{k_1+\cdots+k_r}(-\delta_1)_{k_1}\cdots(-\delta_r)_{k_r}}{(\beta)_{k_1+\cdots+k_r}k_1!\cdots k_r!}, \end{aligned} \tag{9}$$

$$\begin{aligned} \bar{H}_{x_1,\ldots,x_r}(\alpha,\beta) &= \frac{\Gamma(\alpha)\Gamma(\beta+\delta_1+\cdots+\delta_r)}{\Gamma(\beta)\Gamma(\alpha+\delta_1+\cdots+\delta_r)} \\ &= \sum_{k_1,\ldots,k_r=0}^{\infty} \frac{(\beta-\alpha)_{k_1+\cdots+k_r}(-\delta_1)_{k_1}\cdots(-\delta_r)_{k_r}}{(1-\alpha-\delta_1-\cdots-\delta_r)_{k_1+\cdots+k_r}k_1!\cdots k_r!} \end{aligned} \tag{10}$$

$$\left(\delta_j := x_j\frac{\partial}{\partial x_j}, j=1,...,r; r\in\mathbb{N} := \{1,2,3,...\}\right),$$

would lead us easily to several decomposition formulas associated with Humbert's hypergeometric functions $\Phi_1, \Phi_2, \Phi_3, \Psi_1, \Psi_2, \Xi_1$ and Ξ_2. The researchers (see [2], [7], [12], [13], [14], [16]) investigated several decomposition formulas for various hypergeometric functions in several variables.

Based on the operators defined in (9) and (10), we aim in this paper to introduce certain decomposition formulas for the triple hypergeometric functions F_{6a}, F_{8c}, F_{10a}, F_{10b}, F_{10d} and F_{10l}. we also obtain some transformation formulas by using these decomposition formulas.

2. Operator identities

By applying the pairs of symbolic operators (9) and (10), we begin by presenting each of the following operator formulas (11)-(22):

$$\begin{aligned} &F_{6a}\left(a_1,a_2,a_3,a_4,a_5;c_1,c_2;x,y,z\right) \\ &= H_x(a_2,c_1)\left(1-x\right)^{-a_1} F_3\left(a_1,a_4,a_3,a_5;c_2;\frac{y}{1-x},z\right), \end{aligned} \tag{11}$$

$$\begin{aligned} &\left(1-x\right)^{-a_1} F_3\left(a_1,a_4,a_3,a_5;c_2;\frac{y}{1-x},z\right) \\ &= \bar{H}_x(a_2,c_1)F_{6a}\left(a_1,a_2,a_3,a_4,a_5;c_1,c_2;x,y,z\right), \end{aligned} \tag{12}$$

$$\begin{aligned} &F_{8c}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)\\ &=H_z(a_3,c_2)\left(1-z\right)^{-b}F_4\left(a_1,a_2;1-b,c_1;-x(1-z),y\right), \end{aligned} \tag{13}$$

$$\begin{aligned} &\left(1-z\right)^{-b}F_4\left(a_1,a_2;1-b,c_1;-x(1-z),y)\right)\\ &=\bar{H}_z(a_5,c_2)F_{8c}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right), \end{aligned} \tag{14}$$

$$\begin{aligned} &F_{10a}\left(a_1,a_2,a_3,a_4;c_1,c_2,c_3;x,y,z\right)\\ &=H_z(a_4,c_3)\left(1-z\right)^{-a_2}F_2\left(a_1,a_3,a_2;c_1,c_2;x,\frac{y}{1-z}\right), \end{aligned} \tag{15}$$

$$\begin{aligned} &\left(1-z\right)^{-a_2}F_2\left(a_1,a_3,a_2;c_1,c_2;x,\frac{y}{1-z}\right)\\ &=\bar{H}_z(a_4,c_3)F_{10a}\left(a_1,a_2,a_3,a_4;c_1,c_2,c_3;x,y,z\right), \end{aligned} \tag{16}$$

$$\begin{aligned} &F_{10b}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)\\ &=H_x(a_2,c_1)\left(1-x\right)^{-b}H_1\left(b,a_1,a_3;c_2;\frac{y}{1-x},(1-x)z\right), \end{aligned} \tag{17}$$

$$\begin{aligned} &\left(1-x\right)^{-b}H_1\left(b,a_1,a_3;c_2;\frac{y}{1-x},(1-x)z\right)\\ &=\bar{H}_x(a_2,c_1)F_{10b}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right), \end{aligned} \tag{18}$$

$$\begin{aligned} &F_{10d}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)\\ &=H_z(a_3,c_2)\left(1-z\right)^{-a_2}H_1\left(b,a_1,a_2;c_1;x,\frac{y}{1-z}\right), \end{aligned} \tag{19}$$

$$\begin{aligned} &\left(1-z\right)^{-a_2}H_1\left(b,a_1,a_2;c_1;x,\frac{y}{1-z}\right)\\ &=\bar{H}_z(a_3,c_2)F_{10d}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right), \end{aligned} \tag{20}$$

$$\begin{aligned} &F_{10l}\left(a_1,a_2,b_1,b_2;c;x,y,z\right)\\ &=H_x(a_2,c)\left(1-x\right)^{-b_1}G_1\left(a_1,b_2,b_1;\frac{y}{1-x},(1-x)z\right), \end{aligned} \tag{21}$$

$$\begin{aligned} &\left(1-x\right)^{-b_1}G_1\left(a_1,b_2,b_1;\frac{y}{1-x},(1-x)z\right)\\ &=\bar{H}_x(a_2,c)F_{10l}\left(a_1,a_2,b_1,b_2;c;x,y,z\right), \end{aligned} \tag{22}$$

where

$$F_2\left(a,b,c;d,e;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(c)_n}{(d)_m(e)_n}\frac{x^m}{m!}\frac{y^n}{n!}$$

$$F_3\left(a,b,c,d;e;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_m(b)_n(c)_m(d)_n}{(e)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!} \tag{23}$$

and

$$F_4\left(a,b;c,d;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c)_m(d)_n}\frac{x^m}{m!}\frac{y^n}{n!}$$

are Appell's hypergeometric functions (see [1], [8])
and

$$G_1\left(a,b,c;x,y\right)=\sum_{m,n=0}^{\infty}(a)_{m+n}(b)_{n-m}(c)_{m-n}\frac{x^m}{m!}\frac{y^n}{n!}$$

and

$$H_1\left(a,b,c;d;x,y\right)=\sum_{m,n=0}^{\infty}\frac{(a)_{m-n}(b)_{m+n}(c)_n}{(d)_m}\frac{x^m}{m!}\frac{y^n}{n!}$$

are the Horn's hypergeometric functions (see [15], [20]).

Proof. The operator identities (11) to (22) can be easily derived by just following the method in Burchnall and Chaundy [3], [4], (see also [5], [6]). So the details of proofs are omitted.

3. Decomposition formulas

In this section, we follow the method given by Burchnall and Chaundy [3], [4], Chaundy [5], and Choi and Hasanov [6] to use the symbolic operators (9) and (10) to derive the following decomposition formulas for the Gaussian hypergeomeric functions in three variables defined in (1)-(6):

$$\begin{aligned}&F_{6a}\left(a_1,a_2,a_3,a_4,a_5;c_1,c_2;x,y,z\right)=(1-x)^{-a_1}\sum_{i=0}^{\infty}\frac{(-1)^i(a_1)_i(c_1-a_2)_i}{(c_1)_i i!}\\&\times\left(\frac{x}{1-x}\right)^i F_3\left(a_1+i,a_4,a_3,a_5;c_2;\frac{y}{1-x},z\right),\end{aligned}\tag{24}$$

$$\begin{aligned}&(1-x)^{-a_1}F_3\left(a_1,a_4,a_3,a_5;c_2;\frac{y}{1-x},z\right)\\&=\sum_{i=0}^{\infty}\frac{(a_1)_i(c_1-a_2)_i}{(c_1)_i i!}x^iF_{6a}\left(a_1+i,a_2,a_3,a_4,a_5;c_1+i,c_2;x,y,z\right),\end{aligned}\tag{25}$$

$$\begin{aligned}&F_{8c}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)=(1-z)^{-b}\sum_{i=0}^{\infty}\frac{(-1)^i(c_2-a_3)_i(b)_i}{(c_2)_i i!}\\&\times\left(\frac{z}{1-z}\right)^i F_4\left(a_1,a_2;1-b-i,c_1;-x(1-z),y\right),\end{aligned}\tag{26}$$

$$\begin{aligned}&(1-z)^{-b} F_4\left(a_1, a_2; 1-b, c_1; -x(1-z), y\right)\\&=\sum_{i=0}^{\infty}\frac{(c_2-a_3)_i(b)_i}{(c_2)_i i!} z^i F_{8c}\left(a_1, a_2, a_3, b+i; c_1, c_2+i; x, y, z\right),\end{aligned}\tag{27}$$

$$\begin{aligned}&F_{10a}\left(a_1, a_2, a_3, a_4; c_1, c_2, c_3; x, y, z\right)=(1-z)^{-a_2}\sum_{i=0}^{\infty}\frac{(-1)^i(a_2)_i(c_3-a_4)_i}{(c_3)_i i!}\\&\times\left(\frac{z}{1-z}\right)^i F_2\left(a_1, a_3, a_2+i; c_1, c_2; x, \frac{y}{1-z}\right),\end{aligned}\tag{28}$$

$$\begin{aligned}&(1-z)^{-a_2} F_2\left(a_1, a_3, a_2; c_1, c_2; x, \frac{y}{1-z}\right)\\&=\sum_{i=0}^{\infty}\frac{(a_2)_i(c_3-a_4)_i}{(c_3)_i i!} z^i F_{10a}\left(a_1, a_2+i, a_3, a_4; c_1, c_2, c_3+i; x, y, z\right),\end{aligned}\tag{29}$$

$$\begin{aligned}&F_{10b}\left(a_1, a_2, a_3, b; c_1, c_2; x, y, z\right)=(1-x)^{-b}\sum_{i=0}^{\infty}\frac{(-1)^i(c_1-a_2)_i(b)_i}{(c_1)_i i!}\\&\times\left(\frac{x}{1-x}\right)^i H_1\left(b+i, a_1, a_3; c_2; \frac{y}{1-x}, (1-x)z\right),\end{aligned}\tag{30}$$

$$\begin{aligned}&(1-x)^{-b} H_1\left(b, a_1, a_3; c_2; \frac{y}{1-x}, (1-x)z\right)\\&=\sum_{i=0}^{\infty}\frac{(c_1-a_2)_i(b)_i}{(c_1)_i i!} x^i F_{10b}\left(a_1, a_2, a_3, b+i; c_1+i, c_2; x, y, z\right),\end{aligned}\tag{31}$$

$$\begin{aligned}&F_{10d}\left(a_1, a_2, a_3, b; c_1, c_2; x, y, z\right)=(1-z)^{-a_2}\sum_{i=0}^{\infty}\frac{(-1)^i(a_2)_i(c_2-a_3)_i}{(c_2)_i i!}\\&\times\left(\frac{z}{1-z}\right)^i H_1\left(b, a_1, a_2+i; c_1; x, \frac{y}{1-z}\right),\end{aligned}\tag{32}$$

$$\begin{aligned}&(1-z)^{-a_2} H_1\left(b, a_1, a_2; c_1; x, \frac{y}{1-z}\right)\\&=\sum_{i=0}^{\infty}\frac{(a_2)_i(c_2-a_3)_i}{(c_2)_i i!} z^i F_{10d}\left(a_1, a_2+i, a_3, b; c_1, c_2+i; x, y, z\right),\end{aligned}\tag{33}$$

$$
\begin{aligned}
&F_{10l}\left(a_1, a_2, b_1, b_2; c; x, y, z\right) = (1-x)^{-b_1} \sum_{i=0}^{\infty} \frac{(-1)^i (c-a_2)_i (b_1)_i}{(c)_i i!} \\
&\times \left(\frac{x}{1-x}\right)^i G_1\left(a_1, b_2, b_1+i; \frac{y}{1-x}, (1-x)z\right),
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
&(1-x)^{-b_1} G_1\left(a_1, b_2, b_1; \frac{y}{1-x}, (1-x)z\right) \\
&= \sum_{i=0}^{\infty} \frac{(c-a_2)_i (b_1)_i}{(c)_i i!} x^i F_{10l}\left(a_1, a_2, b_1+i, b_2; c+i; x, y, z\right).
\end{aligned} \tag{35}
$$

Proof. we begin by recalling the following formulas (see [19, p.93]):

$$
(-\delta)_n\{f(\xi)\} = (-1)^n \xi^n \frac{d^n}{d\xi^n}\{f(\xi)\}, \tag{36}
$$

$$
(\alpha+\delta)_n\{f(\xi)\} = \xi^{1-\alpha} \frac{d^n}{d\xi^n}\{\xi^{\alpha+n-1} f(\xi)\}, \tag{37}
$$

$$
\left(\delta := \xi \frac{d}{d\xi}; \alpha \in \mathbb{C}; n \in \mathbb{N}_0 := \{0, 1, 2, ...\}\right),
$$

where $f(\xi)$ is analytic function.
Let us prove the expansion formula (24). It is not difficult to show that the equality

$$
(1-x)^{-a_1} F_3\left(a_1, a_4, a_3, a_5; c_2; \frac{y}{1-x}, z\right) = \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n} (a_3)_n (a_4)_p (a_5)_p}{(c_2)_{n+p} m! n! p!} x^m y^n z^p. \tag{38}
$$

By taking into account operator (9) and relation (38), we can write the operaton identity (11) in form

$$
\begin{aligned}
&F_{6a}\left(a_1, a_2, a_3, a_4, a_5; c_1, c_2; x, y, z\right) \\
&= \sum_{i=0}^{\infty} \frac{(c_1-a_2)_i (-\delta_x)_i}{(c_1)_i i!} \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n} (a_3)_n (a_4)_p (a_5)_p}{(c_2)_{n+p} m! n! p!} x^m y^n z^p.
\end{aligned} \tag{39}
$$

Using the differentiation formula for the hypergeometric formula, from (39) we get

$$
\begin{aligned}
&(-\delta_x)_i \sum_{m,n,p=0}^{\infty} \frac{(a_1)_{m+n} (a_3)_n (a_4)_p (a_5)_p}{(c_2)_{n+p} m! n! p!} x^m y^n z^p \\
&= (-1)^i x^i (a_1)_i \sum_{m,n,p=0}^{\infty} \frac{(a_1+i)_{m+n} (a_3)_n (a_4)_p (a_5)_p}{(c_2)_{n+p} m! n! p!} x^m y^n z^p.
\end{aligned} \tag{40}
$$

Now, substituting (40) into (39) and from the definition of function F_3 (23), we have the result (24). Expansion formulas (25)-(35) can be proven in a similar way.

4. Transformation formulas

In this section, we give some transformation formulas for the Gaussian triple functions as follows:

$$\begin{aligned}&F_{6a}\left(a_1,a_2,a_3,a_4,a_5;c_1,c_2;x,y,z\right)\\&=(1-x)^{-a_1}F_{6a}\left(a_1,c_1-a_2,a_3,a_4,a_5;c_1,c_2;\frac{x}{x-1},\frac{y}{1-x},z\right),\end{aligned}\tag{41}$$

$$\begin{aligned}&F_{8c}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)\\&=(1-z)^{-b}F_{8c}\left(a_1,a_2,c_1-a_3,b;c_1,c_2;x(1-z),y,\frac{z}{z-1}\right),\end{aligned}\tag{42}$$

$$\begin{aligned}&F_{10a}\left(a_1,a_2,a_3,a_4;c_1,c_2,c_3;x,y,z\right)\\&=(1-z)^{-a_2}F_{10a}\left(a_1,a_2,a_3,c_3-a_4;c_1,c_2,c_3;x,\frac{y}{1-z},\frac{z}{z-1}\right),\end{aligned}\tag{43}$$

$$\begin{aligned}&F_{10b}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)\\&=(1-x)^{-b}F_{10b}\left(a_1,c_1-a_2,a_3,b;c_1,c_2;\frac{x}{x-1},\frac{y}{1-x},(1-x)z\right),\end{aligned}\tag{44}$$

$$\begin{aligned}&F_{10d}\left(a_1,a_2,a_3,b;c_1,c_2;x,y,z\right)\\&=(1-z)^{-a_2}F_{10d}\left(a_1,a_2,c_2-a_3,b;c_1,c_2;x,\frac{y}{1-z},\frac{z}{z-1}\right),\end{aligned}\tag{45}$$

$$\begin{aligned}&F_{10l}\left(a_1,a_2,b_1,b_2;c;x,y,z\right)\\&=(1-x)^{-b_1}F_{10l}\left(a_1,c-a_2,b_1,b_2;c;\frac{x}{x-1},\frac{y}{1-x},(1-x)z\right).\end{aligned}\tag{46}$$

Proof. The results (41)-(46) follow easily from the expansion formulas (24)-(35). So details of proofs are omitted.

References

[1] P. Appell and J. Kampé de Fériet, *Fonctions Hypergeometriques et Hyperspheriques; Polynomes d'Hermite*, Gauthier - Villars, Paris, 1926.

[2] M. Bin-Saad, A. Hasanov and M. Turaev, *Decomposition formulas of Kampé de Fériet double hypergeometric functions*, Anal. Theory Appl., 34, (2018), 275-292.

[3] J.L. Burchnall and T.W. Chaundy, *Expansions of Appell's double hypergeometric functions*, Quart. J. Math. Oxford Ser. 11 (1940), 249-270.

[4] J.L. Burchnall and T.W. Chaundy, *Expansions of Appell's double hypergeometric functions II*, Quart. J. Math. Oxford Ser. 12 (1941), 112-128.

[5] T.W. Chaundy, *Expansions of hypergeometric functions*, Quart. J. Math. Oxford Ser. 13 (1942), 159-171.

[6] J. Choi and A. Hasanov, *Applications of the operator $H(\alpha,\beta)$ to the Humbert double hypergeometric functions*, Comput. Math. Appl., 61 (2011), 663-671.

[7] J. Choi and A. Hasanov, *Certain decomposition formulas of generalized hypergeometric functions ${}_pF_q$ and some formulas of an analytic continuation of the Clausen function ${}_3F_2$*, Commun. Koren. Math. Soc. , 27(1) (2012), 107-116.

[8] A. Erdélyi , W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions*, Vol. I, McGraw-Hill Book Company, New York, Toronto and London, 1953.

[9] F.I. Frankl, *Selected Works in Gas Dynamics*, Nauka, Moscow, 1973 (in Russian).

[10] A. Hasanov and H.M. Srivastava, *Some decomposition formulas associated with the Lauricella function $F_A^{(r)}$ and other multiple hypergeometric functions*, App. Math. Lett., 19 (2006), 113-121.

[11] A. Hasanov and H.M. Srivastava, *Decomposition formulas associated with the Lauricella multivariable hypergeometric functions*, Comput. Math. Appl., 53 (2007), 1119-1128.

[12] A. Hasanov, H. M. Srivastava and M. Turaev, *Decomposition formulas for some triple hypergeometric functions*, J. Math. Anal. Appl., 324 (2006), 955-969.

[13] A. Hasanov and H.M. Srivastava, *Decomposition formulas associated with the Lauricella multivariable hypergeometric functions*, Comput. Math. Appl., 53 (2007), 1119-1128.

[14] A. Hasanov and M. Turaev, *Decomposition formulas for the double hypergeometric functions G_1 and G_2*, Appl. Math. Comput., 187 (2007), 195-201.

[15] J. Horn, *Ueber die Convergenz der Hypergeometrische Reihen Zweier und Dreier Veränderlichen*, Math. Ann., 34 (1889), 544-600.

[16] Y.S. Kim, A. Hasanov and C.H. Lee, *Some decomposition formulas associated with the Saran function F_E*, Honam Math. J. 32(4) (2010), 581-592.

[17] G. Lohöfer, *Theory of an electromagnetically deviated metal sphere. I: Absorbed power*, SIAM J. Appl. Math., 49 (1989), 567-581.

[18] A.W. Niukkanen, *Generalised hypergeometric series ${}^NF(x_1,...,x_N)$ arising in physical and quantum chemical applications*, J. Phys. A: Math. Gen., 16 (1983), 1813-1825.

[19] E.G. Poole, *Introduction to the Theory of Linear Differential Equations*, Clarendon (Oxford University) Press, Oxford, 1936.

[20] H.M. Srivastava and P.W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood Lt1., Chichester, 1984.

Anvar Hasanov
Institute of Mathematics, Uzbek Academy of Sciences,
29, F. Hodjaev Street, Tashkent 100125, Uzbekistan
Department of Mathematics, Analysis,
Logic and Discrete Mathematics Ghent University,
Ghent, Belgium
email: *anvarhasanov@yahoo.com*

Jihad A. Younis
Aden University
Department of Mathematics
Aden, Khormaksar, P.O.Box 6014,Yemen
email: *jihadalsaqqaf@gmail.com*