

Homology and homotopy groups of the complement of certain family of fibers in a critical point problem

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Abstract

In this paper we improve the results of [5] by showing that the collection of fibers over a closed countable subset of the base space of a differentiable fibration is not CS^∞ -critical, under some different topological conditions on the involved spaces. The example we always have in mind is that of Brieskorn manifolds as total spaces of certain principal fiber bundles.

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1 Basic results.

We start this section by proving that the complement of a closed countable subset of a given n -dimensional manifold has large $n - 1$ rational homology group. The manifold M will be with empty boundary all along the paper.

Let M be an n -dimensional differentiable manifold and $A \subseteq M$ be a closed countable subset of M . Recall that A has uncountably many isolated points, all the other of its points being accumulation points. Assume that $A = I \cup A'$ where $I = \{a_1, a_2, \dots\}$ is the set of isolated points of A and A' is its derived set, namely its set of accumulation points. If $H_{n-1}(M) \simeq 0$, then for each $k \geq 1$ there exists, according to [5, Proposition 2.1], a surjective group homomorphism

$$\delta_k : H_{n-1}(M \setminus A) \rightarrow \mathbf{Z}^{k-1} \oplus H_{n-1}(M \setminus A_k),$$

where $A_k = \{a_{k+1}, a_{k+2}, \dots\} \cup A'$. Moreover, if M is either not compact or compact but not orientable, then $H_{n-1}(M \setminus A) \simeq \mathbf{Z}^k \oplus H_{n-1}(M \setminus A_k)$, for each $k \geq 1$, that is $H_{n-1}(M \setminus A)$ has free abelian subgroups of arbitrarily large rank.

Proposition 1.1 *Let M be an n -dimensional differential manifold $n \geq 2$ and $A = I \cup A'$ be a closed countable subset of M , where $I = \{a_1, a_2, \dots\}$ is the set of isolated points of A and A' is its derived set. If $H_{n-1}(M) \simeq 0$, then the group homomorphism*

$$\delta_k \otimes id_{\mathbf{Q}} : H_{n-1}(M \setminus A) \otimes_{\mathbf{Z}} \mathbf{Q} \rightarrow \mathbf{Q}^{k-1} \oplus (H_{n-1}(M \setminus A_k) \otimes_{\mathbf{Z}} \mathbf{Q})$$

is surjective, it actually being a \mathbf{Q} -linear mapping.

Corollary 1.2 *Let M be an n -dimensional differentiable manifold with $n \geq 2$ and $A = I \cup A'$ be a closed countable subset of M , where $I = \{a_1, a_2, \dots\}$ is the set of isolated points of A and A' is its derived set. If V is a \mathbf{Q} -vector space and $\varphi : V \rightarrow H_{n-1}(M \setminus A) \otimes_{\mathbf{Z}} \mathbf{Q}$ is a surjective \mathbf{Q} -linear mapping, then V is not finite dimensional. In particular $H_{n-1}(M \setminus A) \otimes_{\mathbf{Z}} \mathbf{Q}$ is not finite dimensional.*

Proof. Assume that V is finite dimensional and $\varphi : V \rightarrow H_{n-1}(M \setminus A) \otimes_{\mathbf{Z}} \mathbf{Q}$ is a surjective \mathbf{Q} -linear mapping. It follows that $(\delta_k \otimes id_{\mathbf{Q}}) \circ \varphi : V \rightarrow \mathbf{Q}^{k-1} \oplus (H_{n-1}(M \setminus A_k) \otimes_{\mathbf{Z}} \mathbf{Q})$ is also a surjective \mathbf{Q} -linear mapping for each $k \geq 1$. Consider $q = \dim_{\mathbf{Q}} V$ and observe that $(\delta_{q+2} \otimes id_{\mathbf{Q}}) \circ \varphi$ acts surjectively from V to $\mathbf{Q}^{q+1} \oplus (H_{n-1}(M \setminus A_k) \otimes_{\mathbf{Z}} \mathbf{Q})$. Therefore we have successively

$$\begin{aligned} q = \dim_{\mathbf{Q}} V &= \dim_{\mathbf{Q}} \ker [(\delta_{q+2} \otimes id_{\mathbf{Q}}) \circ \varphi] + \dim_{\mathbf{Q}} \text{Im} [(\delta_{q+2} \otimes id_{\mathbf{Q}}) \circ \varphi] \geq \\ &\geq \dim_{\mathbf{Q}} \text{Im} [(\delta_{q+2} \otimes id_{\mathbf{Q}}) \circ \varphi] = \dim_{\mathbf{Q}} [\mathbf{Q}^{q+1} \oplus (H_{n-1}(M \setminus A_k) \otimes_{\mathbf{Z}} \mathbf{Q})] \geq \\ &\geq \dim_{\mathbf{Q}} \mathbf{Q}^{q+1} = q + 1. \square \end{aligned}$$

Proposition 1.3 ([5]) *If $p : E \rightarrow M$ is a fibration whose base space M is an n -dimensional differentiable manifold and A is a closed countable subset of M , then the pair $(E, E \setminus p^{-1}(A))$ is $(n-1)$ -connected, that is $\pi_q(E, E \setminus p^{-1}(A)) \simeq 0$ for all $q \in \{1, \dots, n-1\}$. In particular we get that $H_q(E, E \setminus p^{-1}(A)) \simeq 0$ for all $q \in \{1, \dots, n-1\}$ and that the natural Hurewicz group homomorphism $\chi_n : \pi_n(E, E \setminus p^{-1}(A)) \rightarrow H_n(E, E \setminus p^{-1}(A))$ is surjective. On the other hand the inclusion $i_{E \setminus p^{-1}(A)} : E \setminus p^{-1}(A) \hookrightarrow E$ is $(n-1)$ -connected, that is the induced group homomorphisms $\pi_q(i_{E \setminus p^{-1}(A)}) : \pi_q(E \setminus p^{-1}(A)) \rightarrow \pi_q(E)$ is an isomorphism for $q \leq n-2$ and it is an epimorphism for $q = n-1$. Hence the morphism χ_n is an isomorphism if E is simply connected and $n \geq 3$.*

Remark 1.4 If M is an n -dimensional differentiable manifold and A is a closed countable subset of M , then, by considering in Proposition 1.3 the particular fibration $id_M : M \rightarrow M$, the pair $(M, M \setminus A)$ is $(n-1)$ -connected, that is $\pi_q(M, M \setminus A) \simeq 0$ for all $q \in \{1, \dots, n-1\}$. In particular we get that $H_q(M, M \setminus A) \simeq 0$ for all $q \in \{1, \dots, n-1\}$ and the natural group homomorphism $\chi_n : \pi_n(M, M \setminus A) \rightarrow H_n(M, M \setminus A)$ is surjective. On the other hand the inclusion $i_{M \setminus A} : M \setminus A \hookrightarrow M$ is $(n-1)$ -connected, that is the induced group homomorphism $\pi_q(i_{M \setminus A}) : \pi_q(M \setminus A) \rightarrow \pi_q(M)$ is an isomorphism for $q \leq n-2$ and it is an epimorphism for $q = n-1$. Hence the morphism χ_n is an isomorphism if M is simply connected and $n \geq 3$.

Corollary 1.5 *Let $p : E \rightarrow M$ be a fibration whose base space M is an n -dimensional differentiable manifold and A be a closed countable subset of M . If the total space E is simply connected, $H_n(E) \otimes_{\mathbf{Z}} \mathbf{Q} \simeq 0 \simeq \pi_{n-1}(E) \otimes_{\mathbf{Z}} \mathbf{Q}$, then the group homomorphism $h_{n-1} \otimes id_{\mathbf{Q}} : \pi_{n-1}(E \setminus p^{-1}(A)) \otimes_{\mathbf{Z}} \mathbf{Q} \rightarrow H_{n-1}(E \setminus p^{-1}(A)) \otimes_{\mathbf{Z}} \mathbf{Q}$, which is actually \mathbf{Q} -linear, is injective, where h_{n-1} is the natural group homomorphism.*

Proof. Indeed, the following ladder with exact rows and commutative rectangles:

$$\begin{array}{ccccccccc}
 \pi_n(E) & \rightarrow & \pi_n(E, E \setminus p^{-1}(A)) & \rightarrow & \pi_{n-1}(E \setminus p^{-1}(A)) & \rightarrow & \pi_{n-1}(E) & \rightarrow & \pi_{n-1}(E, E \setminus p^{-1}(A)) \\
 \downarrow h_n^E & & \downarrow \chi_n & & \downarrow h_{n-1} & & \downarrow h_{n-1}^E & & \downarrow \chi_{n-1} \\
 H_n(E) & \rightarrow & H_n(E, E \setminus p^{-1}(A)) & \rightarrow & H_{n-1}(E \setminus p^{-1}(A)) & \rightarrow & H_{n-1}(E) & \rightarrow & H_{n-1}(E, E \setminus p^{-1}(A))
 \end{array}$$

leads us, simply by tensorizing with \mathbf{Q} on the right, to the new one

$$\begin{array}{ccccccccc}
 \pi_n(E) \otimes \mathbf{Q} & \rightarrow & \pi_n(E, E \setminus p^{-1}(A)) \otimes \mathbf{Q} & \rightarrow & \pi_{n-1}(E \setminus p^{-1}(A)) \otimes \mathbf{Q} & \rightarrow & \pi_{n-1}(E) \otimes \mathbf{Q} & \rightarrow & \pi_{n-1}(E, E \setminus p^{-1}(A)) \otimes \mathbf{Q} \\
 \downarrow h_n^E \otimes id_{\mathbf{Q}} & & \downarrow \chi_n \otimes id_{\mathbf{Q}} & & \downarrow h_{n-1} \otimes id_{\mathbf{Q}} & & \downarrow h_{n-1}^E \otimes id_{\mathbf{Q}} & & \downarrow \chi_{n-1} \otimes id_{\mathbf{Q}} \\
 H_n(E) \otimes \mathbf{Q} & \rightarrow & H_n(E, E \setminus p^{-1}(A)) \otimes \mathbf{Q} & \rightarrow & H_{n-1}(E \setminus p^{-1}(A)) \otimes \mathbf{Q} & \rightarrow & H_{n-1}(E) \otimes \mathbf{Q} & \rightarrow & H_{n-1}(E, E \setminus p^{-1}(A)) \otimes \mathbf{Q}
 \end{array}$$

with exact rows and commutative rectangles as well. The exactness of the rows follows by using the universal-coefficient theorem [7, Theorem 5.2.14]. In the above diagram we have omitted, for simplicity, the notation $\otimes_{\mathbf{Z}}$. Because $H_n(E) \otimes_{\mathbf{Z}} \mathbf{Q} \simeq 0 \simeq \pi_{n-1}(E) \otimes_{\mathbf{Z}} \mathbf{Q}$, it follows that $h_n^E \otimes id_{\mathbf{Q}}$ is surjective and $h_{n-1}^E \otimes id_{\mathbf{Q}}$ is injective. On the other hand $\chi_n \otimes id_{\mathbf{Q}}$ is, according to corollary 1.3, an isomorphism such that, by combining all these facts with the fifth Lemma in [4, pp. 46], one can conclude that $h_{n-1} \otimes id_{\mathbf{Q}}$ is indeed injective. \square

Remark 1.6 Let M be an n -dimensional differentiable manifold and A be a closed countable subset of M . If the natural group homomorphism $h_{n-1}^M : \pi_{n-1}(M) \rightarrow H_{n-1}(M)$ is surjective, then the natural group homomorphism $h_{n-1}^{M \setminus A} : \pi_{n-1}(M \setminus A) \rightarrow H_{n-1}(M \setminus A)$ is also surjective and the \mathbf{Q} -linear mapping $h_{n-1}^{M \setminus A} \otimes id_{\mathbf{Q}} : \pi_{n-1}(M \setminus A) \otimes_{\mathbf{Z}} \mathbf{Q} \rightarrow H_{n-1}(M \setminus A) \otimes_{\mathbf{Z}} \mathbf{Q}$ is surjective as well. In particular, according to corollary 1.2, $\pi_{n-1}(M \setminus A) \otimes_{\mathbf{Z}} \mathbf{Q}$ is not finite dimensional.

2 Application to critical points

In this section we will show that, under some topological conditions on the total space, on the base spaces and on the fiber of a fibration, the collection of fibers over a closed countable subset of the base space is not critical with respect to certain class of special real functions.

Let M, N be differentiable manifolds and $f : M \rightarrow N$ be a differentiable mapping. Denote by $C(f)$ its critical set and by $R(f)$ its set of regular points, while the set of its critical values $f(C(f))$ will be denoted by $B(f)$.

Definition 2.1 We say that a differentiable mapping $f \in C^\infty(M, N)$ separates the critical values by the regular ones if $B(f) \cap f(R(f)) = \emptyset$. Denote by $CS^\infty(M, N)$ the set of all of these mappings. A closed subset C of M is said to be $CS^\infty(M, N)$ - (properly) critical if $C(f) = C$ for some (proper) mapping $f \in CS^\infty(M, N)$.

Remark 2.2 Let $p : \tilde{N} \rightarrow N$ be a covering mapping and $\tilde{f} : M \rightarrow \tilde{N}$ be a differentiable mapping. If $\tilde{f} \notin CS^\infty(M, \tilde{N})$, then $p \circ \tilde{f} \notin CS^\infty(M, N)$. Therefore if $g \in CS^\infty(M, N)$ and $\tilde{g} \in C^\infty(M, \tilde{N})$ is a lifting of g , then $\tilde{g} \in CS^\infty(M, \tilde{N})$.

Indeed, since $\tilde{f} \notin CS^\infty(M, \tilde{N})$, it follows that there exist $x_0 \in C(\tilde{f})$, $x_1 \in R(\tilde{f})$ such that $\tilde{f}(x_0) = \tilde{f}(x_1)$. Because p is a local diffeomorphism, it implies that $x_0 \in$

$C(p \circ \tilde{f})$, $x_1 \in R(p \circ \tilde{f})$ and obviously $(p \circ \tilde{f})(x_0) = (p \circ \tilde{f})(x_1) \in B(p \circ \tilde{f}) \cap (p \circ \tilde{f})(R(p \circ \tilde{f}))$, that is $p \circ \tilde{f} \notin CS^\infty(M, N)$.

The next Proposition can be easily proved and it actually appears in [5].

Proposition 2.3 (i) $f \in CS^\infty(M, N)$ iff $C(f) = f^{-1}(B(f))$.

(ii) If M is a connected differentiable manifold and $f \in CS^\infty(M, \mathbf{R})$ is such that $R(f) = M \setminus C(f)$ is also connected, then $f(R(f)) = (m_f, M_f)$, where $m_f = \inf_{x \in M} f(x)$, $M_f = \sup_{x \in M} f(x)$ and $B(f) \subseteq \{m_f, M_f\} \cap \mathbf{R}$. Moreover, if M is compact, then $m_f, M_f \in \mathbf{R}$ and $B(f) = \{m_f, M_f\}$.

Theorem 2.4 Let $F \hookrightarrow E \xrightarrow{p} M^n$ be a differential fibration with compact total space and commutative fundamental group of the fiber F . If A is a closed countable subset of M , $n \geq 3$, E is simply connected and $H_n(E)$, $H_{n-1}(M)$, $\pi_{n-1}(E) \otimes_{\mathbf{Z}} \mathbf{Q}$ are trivial and $\pi_{n-2}(F)$ is finitely generated, then $p^{-1}(A)$ is neither $CS^\infty(E, \mathbf{R})$ -critical nor $CS^\infty(E, S^1)$ -critical.

Proof. Assume that there exists a mapping $f \in CS^\infty(E, \mathbf{R})$ such that $C(f) = p^{-1}(A)$. This means that $B(f) = \{m_f, M_f\}$ and that its restriction

$$E \setminus C(f) \rightarrow \text{Im} f \setminus B(f) = (m_f, M_f), \quad p \mapsto f(p)$$

is a proper submersion, that is, via Ehresmann's theorem, a locally trivial fibration whose compact fiber we are denoting by \mathcal{F} . Its base space (m_f, M_f) being contractible, it follows that the inclusion $i_{\mathcal{F}} : \mathcal{F} \hookrightarrow E \setminus C(f)$ is a weak homotopy equivalence, namely the induced group homomorphisms $\pi_q(i_{\mathcal{F}}) : \pi_q(\mathcal{F}) \rightarrow \pi_q(E \setminus C(f))$ are all isomorphisms. Consequently, using the Whitehead theorem [3, pp. 167] or [7, pp. 399], it follows that the induced group homomorphisms $H_q(i_{\mathcal{F}}) : H_q(\mathcal{F}) \rightarrow H_q(E \setminus C(f)) = H_q(E \setminus p^{-1}(A))$ are also isomorphisms. Consequently $H_q(i_{\mathcal{F}}) \otimes id_{\mathbf{Q}} : H_q(\mathcal{F}) \otimes_{\mathbf{Z}} \mathbf{Q} \rightarrow H_q(E \setminus C(f)) \otimes_{\mathbf{Z}} \mathbf{Q} = H_q(E \setminus p^{-1}(A)) \otimes_{\mathbf{Z}} \mathbf{Q}$ is a \mathbf{Q} -linear isomorphism. Therefore $H_q(E \setminus p^{-1}(A)) \otimes_{\mathbf{Z}} \mathbf{Q}$ is finite dimensional for all q since $H_q(\mathcal{F})$ is finitely generated as homology group of a compact manifold. The hypothesis $H_{n-1}(M) \simeq 0$ ensures us that $h_{n-1}^M \otimes id_{\mathbf{Q}}$ is obviously surjective and implicitly that $\pi_{n-1}(M \setminus A) \otimes_{\mathbf{Z}} \mathbf{Q}$ is not finite dimensional while the hypothesis $H_n(E) \simeq 0$ and the triviality of $\pi_{n-1}(E) \otimes_{\mathbf{Z}} \mathbf{Q}$ ensure us that the \mathbf{Q} -linear mapping $h_{n-1} \otimes id_{\mathbf{Q}} : \pi_{n-1}(E \setminus p^{-1}(A)) \otimes_{\mathbf{Z}} \mathbf{Q} \rightarrow H_{n-1}(E \setminus p^{-1}(A)) \otimes_{\mathbf{Z}} \mathbf{Q}$ is injective. Consequently $\pi_{n-1}(E \setminus p^{-1}(A)) \otimes_{\mathbf{Z}} \mathbf{Q}$ is a finite dimensional \mathbf{Q} -vector space as a subspace of the finite dimensional \mathbf{Q} -vector space $H_{n-1}(E \setminus p^{-1}(A)) \otimes_{\mathbf{Z}} \mathbf{Q}$.

The exact homotopy sequence of the fibration $F \hookrightarrow E \setminus p^{-1}(A) \xrightarrow{p} M \setminus A$

$$\cdots \rightarrow \pi_{n-1}(E \setminus p^{-1}(A)) \rightarrow \pi_{n-1}(M \setminus A) \xrightarrow{\partial_{n-1}} \pi_{n-2}(F) \rightarrow \cdots$$

produces the exact sequence

$$\cdots \rightarrow \pi_{n-1}(E \setminus p^{-1}(A)) \otimes_{\mathbf{Z}} \mathbf{Q} \rightarrow \pi_{n-1}(M \setminus A) \otimes_{\mathbf{Z}} \mathbf{Q} \rightarrow \pi_{n-2}(F) \otimes_{\mathbf{Z}} \mathbf{Q} \rightarrow \cdots$$

simply by tensoring with \mathbf{Q} . Its exactness follows by using the universal-coefficient theorem. But since $\pi_{n-1}(E \setminus p^{-1}(A)) \otimes_{\mathbf{Z}} \mathbf{Q}$ and $\pi_{n-2}(F) \otimes_{\mathbf{Z}} \mathbf{Q}$ are finite dimensional they force $\pi_{n-1}(E \setminus p^{-1}(A)) \otimes_{\mathbf{Z}} \mathbf{Q}$ to be finite dimensional too, not being the case as

follows from remark 1.6. In order to prove the $CS^\infty(M, S^1)$ -non-criticality of $p^{-1}(A)$ we assume that there exists a mapping $f \in CS^\infty(M, S^1)$ such that $C(f) = p^{-1}(A)$. Consider a lifting $\tilde{f} : M \rightarrow \mathbf{R}$ with respect to the covering mapping $exp : \mathbf{R} \rightarrow S^1$, recall that $\tilde{f} \in CS^\infty(M, \mathbf{R})$ and observe that $C(\tilde{f}) = C(f) = p^{-1}(A)$, such that we have got a contradiction with the $CS^\infty(M, \mathbf{R})$ -non-criticality of $p^{-1}(A)$. \square

Example 2.5. Consider the integers $n \geq 2$ and $d \geq 1$ and the *Brieskorn manifolds* W_d^{2n-1} as the $(2n - 1)$ -dimensional real algebraic submanifolds of \mathbf{C}^{n+1} defined by the equations

$$z_0^d + z_1^2 + \dots + z_n^2 = 0 \text{ and } |z_0|^2 + |z_1|^2 + \dots + |z_n|^2 = 1.$$

The topology of Brieskorn manifolds is mostly known it being simply connected for $n \geq 3$ and a homotopy sphere for both $n \geq 3, d \geq 1$ odd. On the other hand, if $n = 2m$ is even, then W_d^{4m-1} is a rational homology sphere whose only nontrivial integral homology groups are given by $H_{2m-1}(W_d^{4m-1}) \simeq \mathbf{Z}_d$ and $H_0(W_d^{4m-1}) \simeq H_{4m-1}(W_d^{4m-1}) \simeq \mathbf{Z}$, [1, Corollary 5.3, pp. 275], [6].

All manifolds W_d^{2n-1} are invariant under the standard linear action of $O(n)$ on the (z_1, \dots, z_n) -coordinates. If $n = 2m$ is even there is a free circle action on W_d^{4m-1} given by the action of the circle group $S^1 = Z(U(m)) \subset O(2m)$ where Z denotes the center. Moreover if $n = 4m$, then $Sp(1)$ realised as subgroup of $O(4m)$ by the scalar multiplication on $\mathbf{R}^{4m} \simeq \mathbf{H}^m$, acts also freely on W_d^{8m-1} . The quotient manifolds $N_d^{4m-2} := W_d^{4m-1}/S^1$ and $\tilde{N}_d^{8m-4} := W_d^{8m-1}/Sp(1)$ are simply connected and their integral cohomology groups are given by

$$H^k(N_d^{4m-2}) \simeq H^k(\mathbf{CP}^{2m-1}), H^k(\tilde{N}_d^{8m-4}) \simeq H^k(\mathbf{HP}^{2m-1}), \text{ see [6].}$$

Therefore if $F \hookrightarrow E \xrightarrow{p} M^n$ is one of the fibrations

$$S^1 \hookrightarrow W_d^{4m-1} \rightarrow N_d^{4m-2}, Sp(1) \hookrightarrow W_d^{8m-1} \rightarrow \tilde{N}_d^{8m-4},$$

then it satisfies the conditions of theorem 2.4. Consequently for a closed countable subset A of the base space, $p^{-1}(A)$ is neither $CS^\infty(E, \mathbf{R})$ -critical nor $CS^\infty(E, S^1)$ -critical. Indeed we have successively

$$H_{4m-2}(W_d^{4m-1}) \simeq 0 \simeq H^1(\mathbf{CP}^{2m-1}) \simeq H^1(N_d^{4m-2}) \stackrel{*}{\simeq} H_{4m-3}(N_d^{4m-2}),$$

$$H_{8m-4}(W_d^{8m-1}) \simeq 0 \simeq H^1(\mathbf{HP}^{2m-1}) \simeq H^1(\tilde{N}_d^{8m-4}) \stackrel{**}{\simeq} H_{8m-5}(\tilde{N}_d^{8m-4}),$$

the last isomorphisms (*) and (**) being ensured by the Poincaré duality. Finally, the triviality of $\pi_{4m-3}(W_d^{4m-1}) \otimes_{\mathbf{Z}} \mathbf{Q}$ and of $\pi_{8m-5}(W_d^{8m-1}) \otimes_{\mathbf{Z}} \mathbf{Q}$ follows from [2, pp. 123], while the triviality of $\pi_{4m-4}(S^1) \otimes_{\mathbf{Z}} \mathbf{Q}$ and $\pi_{8m-6}(Sp(1)) \otimes_{\mathbf{Z}} \mathbf{Q} = \pi_{8m-6}(S^3) \otimes_{\mathbf{Z}} \mathbf{Q}$ is obvious since $\pi_{4m-4}(S^1) \simeq 0$ and $\pi_{8m-6}(S^3)$ is finite [3, 318].

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