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Series of Inscribed n-Gons and Rank 3 Configurations

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Abstract. In the paper we study configurations which can be represented as families of cyclically inscribed n-gons. The most regular of them arise from quasi difference sets distinguished in a product of two cyclic groups, but some other more general techniques which define series of inscribed n-gons are found and studied. We give conditions which assure the existence of certain automorphisms of the defined configurations. The automorphism groups of configurations arising from quasi difference sets is established.

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1. Introduction, basic notions and definitions

In the paper we study configurations which can be represented as families of cyclically inscribed n-gons.

Ideas are rather simple: we have k copies $W_0, \ldots, W_{k-1}$ of an n-gon. The vertices of W_j are $p_{j,i}$, where $i = 0, \ldots, n-1$, and for every i two vertices $p_{j,i}$ and $p_{j,i+1}$ are joined with a side $l_{j,i}$. To inscribe W_{j+1} into W_j we need a function f_j which assigns to a side $l_{j,i}$ of W_j the vertex $p_{j+1,f_j(i)}$ of W_{j+1}, which we put on this side. The points of the obtained structure are all vertices of the given n-gons, and its "lines" are sets of the form $\{p_{j,i}, p_{j,i+1}, p_{j+1,f_j(i)}\}$. Thus the family $F = \{f_j \colon j = 0, \ldots, k-1\}$ defines a structure, which can be interpreted as a series of cyclically inscribed n-gons. The point is to characterize families F of functions, which yield sufficiently regular partial linear spaces (of line rank 3), and to characterize the obtained structures.

It is more convenient to consider given n-gon as the cyclic group C_n and to define functions f_j in terms of algebraic operations. Then most of the geometric questions can be formulated and solved within simple algebra. Thus, formally, given a sequence $F = (f_j\colon j = 1, \dots, k-1)$ of bijections of C_n we set

$$\circledast_F C_n = \langle C_k \times C_n, \mathcal{L}_F \rangle,$$

where $\mathcal{L}_F = \big\{\{(j,i),(j,i+1),(j+1,f_j(i))\}\colon i \in C_n,\ j \in C_k\big\}$. Let us note an evident

Fact 1.1. *Let f_j be a bijection of C_n for every $j \in C_k$ and let $F = (f_j\colon j \in C_k)$. If $3 \le n, k$ then the structure $\circledast_F C_n$ is a partial linear space with nk lines and nk points, each one of the rank 3.*

A standard example of a configuration, which can be represented in this way is the projective Pappos configuration, defined as $\circledast_F C_3$ with $f_j = id_{C_3}$ for $j \in C_3$ (see [1] and Section 4 for more details).

There are two basic questions investigated in the paper. First – which of the natural transformations of an n-gon W_0 can be extended to an automorphism of the whole configuration? And then, what is the automorphism group of the considered configuration? Many interesting results concerning automorphism groups of configurations are to be found in [2].

In the paper we follow some standard notation of the theory of partial linear spaces. Let $\mathfrak{A} = \langle X, \mathcal{L}\rangle$ with $\mathcal{L} \subseteq \wp(X)$ be a partial linear space. For $a, b \in X$ write $a \sim b$ if a and b are collinear, i.e. if there is $l \in \mathcal{L}$ with $a, b \in l$. If $a \sim b$ and $a \neq b$ we write $\overline{a,b}$ for the (unique!) line which joins a and b.

Given an arbitrary group, we shall follow a standard "multiplicative" notation. If the considered group is abelian, we shall use "additive" notation. Given a group G we denote by τ_a the left translation in G, defined by $\tau_a(x) = a \cdot x$. One more general construction will be needed. Let us consider an arbitrary finite group G and let $D \subset G$. We set

$$\mathcal{L} = \mathcal{L}_{(G,D)} = G/D = \{a \cdot D\colon a \in G\}.$$

Clearly, $\mathcal{L} \subseteq \wp(G)$, and, as the left translation $\tau_a\colon G \ni x \mapsto a \cdot x \in G$ is a bijection, $|a \cdot D| = |D|$ for every $a \in G$. Following this notation we can write $a \cdot D = \tau_a(D)$, and $\mathcal{L}_{(G,D)} = \{\tau_a(D)\colon a \in G\}$. We set

$$\mathbf{D}(G,D) = \langle G, \mathcal{L}_{(G,D)}\rangle.$$

Note that an arbitrary n-gon can be represented as $\mathbf{D}(C_n, \{0,1\})$.

Fact 1.2. *The automorphism group* $\mathrm{Aut}(\mathbf{D}(C_n,\{0,1\}))$ *is the dihedral group D_n, i.e. the group of all the maps f of C_n of the form $f(i) = \varepsilon i + a$, where $\varepsilon = 1, -1$.*

Proposition 1.3. *For every $a \in G$ the translation τ_a is an automorphism of the structure $\mathbf{D}(G,D)$.*

Proof. It suffices to note that $\tau_a(b \cdot D) = (a \cdot b) \cdot D$. □

Proposition 1.3 yields, in particular, that $\mathbf{D}(G,D)=\mathbf{D}(G,q\cdot D)$ for every $q\in G$. Therefore, without loss of generality, we can always assume that $1\in D$.

Proposition 1.4. *If* $\varphi\in\mathrm{Aut}(G)$ *then* $\varphi\in\mathrm{Aut}(\mathbf{D}(G,D))$ *iff* $\varphi(D)=q\cdot D$ *for some* $q\in G$.

Proof. For every $a\in G$ we need to find $b\in G$ with $\varphi(a\cdot D)=\varphi(a)\cdot\varphi(D)=b\cdot D$. This yields $\varphi(D)=((\varphi(a))^{-1}\cdot b)\cdot D$. Conversely, if $\varphi(D)=q\cdot D$, for a given $a\in G$ we set $b=\varphi(a)\cdot q$ and we are through. □

2. Cyclic inscribed series of polygons

In this section we shall develop "series" $\circledast_F C_n$ of n-gons suitably cyclically inscribed one into the previous one. Recall that point $p_{j+1,f_j(i)}$, a point of the $(j+1)$-th polygon, completes the i-th side of a j-th n-gon to the line

$$\overline{p_{j,i},p_{j,i+1}}=\left\{p_{j,i},p_{j,i+1},p_{j+1,f_j(i)}\right\}$$

of the considered configuration. Formally, we deal with $C_k\times C_n$, but here the ring-structure of C_n is more exploited.

Now, we shall pay attention to some special classes of permutations f_j, namely to "linear" maps of the ring C_n defined by

$$f_j(i)=q_j\cdot i+b_j, \tag{1}$$

where $q_j,b_j\in C_n$, with $\mathrm{GCD}(q_j,n)=1$ for all $j\in C_k$; this assumption assures that each one of the maps f_j is a bijection of C_n. In view of 1.1, if F consists of linear bijections then the structure $\circledast_F C_n$ is a partial linear space, in fact – a configuration.

If $q_j=q$, $b_j=b$ are fixed we write $k\circledast_{(q,b)}C_n=\circledast_{F(q,b)}C_n$, where $F(q,b):=(f_j\colon j=0,\dots,k-1)$.

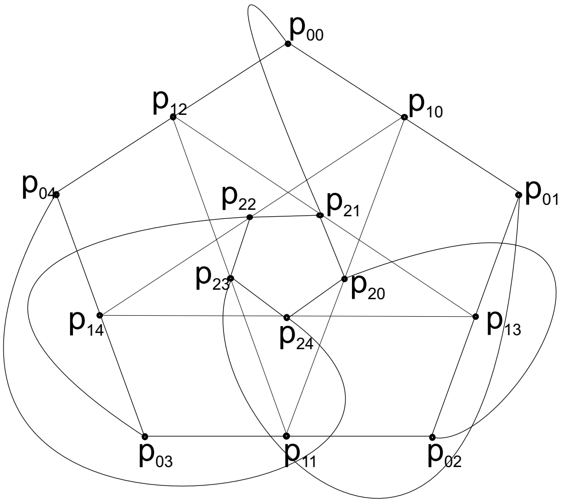

Figure 1: $3\circledast_{(-2,0)}C_5$

In the sequel we shall determine when, given some natural automorphism ψ of an n-gon C_n, the map $(0,i) \mapsto (0,\psi(i))$ of $\{0\} \times C_n \subseteq C_k \times C_n$ can be extended to an automorphism $\widetilde{\psi}$ of $\circledast_F C_n$, where F is a sequence of bijections of C_n. In short we say that ψ can be extended to $\widetilde{\psi}$. Then $\widetilde{\psi}$ is determined by a family of bijections g_j of C_n such that

$$\widetilde{\psi}(j,i) = (j, g_j(i)), \tag{2}$$

where $g_0 = \psi$. If $\widetilde{\psi}$ is an automorphism then it maps each two collinear points (j,i) and $(j,i+1)$ onto collinear ones, so for every j the map g_j is a collineation of C_n and thus from 1.2 there exist $\alpha_j \in \{1,-1\}$ and $c_j \in C_n$ such that

$$g_j(i) = \alpha_j \cdot i + c_j. \tag{3}$$

Lemma 2.1. *If $\widetilde{\psi} \in \operatorname{Aut}(\circledast_F C_n)$ is defined by* (2)*, where g_j are given by* (3) *then the following recursive formula holds*

$$\alpha_{j+1} \cdot f_j(i) + c_{j+1} = \begin{cases} f_j(\alpha_j \cdot i + c_j) & \text{for } \alpha_j = 1 \\ f_j(\alpha_j \cdot i + c_j - 1) & \text{for } \alpha_j = -1 \end{cases} . \tag{4}$$

If the system (4) *determines a periodic solution $\alpha_k = \alpha_0$, $c_k = c_0$ then $\widetilde{\psi}$ defined by* (2), (3) *is an automorphism of $\circledast_F C_n$.*

Proof. Note that the map $\psi = g_0$ uniquely determines $\widetilde{\psi}$. We have

$$L_{j,i} = \overline{(j,i),(j,i+1)} \xrightarrow{\widetilde{\psi}} \overline{(j,\alpha_j i + c_j),(j,\alpha_j(i+1)+c_j)} = L'_{j,i}.$$

The third point of $L_{j,i}$ is $(j+1, f_j(i))$, and then the third point of $L'_{j,i}$ is $(j+1, \alpha_{j+1} \cdot f_j(i) + c_{j+1})$ which, on the other hand is either $(j+1, f_j(\alpha_j i + c_j))$ if $\alpha_j = 1$, or $(j+1, f_j(\alpha_j i + c_j - 1))$ if $\alpha_j = -1$. This proves the claim. □

As an immediate consequence of 2.1 we have

Corollary 2.2. *If elements of F are determined by* (1) *then the following recursive formula characterizes a map $\widetilde{\psi}$ which is defined by* (2), (3) *and extends g_0:*

$$\alpha_{j+1} b_j + c_{j+1} = \begin{cases} q_j \cdot c_j + b_j & \text{for } \alpha_0 = 1 \\ q_j \cdot (c_j - 1) + b_j & \text{for } \alpha_0 = -1 \end{cases} \tag{5}$$

with $\alpha_j = \alpha_0$, for $j \in C_k$.

Proposition 2.3. *Let F consist of maps defined by* (1) *and $a \in C_n$. The following conditions are equivalent:*

(i) *the translation τ_a of C_n can be extended to an automorphism φ of $\circledast_F C_n$,*
(ii) $\prod_{s=0}^{k-1} q_s \cdot a = a$.

If (ii) *holds then φ is defined by*

$$(j,i) \xrightarrow{\varphi} (j, \tau_{a_j}(i)), \text{ where } a_0 = a,\ a_{j+1} = \prod_{s=0}^{j} q_s \cdot a. \tag{6}$$

Proof. Substituting in (5) $\alpha_0 = 1$ and $c_0 = a$, we get $c_{j+1} = q_j\, c_j$, and formulas (2), (3) give (6). By 2.2, we need $c_k = c_0$, i.e. $a = \prod_{s=0}^{k-1} q_s \cdot a$, which is our claim. □

Immediately we obtain

Corollary 2.4. *A translation τ_a of C_n can be extended to an automorphism φ of $k \circledast_{(q,b)} C_n$ iff $aq^k = a$ in C_n. If $aq^k = a$ then φ is defined by*

$$(j, i) \xmapsto{\varphi} (j, \tau_{q^j \cdot a}(i)). \tag{7}$$

With similar methods we prove

Proposition 2.5. *Let F consist of maps defined by* (1)*, and $a \in C_n$. Define recursively the sequence $a_0 = a$, $a_{j+1} = q_j a_j - q_j + 2b_j$ for arbitrary j. The following conditions are equivalent:*

(i) *the symmetry $\sigma_a\colon i \mapsto -i + a$ of C_n can be extended to an automorphism φ of $\circledast_F C_n$,*

(ii) *the equality $a = a_k$ holds in C_n.*

If (ii) *holds then φ is given by*

$$\varphi(j, i) = (j, \sigma_{a_j}(i)). \tag{8}$$

Proof. Let φ be an automorphism of $\circledast_F C_n$ which extends σ_a. After substitution $\alpha_0 = -1$ and $c_0 = a = a_0$, from (5) we get $c_{j+1} - b_j = q_j c_j - q_j + b_j$, which gives (8) with $c_j = a_j$. With $j = k$ from 2.2 we obtain $a = a_k$, which is our claim. □

Corollary 2.6. *Let $a \in C_n$. The symmetry $\sigma_a\colon i \mapsto -i + a$ of C_n can be extended to an automorphism φ of $k \circledast_{(q,b)} C_n$ iff the equality $\sum_{s=1}^{k} q^s - 2b \sum_{s=0}^{k-1} q^s = aq^k - a$ holds in C_n. The automorphism φ is given by*

$$\varphi(j, i) = (j, \sigma_{aq^j - \sum_{s=1}^{j} q^s + 2b \sum_{s=0}^{k-1} q^s}(i)). \tag{9}$$

Proof. It suffices to note that, in accordance with notation of 2.5, $a_j = aq^j - \sum_{s=1}^{j} q^s + 2b \sum_{s=0}^{k-1} q^s$. □

Let f_j be a bijection of C_n for $j = 0, \dots, k-1$ and let $F = (f_0, \dots, f_{k-1})$. In the sequel we shall investigate "spiral" automorphisms of $\mathfrak{F} = \circledast_F C_n$ of the form

$$\varphi(j, i) = (j + 1, g_j(i)), \tag{10}$$

where g_j is a bijection of C_n for $j = 0, \dots, k-1$. Since φ has to be an automorphism, each g_j must be defined by the formula (3), for some $\alpha_j \in \{1, -1\}$ and $c_j \in C_n$.

Lemma 2.7. *If $\varphi \in \operatorname{Aut}(\circledast_F C_n)$ is defined by* (10)*, where g_j are given by* (3) *then the following recursive formulae hold:*

$$f_{j+1}(i) = \begin{cases} \alpha_{j+1} \cdot f_j(i - c_j) + c_{j+1} & \text{for } \alpha_j = 1 \\ \alpha_{j+1} \cdot f_j(-i + c_j - 1) + c_{j+1} & \text{for } \alpha_j = -1 \end{cases} \tag{11}$$

for $j = 0, \dots, k-1$, where $f_{(k-1)+1} = f_k = f_0$.

Proof. With a standard reasoning we have

$$L := \overline{(j,i),(j,i+1)} \overset{\varphi}{\longmapsto} \overline{(j+1,\alpha_j\cdot i+c_j),(j+1,\alpha_j\cdot(i+1)+c_j)} =: L'.$$

The third point of L is $(j+1,f_j(i))$ and if $\alpha_j=1$ then the third point on L' is $(j+2,f_{j+1}(i+c_j))$, if $\alpha_j=-1$ then the third point of L' is $(j+2,f_{j+1}(-i-1+c_j))$. Thus

$$(j+1,f_j(i)) \overset{\varphi}{\longmapsto} \begin{cases} (j+2,f_{j+1}(i+c_j)) & \text{for } \alpha_j=1\\ (j+2,f_{j+1}(-i-1+c_j)) & \text{for } \alpha_j=-1 \end{cases},$$

which yields the required formula. □

Now, with a simple substitution in the equation (11) of 2.7 we obtain

Corollary 2.8. *Let functions f_j be defined by the "linear" formulas* (1) *and let φ be defined by* (10)*, where g_j are given by* (3)*. If $\varphi\in\mathrm{Aut}(\circledast_F C_n)$ then the following recursive formula holds:*

$$q_{j+1} = \begin{cases} \alpha_{j+1}q_j & \textit{if } \alpha_j=1\\ -\alpha_{j+1}q_j & \textit{if } \alpha_j=-1 \end{cases}, \tag{12}$$

and

$$b_{j+1} = \begin{cases} c_{j+1}+\alpha_{j+1}b_j-\alpha_{j+1}q_jc_j & \textit{if } \alpha_j=1\\ c_{j+1}+\alpha_{j+1}b_j+\alpha_{j+1}q_j(c_j-1) & \textit{if } \alpha_j=-1 \end{cases}. \tag{13}$$

Consequently, a map defined by (10), (3) *can be an automorphism of $\circledast_F C_n$ only if $q_{j+1}=\pm q_j=\pm q_0$. Conversely, if* (12), (13) *determine periodic solution $c_k=c_0$, $\alpha_k=\alpha_0$ with $q_k=q_0$, $b_k=b_0$ then φ is an automorphism of $\circledast_F C_n$.*

3. Simple series

From among all the structures $\circledast_F C_n$ we distinguish some more regular with the following observation.

Proposition 3.1. *Let $\mathfrak{F}=\circledast_F C_n$, for a sequence $F=(f_j\colon j=0,\dots,k-1)$ of bijections of C_n. The following conditions are equivalent:*

(i) *If l_1,l_2 are two sides of the n-gon $\{j\}\times C_n$, p_1,p_2 are two vertices of the n-gon $\{j+1\}\times C_n$, p_i is on l_i for $i=1,2$, and l_1,l_2 have a common vertex then p_1,p_2 are collinear.*

(ii) *$f_j\in D_n$ for every $j\in C_k$.*

Proof. Two arbitrary sides l_1,l_2 as above are of the form

$$l_1=\overline{(j,i_1),(j,i_1+1)} \text{ and } l_2=\overline{(j,i_2),(j,i_2+1)},$$

and then $p_1=(j+1,f_j(i_1))$, $p_2=(j+1,f_j(i_2))$. The lines l_1,l_2 have a common point if $i_2-i_1=\pm1$, and p_1,p_2 are collinear if $f_j(i_2)-f_j(i_2)=\pm1$. Thus the map f_j must be a collineation of $\mathbf{D}(C_n,\{0,1\})$ and, by 1.2, $f_j\in D_n$, as required. □

It is seen that if each element f_j of F is in D_n, then there are $b \in C_n$ and $\varepsilon \in \{1, -1\}$ such that $\mathfrak{F} \cong \circledast_{F'} C_n$, where $F = (f'_0, \ldots, f'_{k-1})$ and

$$f'_j(i) = i \text{ for } j = 0, \ldots, k-2, \text{ and } f'_{k-1}(i) = \varepsilon \cdot i + b. \tag{14}$$

Configurations determined by such families of bijections will be referred to as *simple configurations.*

One can notice that the structure $k \circledast_{(q,b)} C_n$, with $q \in \{-1, 1\}$, is a simple configuration and, when constructed up to $(k-1)$-th level, coincides with the structure $\mathbf{D}(C_k \oplus C_n, \mathcal{D})$ defined in Section 4; the only difference lies in the way of labeling points, but not in their geometrical arrangement. More formally, this can be stated as follows.

Proposition 3.2. *Let* $\mathfrak{F} = k \circledast_{(q,b)} C_n$ *with* $q \in \{-1, 1\}$. *Define* $f_j(i) = i$ *for* $i \in C_n$ *and* $j = 0, \ldots, k-2$, *and* $f_{k-1}(i) = \varepsilon i + \widehat{b}$, *where*

- (i) *if* $q = 1$ *then* $\varepsilon = 1$, $\widehat{b} = k \cdot b$,
- (ii) *if* $q = -1$ *and* $k = 2s$ *then* $\varepsilon = 1$, $\widehat{b} = s$, *and*
- (iii) *if* $q = -1$ *and* $k = 2s + 1$ *then* $\varepsilon = -1$, $\widehat{b} = b - s$.

Then $\mathfrak{F} \cong \circledast_F C_n$, *where* $F = (f_j \colon j = 0, \ldots, k-1)$.

Proof. Let $q = 1$. Let us consider the map φ (a new labeling of points) defined by

$$\varphi(j, i) = (j, i - j \cdot b).$$

Clearly, φ is a bijection. Let $L_{j,i} = \overline{(j,i), (j, i+1)} = \{(j,i), (j, i+1), (j+1, i+b)\}$ be a line of $\mathfrak{F}$ with $j < k-1$. Note that the following holds:

Comp: if (j, i'), $(j, i'+1)$ are new labels of the points (j,i), $(j, i+1)$, and $(j+1, i'')$ is on the line $L_{j,i}$ then its new label is $(j+1, i')$,

as required.

Then we have $\varphi(k-1, i) = (k-1, i-(k-1)b)$, and $\varphi(k-1, i+1) = (k-1, i-(k-1)b+1)$. The third point of $\overline{(k-1, i), (k-1, i+1)}$ is $(0, i+b)$. This gives $f_{k-1}(i - (k-1)b) = i + b$ in the new labeling. From this we calculate $f_{k-1}(i) = i + kb$.

Now, let $q = -1$. We define a bijection ψ with the formula

$$\psi \colon C_k \times C_n \ni (j, i) \mapsto \begin{cases} (j, i - s) & \text{for } j = 2s \text{ – even} \\ (j, -i + b - s) & \text{for } j = 2s + 1 \text{ – odd.} \end{cases}$$

Take a line $L = L_{j,i} = \{(j,i), (j, i+1), (j+1, -i+b)\}$ of $\mathfrak{F}$. Let j be even, $j = 2s$. Then ψ maps L onto the set

$$\psi(L) = \{(j, i-s), (j, i+1-s), (j+1, -(-i+b)+b-s)\} = (j, i-s) + \mathcal{D}.$$

If $j = 2s+1$ is odd then $j + 1 = 2(s+1)$ and we have

$$\psi(L) = \{(j, -i+b-s), (j, -(i+1)+b-s), (j+1, (-i+b)-(s+1))\} = (j, -i+b-s-1) + \mathcal{D}.$$

Clearly, Comp holds for the re-labeling defined by ψ.

To determine f_{k-1} we consider two cases. If $k-1=2s$ then the new labels of the points $(k-1,i)$ and $(k-1,i+1)$ are $(k-1,i-s)$ and $(k-1,i-s+1)$ resp., from which we obtain $f_{k-1}(i-s)=-i+b$. This yields $f_{k-1}(i)=-i+b-s$.

Analogously, if $k-1=2s+1$ (i.e. $k=2(s+1)$) we infer $f_{k-1}(-i+b-s-1)=-i+b$, which gives $f_{k-1}(i)=i+(s+1)$. $\square$

Let us take a look into the three 9-points configurations 9_3 (cf. [1]) for a moment. One can note that the Pappos configuration $(9_3)_1$ is represented as $3 \circledast_{(1,0)} C_3$, the configuration $(9_3)_3$ is represented as $3 \circledast_{(-1,2)} C_3$, or – in view of 3.2 – as $\circledast_F C_3$, where $f_0 = f_1 = id_{C_3}$ and $f_2(i) = -i+1$, and the configuration $(9_3)_2$ is represented as $\circledast_F C_3$ with $f_0 = f_1 = id_{C_3}$, $f_2(i) = i+1$.

Immediately from 2.3 and 2.5 we find conditions which assure that a map in D_n, i.e. a translation or a symmetry of C_n, can be extended to a collineation of a simple configuration which arises as $\circledast_F C_n$.

Proposition 3.3. *Let $\mathfrak{F} = \circledast_F C_n$, where F consists of functions f'_j defined by* (14)*, and let $\varepsilon \in \{1,-1\}$.*

- (i) *A translation τ_a of C_n can be extended to an automorphism of $\mathfrak{F}$ iff $a = \varepsilon \cdot a$ holds in C_n.*
- (ii) *A symmetry σ_a of C_n can be extended to an automorphism of $\mathfrak{F}$ iff the equality $(1-\varepsilon)\cdot a = 2b - \varepsilon \cdot k$ holds in C_n.*

Proof. (i) From (14) we obtain $\prod_{s=0}^{k-1} q_s = \varepsilon$, so the claim follows directly from 2.3.

(ii) In accordance with notation of 2.5 we obtain $a_0 = a = a - 0$, $a_{j+1} = a_j - 1 + 0 = a - j$ for $j < k-1$, and $a_k = \varepsilon(a-(k-1)) - \varepsilon + 2b$. Thus $a = a_k$ iff $(1-\varepsilon)a = 2b - \varepsilon k$, as required. $\square$

In particular we can find conditions for extending translations and symmetries to collineations of the structures $k \circledast_{(q,b)} C_n$ with $q = -1, 1$. Note that in accordance with the criterion 2.3 every translation of C_n can be extended to an automorphism of $k \circledast_{(1,b)} C_n$ – in this case we have, evidently, $a1^k = a$ for every $a \in C_n$ and every k.

Corollary 3.4. *A symmetry $\sigma_a \colon (0,y) \mapsto (0,-y+a)$ of C_n can be extended to an automorphism of $k \circledast_{(1,b)} C_n$ iff $k(1-2b) = 0 \mod n$.*

Proof. In accordance with 3.2 and 3.3 we need $(1-1)a = 2kb - 1 \cdot k$, which is the claim. $\square$

Corollary 3.5. *If k is even then every translation $\tau_a \colon (0,y) \mapsto (0,y+a)$ of C_n can be extended to an automorphism of $k \circledast_{(-1,b)} C_n$. If k is odd then τ_a can be extended as above iff $2a = 0 \mod n$.*

Proof. Substituting $q=-1$ into the conditions of 2.3 we obtain the condition $a(-1)^k = a$. If k is even then this is a tautology, if k is odd we need $2a = 0 \mod n$. $\square$

Corollary 3.6. *If k is even then every symmetry $\sigma_a \colon (0,y) \mapsto (0,-y+a)$ of C_n can be extended to an automorphism of $k \circledast_{(-1,b)} C_n$; if k is odd then σ_a can be extended as above iff $2(a-b) = 1 \mod n$.*

Proof. Let $q = -1$. If $k = 2s$ then in accordance with 3.2 we have $\varepsilon = 1$ and $\widehat{b} = s$. Then using 3.3 we should require $(1-1)a = 2\widehat{b} - 1 \cdot k$, which is a tautology. If $k = 2s+1$ we have $\varepsilon = -1$ and $\widehat{b} = b - s$; the requirement $(1-(-1))a = 2(b-s) - (-1) \cdot k$ of 3.3 gives the claim. □

Now, we shall find "spiral" automorphisms of simple configurations.

Proposition 3.7. *Let* $\mathfrak{F} = \circledast_F C_n$, *where* F *consists of functions* f'_j *defined by* (14) *with* $\varepsilon \in \{1, -1\}$. *Then the map* $(0, i) \mapsto (1, \alpha_0 \cdot i + c_0)$ *can be extended to an automorphism* φ *of* $\mathfrak{F}$ *iff one of the following holds:*

(i) *if* $\alpha_0 = 1$ *then* $\varepsilon = 1$, *or* $\varepsilon = -1$, $2c_0 = -1$;

(ii) *if* $\alpha_0 = -1$, *then* $\varepsilon = 1$, $2b = k$, *or* $\varepsilon = -1$, $2b = 2c_0 - k + 1$.

In such a case φ *is defined by* (10), (3) *with*

$$\alpha_j = \alpha_0 \quad \textit{and} \quad c_j = \begin{cases} c_0 & \textit{if } \alpha_0 = 1 \\ c_0 - j & \textit{if } \alpha_0 = -1 \end{cases} \quad \textit{for } j < k-1,$$

$$\alpha_{k-1} = \varepsilon\alpha_0 \quad \textit{and} \quad c_{k-1} = \begin{cases} b + \varepsilon c_0 & \textit{if } \alpha_0 = 1 \\ b - \varepsilon(c_0 - k + 1) & \textit{if } \alpha_0 = -1 \end{cases}.$$

Proof. We substitute $q_0 = \dots = q_{k-2} = 1$, $b_0 = \dots = b_{k-2} = 0$, $q_{k-1} = \varepsilon$, $b_{k-1} = b$ in the equations (12) and (13) of 2.8. The required map φ exists if (12) and (13) yield $q_k = 1 = q_0$ and $b_k = 0 = b_0$, with $\alpha_k = \alpha_0$, $c_k = c_0$.

We obtain, consecutively, for $j+1 < k-1$: $\alpha_{j+1} = \alpha_j = \dots = \alpha_0$, and then $c_{j+1} = c_j = c_0$ if $\alpha_0 = 1$, or $c_{j+1} = c_j - 1 = c_0 - (j+1)$ if $\alpha_0 = -1$.

Set $\alpha_0 = 1$. For $j+1 = k-1$ from (12) we get $\alpha_{k-1} = \varepsilon$, and then (13) yields $c_{k-1} = b + \varepsilon c_0$.

Finally, we apply 2.8 for $j = k-1$ (now $j+1 = 0$). Then from (12) we obtain $q_k = 1 = q_0$, as required. Let $\varepsilon = 1$, then (13) with $b_k = b_0$ yield a tautology. Let $\varepsilon = -1$; then (13) with $b_k = b_0 = 0$ gives $0 = 2c_0 + 1$.

Now, suppose that $\alpha_0 = -1$. For $j+1 = k-1$ the formula (12) gives $\alpha_{k-1} = -\varepsilon$ and (13) gives $c_{k-1} = b - \varepsilon(c_0 - k + 1)$.

For $j+1 = k$ from (12) we have $q_k = 1 = q_0$, as required. If $\varepsilon = 1$ then $\alpha_{k-1} = -1$, so for $j+1 = k$ from (13) we obtain $b_k = k - 2b$. Analogously, if $\varepsilon = -1$ with (13) we obtain $b_k = 2c_0 - 2b - k + 1$. □

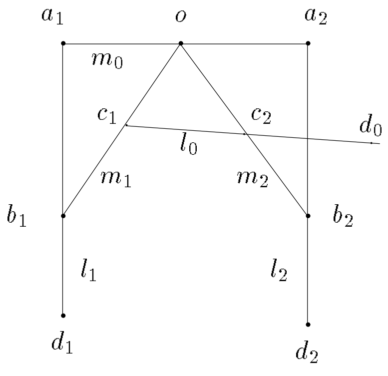

Figure 2: Neighborhood of a point o

Proposition 3.8. *Let* $\mathfrak{F} = \circledast_F C_n$ *be a simple configuration, let* $3 < k$*, and let* $o = (j, i)$ *be its point. Then the structure determined by points which are collinear with* o *can be visualized on the Figure* 2*. If* $n > 3$ *then no other incidence besides those indicated in the figure holds. If* $n = 3$ *then* $d_1 = d_2$ *and points* b_1, b_2 *are collinear.*

Proof. Set $b_1 = (j, i-1)$, $b_2 = (j, i+1)$, $c_1 = (j+1, \varepsilon_j(i-1) + a_j)$, $c_2 = (j+1, \varepsilon_j i + a_j)$. Then o, b_1, c_1 are on a line m_1 and o, b_2, c_2 are on a line m_2 of $\mathfrak{F}$. The points c_1 and c_2 are collinear and the third point of the line $l_0 = \overline{c_1, c_2}$ is d_0,

$$d_0 = \begin{cases} (j+1, \varepsilon_{j+1}(\varepsilon_j i + a_j - 1) + a_{j+1}) & \text{if } \varepsilon_j = 1 \\ (j+1, \varepsilon_{j+1}(\varepsilon_j i + a_j) + a_{j+1}) & \text{if } \varepsilon_j = -1 \end{cases}.$$

Let $i = \varepsilon_{j-1} i' + a_{j-1}$, then o is on the third line m_3, which has points o, $a_1 = (j-1, i')$, and $a_2 = (j-1, i'+1)$. Consider points $d_1 = (j-1, i'-1)$ and $d_2 = (j-1, i'+2)$, and lines $l_1 = \overline{d_1, a_1}$ and $l_2 = \overline{d_2, a_2}$. Let p_1, p_2 be the third points of the corresponding lines: $p_1 = (j, \varepsilon_{j-1}(i'-1) + a_{j-1})$ and $p_2 = (j, \varepsilon_{j-1}(i'+1) + a_{j-1})$. Note $\varepsilon_{j-1}(i'-1) + a_{j-1} = i - \varepsilon_{j-1}$ and $\varepsilon_{j_1}(i'+1) + a_{j-1} = i + \varepsilon_{j-1}$. Thus if $\varepsilon_{j-1} = 1$ we get $p_1 = b_1$ and $p_2 = b_2$; if $\varepsilon_{j-1} = -1$ then $p_1 = b_2$ and $p_2 = b_1$.

If $3 < n$ then no other collinearity can appear. If $n = 3$ then $i'_1 = i' + 2$ and thus $d_1 = d_2$. Moreover, in this case b_1, b_2 are collinear and the third point of the line $\overline{b_1, b_2}$ is $(j+1, \varepsilon_j(i+1) + a_j)$. □

From now on we assume that $3 < k, n$. Note that the following holds in every simple configuration:

Lemma 3.9. *If* q *is a point collinear with* o*,* $o \neq q$ *then there is exactly one point* q' *collinear with* o *such that* $q' \neq o$ *and* (o, q, q') *is a triangle.*

From this we get the rigidity of the automorphism group of the investigated configuration:

Proposition 3.10. *Let* $f \in \mathrm{Aut}(\mathfrak{F})$*, where* $\mathfrak{F}$ *is a simple configuration. If* $f \restriction m = id_m$ *for some line* m *of* $\mathfrak{F}$ *then* $f = id$*.*

Proof. Let us take into account the schema 2, and assume that $f \restriction m_0 = id_{m_0}$, so $f(o) = o$, and $f(a_i) = a_i$ for $i = 1, 2$. With the help of 3.9 we obtain $f(b_i) = b_i$ for $i = 1, 2$, and then $f(c_i) = c_i$. Note that, consequently, $f(d_i) = d_i$ for $i = 0, 1, 2$. Thus we proved that $f(q) = q$ for all the points collinear with o, and for every such a point q there is a line l through q such that $f \restriction l = id_l$. Then, inductively, we get $f(q) = q$ for every point of the configuration. □

As an immediate consequence we infer that if f, g are collineations and $f(q_i) = g(q_i)$ for $i = 0, 1, 2$, for three distinct and collinear points q_0, q_1, q_2, then $f = g$.

The structure formed by the points collinear with a given point o in a simple configuration, visualised in Figure 2, can be, formally, defined by the following incidence matrix:

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
 & o & a_1 & a_2 & b_1 & b_2 & c_1 & c_2 & d_0 & d_1 & d_2 \\ \hline
m_0 & \times & \times & \times & & & & & & & \\ \hline
m_1 & \times & & & \times & & \times & & & & \\ \hline
m_2 & \times & & & & \times & & \times & & & \\ \hline
l_0 & & & & & & \times & \times & \times & & \\ \hline
l_1 & & \times & & \times & & & & & \times & \\ \hline
l_2 & & & \times & & \times & & & & & \times
\end{array}\right]$$

This – small – configuration has a very regular automorphism group. Let us write Z_o for the set of all points collinear with o and distinct from o. With a careful use of 3.9 and 3.10 we can calculate

Proposition 3.11. *The group $\mathfrak{O}_{(o)}$ of collineations of the incidence structure visualized in Figure* 2 *consists of the maps defined in Table* 1. *The group $\mathfrak{O}_{(o)}$ is isomorphic to S_3: each transformation f given in Table* 1 *is uniquely determined by images of the points d_0, d_1, d_2, and by images of the lines m_0, m_1, m_2 (or the lines l_0, l_1, l_2) as well. On the other hand, f is also determined by an image $f(q)$ of just one point $q \in Z_o$.*

map	a_1	a_2	b_1	b_2	c_1	c_2	d_1	d_2	d_0	m_0	m_1	m_2	l_0	l_1	l_2
id	a_1	a_2	b_1	b_2	c_1	c_2	d_1	d_2	d_0	m_0	m_1	m_2	l_0	l_1	l_2
σ'	a_2	a_1	b_2	b_1	c_2	c_1	d_2	d_1	d_0	m_0	m_2	m_1	l_0	$_2$	l_1
σ''	c_2	b_2	c_1	a_2	b_1	a_1	d_0	d_2	d_1	m_2	m_1	m_0	l_1	l_0	l_2
σ'''	b_1	c_1	a_1	c_2	a_2	b_2	d_1	d_0	d_2	m_1	m_0	m_2	l_2	l_1	l_0
ρ'	c_1	b_1	c_2	a_1	b_2	a_2	d_0	d_1	d_2	m_1	m_2	m_0	l_2	l_0	l_1
ρ''	b_2	c_2	a_2	c_1	a_1	b_1	d_2	d_0	d_1	m_2	m_0	m_1	l_1	l_2	l_0

Table 1: Collineations of a neighborhood of a point o

Clearly, if f is a collineation of a simple configuration which fixes a point o then $f \restriction Z_{(o)} \in \mathfrak{O}_{(o)}$. In the sequel we shall determine which of the maps $\sigma', \sigma'', \sigma''', \rho', \rho''$ defined in Table 1 can be extended to a collineation of the investigated simple configurations for various points o and labelling of the points collinear with o.

4. Quasi difference sets and associated configurations

Let G be a finite group and $D \subseteq G$. The technique of difference sets (cf. [3]) can be successfully used to produce partial linear spaces, not necessarily being linear spaces.

Proposition 4.1. *Let $n = |D|$. The structure $\mathbf{D}(G, D)$ is a partial linear space iff for every $c \in G \setminus \{1\}$, either*

C1: *there is no pair $(a, b) \in D \times D$ with $ab^{-1} = c$, or*

C2: *there is the unique pair $(a, b) \in D \times D$ with $ab^{-1} = c$, or*

C3: *there are exactly n pairs $(a, b) \in D \times D$ with $ab^{-1} = c$.*
If this is the case then the number v of points is $v = |G|$, the number b of lines is $b = \frac{|G|}{|G_D|}$, where $G_D = \{q \in G\colon q \cdot D = D\}$ is the stabilizer of D in G, the rank $\varkappa$ of each line is $\varkappa = |D|$, and the rank λ of each point is $\lambda = \frac{|D|}{|G_D|}$.

Proof. Set $\mathfrak{D} = \mathbf{D}(G, D)$. In view of 1.3, it suffices to give conditions which assure that $|D \cap (q \cdot D)| \geq 2$ yields $D = q \cdot D$. Note that $a \in D \cap (q \cdot D)$ means that $a' = q \cdot a \in D$, so every point $a \in D \cap (q \cdot D)$ corresponds to a pair $a, a' \in D$ with $a'a^{-1} = q$. Since $\mathfrak{D}$ should be a partial linear space, each set $D \cap (q \cdot D)$ must be empty, a one-element set or be the set D. The values of the parameters of $\mathfrak{D}$ are evident. □

In the sequel we are mainly interested in *configurations*, i.e. in partial linear spaces with constant and equal point rank and line rank ($\varkappa = \lambda$ and $v = b$). In view of 4.1, to this aim we need $|G_D| = 1$.

Proposition 4.2. *Let $\mathfrak{D} = \mathbf{D}(G, D)$. The following conditions are equivalent:*

(i) *$\mathfrak{D}$ is a configuration.*

(ii) *For every $c \in G \setminus \{1\}$ there is at most one pair $(a, b) \in D \times D$ with $ab^{-1} = c$.*

If G is abelian then (ii) *is necessary and sufficient for $\mathfrak{D}$ to be nontrivial in the following sense: through each point there pass at least two lines.*

Proof. Clearly, (ii) implies that (C1) or (C2) of 4.1 holds for every $c \in G \setminus \{1\}$, which yields: (ii) $\Longrightarrow$ (i). Assume (C3) holds for some $c \neq 1$ and consider two pairs $(a, a'), (b, b') \in D \times D$ with $c = a'a^{-1} = b'b^{-1}$. Then $D = c \cdot D$ so $|G_D| > 1$ and $\mathfrak{D}$ is not a configuration. This proves (i) $\Longrightarrow$ (ii).

Now, let G be abelian. Clearly, if $\mathfrak{D}$ is a configuration then it is nontrivial. Assume that $\mathfrak{D}$ is nontrivial, let $D = \{a_1, a_2, \ldots, a_n\}$. Suppose that (C3) holds for some $c \neq 1$, then $D = c \cdot D$, so for every $i = 1, \ldots, n$ there is j_i with $a_i = c \cdot a_{j_i}$ (note: $a_i \neq a_{j_i}$!) Consider $q_i = a_1 {a_i}^{-1}$ for $i = 1, \ldots, n$. Then $q_i = a_{j_1} {a_{j_i}}^{-1}$, so from 4.1 we get $D = q_i \cdot D$. Thus $|G_D| \geq n$, and $\lambda \leq 1$, which contradicts assumptions. □

As a tricky consequence we obtain

Corollary 4.3. *If G is an abelian group then $\mathbf{D}(G, D)$ is a partial linear space with point rank at least two iff it is a configuration.*

Note that the condition (ii) of 4.2 can be, informally, read as follows: all the differences ab^{-1} for $a, b \in D$, $a \neq b$ are pairwise distinct.

As known examples of configurations which are defined in this way we can quote Fano Configuration $\mathbf{D}(C_7, \{0, 1, 3\})$ (cf. [3] or [2]) which can be represented as a self-inscribed 7-gon, and Pappos Configuration $\mathbf{D}(C_3 \oplus C_3, \{(0, 0), (0, 1), (1, 0)\})$ (cf. [1], [2], and Section 5), which can be represented as three cyclically inscribed triangles.

5. Series determined by quasi difference sets

Now, let us consider the group G being the direct sum of two cyclic groups $G = C_k \oplus C_n$, let $\mathcal{D} = \{(0,0),(0,1),(1,0)\}$. Set $\mathfrak{W} = \mathbf{D}(G,\mathcal{D})$. Evidently, $\mathfrak{W} = k \circledast_{(1,0)} C_n$ and $\mathfrak{W} \cong \mathbf{D}(C_n \oplus C_k,\mathcal{D}) = n \circledast_{(1,0)} C_k$. This yields

Fact 5.1. *The structure $\mathfrak{W}$ can be considered as an n-gon k times inscribed cyclically into itself, or a k-gon inscribed into itself n times.*

Note that the line $l_{j,i} = \overline{(j,i),(j,i+1)}$ of $\mathfrak{W}$ coincides with $(j,i)+\mathcal{D}$.

Proposition 5.2. *The structure $\mathfrak{W}$ is self-dual, its involutory correlation can be defined by the formula*

$$(j,i) \mapsto l_{-j,-i}.$$

Proof. If $(r,s) \in l_{j,i}$ then $(r,s) = (j,i)+d$ for some $d \in \mathcal{D}$. Then $(-j,-i) = (-r,-s)+d$, which proves the claim. □

Lemma 5.3. *Every point (j,i) of $\mathfrak{W}$ is collinear with exactly 6 distinct points $(j+1,i)$, $(j,i+1)$, $(j-1,i)$, $(j-1,i+1)$, $(j,i-1)$, and $(j+1,i-1)$. Two points $a=(j,i)$ and $b=(r,s)$ of $\mathfrak{W}$ are collinear iff $a-b$ is one of the following: $(0,0)$, $(0,\pm1)$, $(\pm1,0)$, $(1,-1)$, or $(-1,1)$.*

Proof. Since (j,i) lies on three distinct lines $l_{j,i}$, $l_{j,i-1}$, $l_{j-1,i}$, and each one of them contains 2 points distinct from $p_{i,j}$, we have 6 points collinear with (j,i). Directly from definition, points on these three lines are of the form $(i,j)+g$, where $g = (0,0),(0,1),(1,0),(0,-1),(1,-1),(-1,0),(-1,1)$. Thus the second claim holds. □

Evidently, for every $a=(r,s) \in G$ the translation τ_a is an automorphism of $\mathfrak{W}$. In particular, the translation $\tau_{(0,1)}$ can be interpreted as a rotation of the corresponding n-gons, and $\tau_{(1,0)}$ is a "jumping" between n-gons. Note that the map $\tau_{(1,1)}$ can be interpreted as a "spiral": it changes an n-gon and, at the same time, rotates it. The map $\mu\colon x \mapsto -x$ (or, more generally, every map $\mu\tau_a$) can be considered in the structure $\mathbf{D}(C_n,\{0,1\})$ as an axial symmetry of the corresponding n-gon. Symmetries of this kind "cannot be extended" to automorphisms of $\mathfrak{W}$. This means that though translations of the group $C_n \oplus C_k$ are automorphisms of $\mathfrak{W}$, formal symmetries of this group are not geometrical automorphisms. Clearly, with 1.4 we have

Remark 5.4. If $n=k$ then the map $\jmath\colon G \ni (a,b) \mapsto (b,a)$ is an involutory automorphism of $\mathfrak{W}$.

It seems that the set $\mathcal{D} \subseteq C_n \oplus C_k$ gives the most interesting and "intuitive" way of defining series of cyclically inscribed n-gons. Clearly, defining $\mathbf{D}(G,B)$ we can always assume that $(0,0),(0,1) \in B$, since we are, in fact, interested in series of n-gons – and this is the first side of the first of them. Let us make the following trivial observations.

Remark 5.5. *Let g_1 be a generator of the group C_k and g_2 be a generator of the group C_n. Set $B = \{(0,0),(g_1,0),(0,g_2)\}$. Then $\mathfrak{W} \cong \mathbf{D}(C_k \oplus C_n, B)$.*

Proof. Under assumptions, the map f defined by $f(j,i) = (g_1 \cdot j, g_2 \cdot i)$ is an automorphism of the group $C_k \oplus C_n$, and $f(\mathcal{D}) = B$. Thus f is a required isomorphism. □

Remark 5.6. *If r is the rank of $b \in C_k$, $r < k$, and $B = \{(0,0),(0,1),(b,0)\}$ then $\mathbf{D}(C_k \oplus C_n, B)$ is a union of $\frac{k}{r}$ pairwise disjoint copies of $\mathbf{D}(C_r \oplus C_n, \mathcal{D})$.*

Proof. Consider an arbitrary point $(x,y) \in C_k \oplus C_n$. We note that points which can be joined with (x,y) by a polygonal path in $\mathbf{D}(C_k \oplus C_n, B)$ are of the form $(x + s_1 \cdot b, y + s_2 \cdot 1)$, which suffices as an argument. □

Remark 5.7. Let $B = \{(0,0),(0,1),(a,b)\}$ be a subset of $G = C_k \oplus C_n$ such that $\mathfrak{B} = \mathbf{D}(G,B)$ is a partial linear space. If $a = 0$ then $\mathfrak{B}$ is a disjoint union of n copies of $\mathbf{D}(C_k, \{0,1,b\})$.

In accordance with the geometrical intuitions, we assume that the third point of B is $(1,b)$ ($(i+1)$-th k-gon inscribed into the i-th one). Set $B_b = \{(0,0),(0,1),(1,b)\}$. One can see that $\mathbf{D}(C_k \oplus C_n, B_b) = k \circledast_{(1,b)} C_n$. As an immediate consequence of 4.2 (or 1.1) we obtain

Proposition 5.8. *For every $b \in C_k$ the structure $\mathbf{D}(G, B_b) =: \mathfrak{B}_b$ is a partial linear space, in fact – a configuration.*

Note that the structures obtained in this way need not to be distinct. In the terminology proposed now, $\mathcal{D} = B_0$. Let the maps η_1, η_2 be defined over $C_n \oplus C_n$ by

$$\eta_1(x,y) = (x, y+x) \text{ and } \eta_2(x,y) = (x+y, y).$$

Clearly, η_1 and η_2 are group automorphisms, and $\eta_1(B_b) = B_{b+1}$. Thus $(\eta_1)^b(\mathcal{D}) = B_b$, and thus $(\eta_1)^b$ is an isomorphism of $\mathbf{D}(C_n \oplus C_n, \mathcal{D})$ onto $\mathbf{D}(C_n \oplus C_n, B_b)$.

Corollary 5.9. *All structures of the form $\mathbf{D}(C_n \oplus C_n, B)$ which correspond to series of inscribed polygons, are pairwise isomorphic.*

The structure $\mathbf{D}(C_n \oplus C_n, B)$ has a lot of automorphisms. We shall briefly discuss them here. A more general case is discussed in the next section.

Proposition 5.10. *Let $n = k$, define the maps σ_1, σ_2 by the formulas*

$$\sigma_1(a,b) = (a, (-a) - b) \text{ and } \sigma_2(a,b) = ((-b) - a, b).$$

Then σ_1 and σ_2 are involutory automorphisms of $\mathfrak{W}$.

Proof. Clearly, $\sigma_i \in \mathrm{Aut}(G)$. Thus in view of 1.4 it remains to note that $\sigma_1(\mathcal{D}) = (0,-1) + \mathcal{D}$ and $\sigma_2(\mathcal{D}) = (-1,0) + \mathcal{D}$. □

Let us calculate

$$(x,y) \xmapsto{\sigma_1} (x, -(x+y)) \xmapsto{\sigma_2} (y, -(x+y)) \xmapsto{\sigma_1} (y,x)$$

and

$$(x,y) \xmapsto{\sigma_2} (-(x+y), y) \xmapsto{\sigma_1} (-(x+y), x) \xmapsto{\sigma_2} (y,x).$$

From this we obtain

$$(\sigma_1)^{\sigma_2} = \jmath = (\sigma_2)^{\sigma_1}. \tag{15}$$

Let $u = (a, b) \in G$. Analogously, we have

$$(x, y) \xmapsto{\sigma_1} (x, -(x+y)) \xmapsto{\tau_u} (x+a, -(x+y)+b) \xmapsto{\sigma_1} (x+a, y-(a+b)).$$

This can be written as follows (the dual formula is obtained analogously):

$$(\tau_u)^{\sigma_i} = \tau_{\sigma_i(u)} \text{ for } u \in G \text{ and } i = 1, 2. \tag{16}$$

Proposition 5.10 gives a method of constructing some more symmetries of $\mathfrak{W}$.

Proposition 5.11. *The composition* $\tau_{(a,b)} \circ \sigma_1$ *is an involution iff* $a = 0$ *and* $\tau_{(a,b)} \circ \sigma_2$ *is an involution iff* $b = 0$.

Proof. Let $\tau = \tau_{(a,b)}$. Clearly, $\tau \circ \sigma_i$ is an involution iff $\tau^{\sigma_i} = \tau^{-1}$. In view of (16), we need $\sigma_i(a, b) = -(a, b)$. For $i = 1$ this yields $a = 0$, and for $i = 2$ we get $b = 0$. ☐

Remark 5.12. *The map* σ_1 *is an automorphism of* $\mathbf{D}(C_k \oplus C_n, \mathcal{D})$ *if* $n|k$, *and* σ_2 *is an automorphism if* $k|n$. *The formulas* (16) *hold in corresponding cases.*

Proof. Note that to have definition of σ_1 meaningful, the conditions $i_1 = i_2 \mod n$ and $j_1 = j_2 \mod k$ should imply $i_1 + j_1 = i_2 + j_2 \mod n$. ☐

On the other hand from 3.4, we obtain immediately

Proposition 5.13. *A symmetry* $\sigma_a \colon (0, y) \mapsto (0, -y + a)$ *of* C_n *can be extended to an automorphism of* $\mathbf{D}(C_k \oplus C_n, \mathcal{D})$ *iff* $n|k$.

6. Automorphisms

Now, we shall find automorphisms of some of the incidence configurations defined earlier, arising from series of mutually inscribed n-gons.

First, we shall determine the structure of points collinear with a given one in the structure $\mathbf{D}(C_k \oplus C_n, \mathcal{D})$. Since it is a simple configuration, structure of incidence formed by these points was visualized in Figure 2 and described in 3.8.

The corresponding points are:
in $\mathbf{D}(C_k \oplus C_n, \mathcal{D})$: $o = (j, i)$, $a_1 = (j-1, i)$, $a_2 = (j-1, i+1)$, $b_1 = (j, i-1)$, $b_2 = (j, i+1)$, $c_1 = (j+1, i-1)$, $c_2 = (j+1, i)$, $d_0 = (j+2, i-1)$, $d_1 = (j-1, i-1)$, $d_2 = (j-1, i+2)$.
From now on we assume $3 < k, n$. In the following lemmas we assume that $o = (j, i)$, points $a_1, a_2, b_1, b_2, c_1, c_2$ are taken in accordance with the above list, and f is a collineation of $\mathfrak{W}$.

Lemma 6.1. *Assume that* $f(o) = o$ *and* $f \restriction Z_o = \sigma'$. *Then* f *yields a symmetry with the center* o *of the* n*-gon* $\{j\} \times C_n$. *Consequently, such a symmetry must be extendable to an automorphism of* $\mathfrak{D}$.

Proof. In view of 3.11 $f(a_1) = a_2$ and $f(b_s) = b_{3-s}$, i.e. $f(j, i-1) = (j, i+1)$. Note that our assumptions determine images under f of all the points on two lines l_1, m_1 through b_1; from this we infer $f(j, i-2) = (j, i+2)$. Then, inductively, we get $f(i, j-s) = (i, j+s)$, so f is a symmetry, as required. □

Lemma 6.2. *Assume that $f(o) = o$ and $f \restriction Z_o = \sigma''$. Then f yields a symmetry of the k-gon $C_k \times \{i-1\}$. Consequently, such a symmetry must be extendable to an automorphism of $\mathfrak{D}$.*

Proof. It suffices to interchange the role of "coordinates" i, j and make use of 3.11 to get $f(j-s, i-1) = (-j+1+s, i-1)$. □

Lemma 6.3. *Assume that $f(o) = o$ and $f \restriction Z_o = \sigma'''$. Take $i = 0 = j$. Then points (s, s) are fixed under f, and f is defined by the formula $f(j, i) = (i, j)$, so this map must be a collineation of $\mathfrak{D}$.*

Proof. In view of 3.11 f interchanges the following pairs of points: a_2, c_1; a_1, b_1; d_2, d_0, and d_1 is fixed. Thus f, $\jmath$, and σ''' coincide on the neighbor of the point o. Inductively, f and $\jmath$ coincide on every point of $\mathfrak{D}$, which yields the result. □

Lemma 6.4. *Assume that $f(o) = o$ and $f \restriction Z_o = \rho'$. Take $i = 0 = j$. Then f is defined by the formula $f(j, i) = (-(i+j), j))$, so this map must be a collineation of $\mathfrak{D}$.*

Proof. We note that the maps f, ρ', and $\sigma_1 \circ \sigma_2$ (cf. 5.10) coincide on the neighbor of the point o. By inductive argument we get our claim. □

Elementary properties of the group S_3 give that the only subgroups of the group $\mathrm{Aut}(\mathbf{D}(C_k \oplus C_n, \mathcal{D}))_o$ can be: the group $G = \{id, \sigma', \sigma'', \sigma''', \rho', \rho''\}$, a trivial group $\{id\}$, a C_2 group $\{id, \sigma^t\}$, where $t \in \{', '', '''\}$, or an alternating group $\{id, \rho', \rho''\}$, which is generated by any ρ^t with $t \in \{', ''\}$ (cf. Table 1).

As a consequence of Lemmas 6.1–6.4 we can formulate a characterization of the automorphism group $\mathrm{Aut}(\mathbf{D}(C_k \oplus C_n, \mathcal{D}))$. Recall that $C_2 \cong S_2$.

Proposition 6.5. *Let $\mathfrak{D} = \mathbf{D}(C_k \oplus C_n, \mathcal{D})$ and $\mathfrak{G} = \mathrm{Aut}(\mathfrak{D})$.*

(i) *If $n \nmid k$ and $k \nmid n$ then $|\mathfrak{G}| = nk$ and $\mathfrak{G} \cong C_k \oplus C_n$.*

(ii) *If $n | k$ and $k \neq n$ then $|\mathfrak{G}| = 2nk$.*

(iii) *] Dually, if $k | n$ and $n \neq k$ then $|\mathfrak{G}| = 2nk$.*

(iv) *In both cases* (ii) *and* (iii) *the group $\mathfrak{G}$ is the semidirect product $S_2 \rtimes (C_k \oplus C_n)$. Moreover, in the case* (ii) *if $2 \nmid k$ then $\mathfrak{G} \cong C_k \oplus D_n$, and in the case* (iii) *if $2 \nmid n$ then and $\mathfrak{G} \cong D_k \oplus C_n$.*

(v) *If $n = k$ then $|\mathfrak{G}| = 6nk$ and $\mathfrak{G}$ is the semidirect product $S_3 \rtimes (C_n \oplus C_n)$.*

Proof. Note that $\mathfrak{G}$ contains a transitive subgroup of translations, so every $F \in \mathfrak{G}$ can be written in the form $F = \tau_a \circ f$, where $f \in \mathfrak{G}_o$ fixes an arbitrary chosen point o. Thus $|\mathfrak{G}| = nk|\mathfrak{G}_o|$. Without loss of generality we can take $o = (0, 0)$.

Assume there is $f \in \mathfrak{G}_o$ such that $f \restriction Z_0 \in \{\sigma''', \rho', \rho''\}$ (cf. Table 1). From 6.3 and 6.4 we obtain $n = k$ and then $|\mathfrak{G}_o| = 6$ and elements of $\mathfrak{G}_o$ exhaust all the maps of the Table 1.

Let $n \neq k$, so $f \restriction Z_o \neq \sigma''', \rho', \rho''$. Assume there is $f \in \mathfrak{G}_o$, with $f \neq id$ and $f \restriction Z_o = \sigma'$ or $f \restriction Z_o = \sigma''$. In view of 6.1 and 6.2, by 5.13, $n|k$ or $k|n$. In the first case $f = \sigma_1$, and in the second one $f = \sigma_2$ (cf. 5.13). In both cases $|\mathfrak{G}_o| = 2$.

Finally, if $n \nmid k$ and $k \nmid n$ then $\mathfrak{G}_o = \{id\}$ and (i) holds.

In each one of the corresponding cases an arbitrary automorphism f of $\mathfrak{D}$ can be written in the form $f = \tau_u \circ \varphi$, where τ_u with $u \in C_k \oplus C_n$ is a translation, and $\varphi \in \Pi$, where Π is a group as follows:

- in the case (ii) – $\Pi = \{id, \sigma_1\} \cong S_2$,
- in the case (iii) – $\Pi = \{id, \sigma_2\} \cong S_2$,
- in the case (v) – $\Pi = \{id, \sigma_1, \sigma_2, \jmath, \sigma_1\sigma_2, \sigma_2\sigma_1\} \cong S_3$.

Thus $f = \tau_u\varphi$ can be identified with the pair $(u, \varphi) \in (C_k \oplus C_n) \times \Pi$. With the formula (16) we obtain $(\tau_{u_1}\varphi_1)(\tau_{u_2}\varphi_2) = (\tau_{u_1}\tau_{\varphi_1(u_2)})(\varphi_1\varphi_2) = \tau_{u_1+\varphi_1(u_2)}(\varphi_1\varphi_2)$. This proves $\mathfrak{G} \cong \Pi \ltimes (C_k \oplus C_n)$.

To close the proof note that, by the above, every automorphism f of $\mathfrak{D}$ can be given in one of the following forms:

$$\begin{aligned} f(j,i) &= (j+a, i+b) = \tau_{(a,b)}(j,i) && (17)\\ f(j,i) &= (j+a, -i-j+b) = \mu'_{(a,b)}(j,i) && (18)\\ f(j,i) &= (-j-i+a, i+b) = \mu''_{(a,b)}(j,i) && (19)\\ f &= \jmath \circ g \text{ where } g = \tau_{(a,b)}, \mu'_{(a,b)}, \mu''_{(a,b)}, && (20) \end{aligned}$$

where $a \in C_k$ and $b \in C_n$ are arbitrary. Automorphisms of the type (17) are in $\mathfrak{G}$ in all the cases (i)–(v).

Let us consider the case (ii). Then $\mathfrak{G}$ consists of all the maps defined with formulas (17) and (18). Note that the group $C_k \oplus D_n$ can be considered as the family of all the maps defined with the formula (17) and the following one:

$$f(j,i) = (j+a, -i+b) = \sigma_{(a,b)}(j,i). \tag{21}$$

Define maps $\eta_{(a,b)}$ and $\nu_{(a,b)}$ with the conditions:

$$\eta_{(a,b)}(j,i) = (j+2a, i+b-a) \text{ and } \nu_{(a,b)}(j,i) = (j+2a, -i-j+b-a) \tag{22}$$

(warning: $i, b, i+b-a \in C_n$, but $a \in C_k$!). Then we get the following equalities:

$$\begin{aligned} \tau_{(c,d)} \circ \tau_{(a,b)} &= \tau_{(c+a,d+b)} &\text{and}\quad \eta_{(c,d)} \circ \eta_{(a,b)} &= \eta_{(c+a,d+b)},\\ \sigma_{(c,d)} \circ \sigma_{(a,b)} &= \tau_{(c+a,d-b)} &\text{and}\quad \nu_{(c,d)} \circ \nu_{(a,b)} &= \eta_{(c+a,d-b)},\\ \sigma_{(c,d)} \circ \tau_{(a,b)} &= \sigma_{(a+c,d-b)} &\text{and}\quad \nu_{(c,d)} \circ \eta_{(a,b)} &= \nu_{(a+c,d-b)},\\ \tau_{(c,d)} \circ \sigma_{(a,b)} &= \sigma_{(a+c,b+d)} &\text{and}\quad \eta_{(c,d)} \circ \nu_{(a,b)} &= \nu_{(a+c,b+d)}. \end{aligned}$$

Let us define the map F by $F\colon \tau_{(a,b)} \mapsto \eta_{(a,b)}$, $F\colon \sigma_{(a,b)} \mapsto \nu_{(a,b)}$. By the above, F is an isomorphism of the group of transformations defined by (17) and (21) (which is, clearly, isomorphic with the group $C_k \oplus D_n$) with the group of maps defined by (22). Now, it suffices to note that $\tau_{(a,b)} = \eta_{(\frac{a}{2},\frac{a}{2}+b)}$, $\eta_{(a,b)} = \tau_{(2a,b-a)}$, $\mu'_{(a,b)} = \nu_{(\frac{a}{2},\frac{a}{2}+b)}$, and $\nu_{(a,b)} = \mu'_{(2a,b-a)}$ to see that the group defined by (22) coincides with the group defined by (17) and (18). Thus F yields an automorphism of $C_k \oplus D_n$ and $\mathfrak{G}$.

Analogously, in the case (iii) we prove $\mathfrak{G} \cong D_k \oplus C_n$. □

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