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THE DIFFERENTIABLE H-CONVEX FUNCTIONS INVOLVING THE BULLEN INEQUALITY

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Abstract. In this article, the author established a new identity for differentiable functions and obtained some new inequalities for differentiable functions based on $h-$convex functions including Bullen-type inequalities. Also, some applications are given special means for arbitrary positive numbers.

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1. Introduction

"A function $\varphi : I \to R$, where $I \subseteq R$ is an interval, is said to be a convex function on I if

$$\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0,1]$. If the reversed inequality in (1) holds, then φ is concave [5].

Let $\varphi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following double inequalities:

$$\varphi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \varphi(x)\,dx \leq \frac{\varphi(a)+\varphi(b)}{2} \tag{HH}$$

hold. This double inequalities are known in the literature as the Hermite-Hadamard inequality for convex functions.Many important inequalities are established for the class of convex functions, but one of the most famous is so called Hermite-Hadamard's inequality (or Hadamard's inequality). For the development and use of this inequality, in recent years many authors established several inequalities connected to this fact. For recent results, refinements, counterparts, generalizations and new Hermite-Hadamard type inequalities see [1]-[14].

The following inequality is well known in the literature as Bullen's inequality [8];

$$\frac{1}{b-a}\int_a^b \varphi(\varkappa)\,d\varkappa \leq \frac{1}{2}\left[\frac{\varphi(a)+\varphi(b)}{2}+\varphi\left(\frac{a+b}{2}\right)\right], \tag{B}$$

provided that $\varphi:[a,b]\to\mathbb{R}$ is a convex function on $[a,b]$.

[5, 15] We say that $\varphi: I\to\mathbb{R}$ is a P-function or that φ belongs to the class $P(I)$ if φ is nonnegative and for all $x,y\in I$ and $t\in[0,1]$, we have

$$\varphi(tx+(1-t)y) \leq \varphi(x)+\varphi(y) \tag{2}$$

Let $s\in(0,1]$. A function $\varphi:(0,\infty]\to[0,\infty]$ is said to be s-convex in the first and second sense, respectively, if

$$\begin{aligned} \varphi(tx+(1-t)y) &\leq t^s\varphi(x)+(1-t^s)\varphi(y), \\ \varphi(tx+(1-t)y) &\leq t^s\varphi(x)+(1-t)^s\varphi(y), \end{aligned} \tag{3}$$

for all $x,y\in(0,\infty]$ and $t\in[0,1]$. The classes of s-convex functions is usually denoted by K_s^1, K_s^2 [4, 9].

[1, 16]Let $h:J\to\mathbb{R}$ be a nonnegative function, $h\not\equiv 0$. We say that $\varphi: I\to\mathbb{R}$ is an h-convex function, or that φ belongs to the class $SX(h,I)$, if φ is nonnegative and for all $x,y\in I$ and $t\in(0,1)$ we have

$$\varphi(tx+(1-t)y) \leq h(t)\varphi(x)+h(1-t)\varphi(y). \tag{4}$$

If inequality (4) is reversed, then φ is said to be h-concave, i.e. $\varphi\in SV(h,I)$. Obviously, if $h(t)=t$, then all nonnegative convex functions belong to $SX(h,I)$ and all nonnegative concave functions belong to $SV(h,I)$; if $h(t)=1$, then $SX(h,I)\supseteq P(I)$; and if $h(t)=t^s$, where $s\in(0,1)$, then $SX(h,I)\supseteq K_s^2$.

If φ is integrable on $[a,b]$, then the average value of φ on $[a,b]$ is

$$\frac{1}{b-a}\int_a^b \varphi(x)dx. \tag{5}$$

[3]$\varphi: I\subset R^+\to R$ be an differentiable function on I° with $0\leq \imath_1<\imath_2$ and $q\geq 1$. If $|\varphi'|^q$ be a convex function on I, then the following inequality holds:

$$\begin{aligned} &\left|\frac{1}{\imath_2-\imath_1}\int_{\imath_1}^{\imath_2}\varphi(x)\,dx-\frac{1}{2}\left[\frac{\varphi(\imath_1)+\varphi(\imath_2)}{2}+\varphi\left(\frac{\imath_1+\imath_2}{2}\right)\right]\right| \\ \leq\ & \frac{\imath_2-\imath_1}{8}\left(\frac{1}{2}\right)^{1+\frac{1}{q}}\times\left[\left(\left(\varphi'(\imath_1)\right)^q+\left(\varphi'\left(\frac{\imath_1+\imath_2}{2}\right)\right)^q\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\left(\varphi'\left(\frac{\imath_1+\imath_2}{2}\right)\right)^q+\left(\varphi'(\imath_2)\right)^q\right)^{\frac{1}{q}}\right]. \end{aligned} \tag{6}$$

In the present paper, Bullen inequality is developed by the means of using h-convexity. In the following section, the new identity is established for differentiable functions and new inequalities are obtained for h-convex functions involving Bullen inequality. Then some corollaries and remarks are given for a different type of convex functions. In the application section; it is given applications of the results from the main section for some special means."

2. Main Results

Lemma 1. *Let $\varphi : I \to \mathbb{R}$, $I \subset \mathbb{R}$ be a differentiable mapping on I°, and $a, b \in I, a < b$. If $\varphi' \in L[a.b]$, $t \in [0,1]$, $c \in (0,\infty)$ and $t \leq c$ then*

$$\begin{aligned} & \frac{4c^2}{b-a}\left(\frac{\varphi(a)+\varphi(b)}{2}+\varphi\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b \varphi(x)\,dx\right) \qquad (7) \\ = & \int_0^c (c-2t)\left[\left(\varphi'\left(\frac{t}{c}a+\left(1-\frac{t}{c}\right)\frac{a+b}{2}\right)+\varphi'\left(\frac{t}{c}\frac{a+b}{2}+\left(1-\frac{t}{c}\right)b\right)\right)\right]dt. \end{aligned}$$

Here I° denotes the interior of I.

Proof. It is obvious by utilizing integrating by parts.

Theorem 2. *Let $I \subset [0,\infty)$, $\varphi : I \to R$ be a differentiable function on I° such that $\varphi^{'} \in L^1[a,b]$, where $a, b \in I$ with $a < b$, $t \in [0,1]$, $c \in (0,\infty)$ and $t \leq c$. If $|\varphi'|^q$ is h-convex on $[a,b]$ for $q \geq 1$, then the following inequalities hold*

$$\begin{aligned} & \frac{4c^2}{b-a}\left|\frac{\varphi(a)+\varphi(b)}{2}+\varphi\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b \varphi(x)\,dx\right| \\ \leq & \left(\frac{c^2}{2}\right)^{1-\frac{1}{q}} \\ & \times\left[\left(\left|\varphi'(a)\right|^q\int_0^c |c-2t|\,h\left(\frac{t}{c}\right)dt+\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q\int_0^c |c-2t|\,h\left(1-\frac{t}{c}\right)dt\right)^{\frac{1}{q}}\right. \\ & \left.+\left(\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q\int_0^c |c-2t|\,h\left(\frac{t}{c}\right)dt+\left|\varphi'(b)\right|^q\int_0^c |c-2t|\,h\left(1-\frac{t}{c}\right)dt\right)^{\frac{1}{q}}\right]. \end{aligned}$$

Proof. By Lemma 1 and using the power-mean inequality, we have

$$\begin{aligned} & \frac{4c^2}{b-a}\left|\frac{\varphi(a)+\varphi(b)}{2}+\varphi\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b \varphi(x)\,dx\right| \\ = & \left|\int_0^c (c-2t)\left[\left(\varphi'\left(\frac{t}{c}a+\left(1-\frac{t}{c}\right)\frac{a+b}{2}\right)+\varphi'\left(\frac{t}{c}\frac{a+b}{2}+\left(1-\frac{t}{c}\right)b\right)\right)\right]dt\right|. \end{aligned}$$

$$\leq \int_0^c |c-2t| \left| \varphi'\left(\frac{t}{c}a + \left(1-\frac{t}{c}\right)\frac{a+b}{2}\right) + \varphi'\left(\frac{t}{c}\frac{a+b}{2} + \left(1-\frac{t}{c}\right)b\right)\right| dt$$

$$\leq \left(\int_0^c |c-2t|\, dt\right)^{1-\frac{1}{q}} \left(\int_0^c |c-2t| \left|\varphi'\left(\frac{t}{c}a + \left(1-\frac{t}{c}\right)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+ \left(\int_0^c |c-2t|\, dt\right)^{1-\frac{1}{q}} \left(\int_0^c |c-2t| \left|\varphi'\left(\frac{t}{c}\frac{a+b}{2} + \left(1-\frac{t}{c}\right)b\right)\right|^q dt\right)^{\frac{1}{q}} \Bigg\}$$

$$\leq \left(\frac{c^2}{2}\right)^{1-\frac{1}{q}}$$

$$\times \left[\left(\left|\varphi'(a)\right|^q \int_0^c |c-2t|\, h\left(\frac{t}{c}\right) dt + \left|\varphi'\left(\frac{a+b}{2}\right)\right|^q \int_0^c |c-2t|\, h\left(1-\frac{t}{c}\right) dt\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q \int_0^c |c-2t|\, h\left(\frac{t}{c}\right) dt + \left|\varphi'(b)\right|^q \int_0^c |c-2t|\, h\left(1-\frac{t}{c}\right) dt\right)^{\frac{1}{q}}\right],$$

where

$$\int_0^c |c-2t|\, dt = \frac{c^2}{2}.$$

Corollary 3. *Let $I \subset [0,\infty)$, $\varphi : I \to R$ be a differentiable function on I° such that $\varphi' \in L^1[a,b]$, where $a,b \in I$ with $a<b$. If $|\varphi'|^q$ is h-convex on $[a,b]$ for some fixed $t \in [0,1]$ and $q=1$, then the following inequalities hold*

$$\begin{aligned} &\frac{4}{b-a}\left|\frac{\varphi(a)+\varphi(b)}{2} + \varphi\left(\frac{a+b}{2}\right) - \frac{2}{b-a}\int_a^b \varphi(x)\,dx\right| \\ \leq\ & \left(\left|\varphi'(a)\right| + \left|\varphi'\left(\frac{a+b}{2}\right)\right|\right)\int_0^1 |1-2t|\, h(t)\, dt \\ &+ \left(\left|\varphi'(b)\right| + \left|\varphi'\left(\frac{a+b}{2}\right)\right|\right)\int_0^1 |1-2t|\, h(1-t)\, dt. \end{aligned}$$

for $c=1$.

Proof. Corollary 3 is immediately deduced by setting $q=1$ in Theorem 2 for $c=1$.

Corollary 4. *If $|\varphi'|^q$ is s-convex in the second sense on $[a,b]$ for some fixed $s \in (0,1]$*

and $q \geq 1$, then the following inequality holds:

$$
\begin{aligned}
& \frac{4}{b-a}\left|\frac{\varphi(a)+\varphi(b)}{2}+\varphi\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b \varphi(x)\,dx\right| \\
\leq\ & \left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{s}{(s+1)(s+2)}\right)^{\frac{1}{q}} \\
& \times\left[\left(\left|\varphi'(a)\right|^q+\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q+\left|\varphi'(b)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Proof. By Lemma 1 and using the power-mean inequality, we have

$$
\begin{aligned}
& \frac{4c^2}{b-a}\left|\frac{\varphi(a)+\varphi(b)}{2}+\varphi\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b \varphi(x)\,dx\right| \\
=\ & \left|\int_0^c (c-2t)\left(\varphi'\left(\frac{t}{c}a+\left(1-\frac{t}{c}\right)\frac{a+b}{2}\right)+\varphi'\left(\frac{t}{c}\frac{a+b}{2}+\left(1-\frac{t}{c}\right)b\right)\right)dt\right| \\
\leq\ & \int_0^c |c-2t|\left|\varphi'\left(\frac{t}{c}a+\left(1-\frac{t}{c}\right)\frac{a+b}{2}\right)+\varphi'\left(\frac{t}{c}\frac{a+b}{2}+\left(1-\frac{t}{c}\right)b\right)dt\right| \\
\leq\ & \left(\int_0^c |c-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^c |c-2t|\left|\varphi'\left(\frac{t}{c}a+\left(1-\frac{t}{c}\right)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}} \\
& +\left(\int_0^c |c-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^c |c-2t|\left|\varphi'\left(\frac{t}{c}\frac{a+b}{2}+\left(1-\frac{t}{c}\right)b\right)\right|^q dt\right)^{\frac{1}{q}} \\
=\ & \left(\frac{c^2}{2}\right)^{1-\frac{1}{q}}\left(\int_0^c |c-2t|\left(\left(\frac{t}{c}\right)^s\left|\varphi'(a)\right|^q+\left(1-\frac{t}{c}\right)^s\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q\right)dt\right)^{\frac{1}{q}} \\
& +\left(\frac{c^2}{2}\right)^{1-\frac{1}{q}}\left(\int_0^c |c-2t|\left(\left(\frac{t}{c}\right)^s\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q+\left(1-\frac{t}{c}\right)^s\left|\varphi'(b)\right|^q\right)dt\right)^{\frac{1}{q}} \\
=\ & \left(\frac{c^2}{2}\right)^{1-\frac{1}{q}}\left(\frac{c^2 s}{s^2+3s+2}\left|\varphi'(a)\right|^q+\frac{c^2 s}{s^2+3s+2}\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} \\
& +\left(\frac{c^2}{2}\right)^{1-\frac{1}{q}}\left(\frac{c^2 s}{s^2+3s+2}\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q+\frac{c^2 s}{s^2+3s+2}\left|\varphi'(b)\right|^q\right)^{\frac{1}{q}},
\end{aligned}
$$

where

$$
\begin{aligned}
\int_0^c |c-2t|\,dt &= \frac{1}{2}c^2, \\
\int_0^c |c-2t|\left(\frac{t}{c}\right)^s dt &= \int_0^c |c-2t|\left(1-\frac{t}{c}\right)^s dt = \frac{c^2 s}{s^2+3s+2}.
\end{aligned}
$$

Remark 1. *If we take $s = 1$ in Corollary (4), then we get a Bullen type inequality*

$$\begin{aligned} &\frac{8}{b-a}\left|\frac{\varphi(a)+\varphi(b)}{2}+\varphi\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b \varphi(x)\,dx\right| \qquad (8) \\ \leq &\left(\frac{1}{3}\right)^{\frac{1}{q}}\left[\left(\left|\varphi'(a)\right|^q+\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q+\left|\varphi'(b)\right|^q\right)^{\frac{1}{q}}\right]. \end{aligned}$$

Remark 2. *Inequality (8) gives better results than inequality (6). In other words, the inequality (8) has a smaller upper bound than the inequality (6) in terms of determining the upper bound.*

Corollary 5. *Under the assumption of Theorem 2, if $|\varphi'|^q$ is $P(I)$, the following inequality holds:*

$$\begin{aligned} &\frac{8}{b-a}\left|\frac{\varphi(a)+\varphi(b)}{2}+\varphi\left(\frac{a+b}{2}\right)-\frac{2}{b-a}\int_a^b \varphi(x)\,dx\right| \\ \leq &\left(\left|\varphi'(a)\right|^q+\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|\varphi'\left(\frac{a+b}{2}\right)\right|^q+\left|\varphi'(b)\right|^q\right)^{\frac{1}{q}}. \end{aligned}$$

where

$$\int_0^c |c-2t|\,dt = \frac{c^2}{2}.$$

Proof. Proof of Corollary 5 is explicit by choosing $h(t) = 1$ in Theorem 2.

3. Applications to Some Special Means

We consider the means for arbitrary positive numbers a, b $(a \neq b)$ as follows;

The arithmetic mean:

$$A(a,b) = \frac{a+b}{2}$$

The generalized log-mean:

$$L_p(a,b) = \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}},\ p \in \mathbb{R} \setminus \{-1,0\}.$$

Now, by using the result of second Section, we give some applications to some special means of real numbers.

Proposition 1. *Let* $0 < a < b$, $s \in (0,1)$. *Then the following inequalities hold:*

$$\begin{aligned} & \left| A\left(a^s, b^s\right) + A^s\left(a,b\right) - 2L_s^s\left(a,b\right) \right| \\ \leq \; & 2^{\frac{1}{q}-3}\left(b-a\right)\left(\frac{s^{q+1}}{\left(s+1\right)\left(s+2\right)}\right)^{\frac{1}{q}} \\ & \times \left[\left(\left|a^{s-1}\right|^q + \left| \left(\frac{a+b}{2}\right)^{s-1} \right|^q \right)^{\frac{1}{q}} + \left(\left| \left(\frac{a+b}{2}\right)^{s-1} \right|^q + \left|b^{s-1}\right|^q \right)^{\frac{1}{q}} \right]. \end{aligned}$$

Proof. The inequalities are derived from Corollary 4 applied to the s-convex functions $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi\left(x\right) = x^s$, $s \in (0,1)$, $x \in [a,b]$ and $\varphi'\left(x\right) = sx^{s-1}$, $s \in (0,1)$, $x \in [a,b]$. The details are disregarded.

Proposition 2. *Let* $0 < a < b$. *Then the following inequalities hold:*

$$\begin{aligned} & \left| A\left(a^n, b^n\right) + A^n\left(a,b\right) - 2L_n^n\left(a,b\right) \right| \\ \leq \; & \frac{n\left(b-a\right)}{8}\left(\left|a^{n-1}\right|^q + \left| \left(\frac{a+b}{2}\right)^{n-1} \right|^q \right)^{\frac{1}{q}} + \left(\left| \left(\frac{a+b}{2}\right)^{n-1} \right|^q + \left|b^{n-1}\right|^q \right)^{\frac{1}{q}}. \end{aligned}$$

Proof. In Corollary 5, if we choose $\varphi(x) = x^n$, $\varphi^{'}(x) = nx^{n-1}$, the proof is obvious. The details are disregarded.

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