Conformal vector fields on tangent bundle of Finsler manifolds

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Abstract. Let (M,g) be a Finsler manifold, TM its tangent bundle and $\tilde{g}$ a Riemannian metric on TM derived from g. Then every complete lift conformal vector field on M is homothetic.

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Introduction.

Let (M, g) be an n-dimensional Riemannian manifold and ϕ a transformation on M. Then ϕ is called a *conformal* transformation, if it preserves the angles. Let V be a vector field on M and $\{\varphi_t\}$ be the local one-parameter group of local transformations on M generated by V. Then V is called a *conformal vector field* on M if each φ_t is a local conformal transformation of M. It is well known that V is a *conformal vector field* on M if and only if there is a scalar function ρ on M such that $\pounds_V g = 2\rho g$ where $\pounds_V$ denotes Lie derivation with respect to the vector field V. Specially V is called *homothetic* if ρ is constant and it is called an *isometry* or *Killing vector field* when ρ vanishes.

There are some lift metrics on $TM = \cup_{x\in M} T_x M$ as follows: *complete* lift metric or g_2, *diagonal* lift metric or $g_1 + g_3$, lift metric $g_2 + g_3$ and lift metric $g_1 + g_2$, where $g_1 := g_{ij} dx^i \otimes dx^j$, $g_2 := 2g_{ij} dx^i \otimes \delta y^j$ and $g_3 := g_{ij} \delta y^i \otimes \delta y^j$ are all bilinear differential forms defined globally on TM.

In the study of Finsler geometry the complete lift vector fields have a great significance. More precisely let V be a vector field on the Finsler manifold $(M, g(x, y))$ and X^c be the complete lift of V. Then V is called a *conformal vector field of Finsler manifold* (M, g) if there is a scalar function[1] Ω on TM which satisfies $\pounds_{X^c} g = 2\Omega g$.

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[1]By a simple calculation and vertical partial derivative using commutative property of Lie derivative one can show that Ω is a function of x alone [1].

For the complete lift vector fields the following results are well known:

Theorem A. *[9]:Let (M, g) be a Riemannian manifold, X a vector field on M and X^C complete lifts of X to TM. If we consider TM with metric g_2 then X^C is a conformal vector field on TM if and only if X is homothetic on M.*

Theorem B. *[10]:Let (M, g) be a Riemannian manifold. If we consider TM with metric $g_1 + g_3$ then X^C is a conformal vector field on TM if and only if X is homothetic.*

In a recent work we introduced a new Riemannian and pseudo-Riemannian lift metrics on TM, $\widetilde{g} = ag_1 + bg_2 + cg_3$ where a, b and c are certain constant real numbers. That is a combination of diagonal lift, and complete lift metrics, which is in some senses more general than those who are used previously. We have replaced the cited lift metrics in Theorems A and B by $\widetilde{g}$. More precisely, we have proved Theorem C in [3] as follows.

Theorem C. *Let M be an n-dimensional Riemannian manifold and let TM be its tangent bundle with metric $\widetilde{g}$. Then every complete lift conformal vector field on TM is homothetic.*

In the present work we replace the Riemannian metric on M by a Finsler metric endowed with a Cartan connection and prove the following theorem.

Theorem 1*: Let (M,g) be a C^∞ connected Finsler manifold, TM its tangent bundle and $\tilde{g}$ the Riemannian (or Pseudo-Riemannian) metric on TM derived from g. Then every complete lift conformal vector field on TM is homothetic.*

1 Preliminaries.

Let M be a real n-dimensional manifold of class C^∞. We denote by $TM \to M$ the bundle of tangent vectors and by $\pi : TM_0 \to M$ the fiber bundle of non-zero tangent vectors. A *Finsler structure* on M is a function $F : TM \to [0, \infty)$, with the following properties: (I) F is differentiable (C^∞) on TM_0; (II) F is positively homogeneous of degree one in y, i.e. $F(x, \lambda y) = \lambda F(x, y), \forall \lambda > 0$, where we denote an element of TM by (x, y). (III) The Hessian matrix of F^2 is positive definite on TM_0; $(g_{ij}) := \left(\frac{1}{2}\left[\frac{\partial^2}{\partial y^i \partial y^j} F^2\right]\right)$. A *Finsler manifold* is a pair of a differentiable manifold M and a Finsler structure F on M. The tensor field $g = (g_{ij})$ is called the *Fundamental Finsler tensor* or *Finsler metric tensor*. Here, we denote a Finsler manifold by (M, g).

Let $V_vTM = ker\pi_*^v$ be the set of the vectors tangent to the fiber through $v \in TM_0$. Then a *vertical vector bundle* on M is defined by $VTM := \bigcup_{v \in TM_0} V_vTM$. A *non-linear connection* or a *horizontal distribution* on TM_0 is a complementary distribution HTM for VTM on TTM_0. Therefore we have the decomposition

$$TTM_0 = VTM \oplus HTM. \tag{1.1.1}$$

HTM is a *vector bundle* completely determined by the non-linear differentiable functions $N_i^j(x,y)$ on TM, called coefficients of the non-linear connection. Let HTM be a non- linear connection on TM and ∇ a linear connection on VTM, then the pair (HTM,∇) is called a *Finsler connection* on the manifold M.
Using the local coordinates (x^i,y^i) on TM we have the local field of frames $\{\frac{\partial}{\partial x_i},\frac{\partial}{\partial y_i}\}$ on TTM. It is well known that we can choose a local field of frames $\{\frac{\delta}{\delta x_i},\frac{\partial}{\partial y_i}\}$ adapted to the above decomposition i.e. $\frac{\delta}{\delta x_i}\in\Gamma(HTM)$ and $\frac{\partial}{\partial y_i}\in\Gamma(VTM)$ set of vector fields on HTM and VTM, where

$$\frac{\delta}{\delta x_i}=\frac{\partial}{\partial x_i}-N_i^j\frac{\partial}{\partial y_j}, \tag{1.1.2}$$

and where we use the *Einstein summation convention*.

Here, in this paper, all manifolds are supposed to be connected.

Let $(M,g(x,y))$ be a Finsler manifold then a Finsler connection is called a *metric Finsler connection* if g is parallel with respect to ∇. According to the Miron terminology this means that g is both horizontally and vertically metric. The *Cartan connection* is a metric Finsler connection for which the Deflection , horizontal and vertical torsion tensor fields vanishes.

Let $(M,g(x,y))$ be a Finsler manifold with metric Finsler connection the *Curvature tensors* of M are defined by

$$R(X,Y)Z=\{[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}\}Z,$$

where $X,Y,Z\in\mathcal{X}(TM)$
They are called accordingly to the choice of X and Y in HTM or VTM horizontal or vertical curvature tensors of Finsler manifold.

Let M be a Finsler manifold and ∇ a Finsler connection on M, then we have [6]

$$R_{k\ ji}^{\ h}=\delta_iF_{k\ j}^{\ h}-\delta_jF_{k\ i}^{\ h}+F_{k\ j}^{\ m}F_{m\ i}^{\ h}-F_{k\ i}^{\ m}F_{m\ j}^{\ h}+C_{k\ m}^{\ h}R_{j\ i}^{\ m},$$

$R^h_{\ ij}=\delta_jN_i^h-\delta_iN_j^h$, where we have put $\partial_i=\frac{\partial}{\partial x^i},\dot\partial_i=\frac{\partial}{\partial y^i}$, $\delta_i=\partial_i-N_i^m\dot\partial_m$.
If ∇ is a Cartan connection then $N_i^h=y^mF_{m\ i}^{\ h}$.

Proposition 1. *[5] Let M be an n-dimensional Finsler space with a Cartan connection, then we have the following equations*

(1) $F_{i\ j}^{\ h}=\frac{1}{2}g^{hm}(\delta_ig_{mj}+\delta_jg_{im}-\delta_mg_{ij})$.
(2) $C_{ijk}=\frac{1}{2}\dot\partial_kg_{ij}$ *where* $C_{ijk}=C_{i\ k}^{\ m}g_{jm}$.
(3) $y^mC_{mij}=0$.
(4) $R^h_{\ ij}=y^mR_{m\ ij}^{\ h}$.

The Cartan horizontal and vertical covariant derivative of a tensor field of type $\binom{1}{2}$ are given locally as follows:

(1.1.3) $$\nabla_j T_k{}^h{}_i := T_k{}^h{}_{i|j} = \delta_j T_k{}^h{}_i + F_m{}^h{}_j T_k{}^m{}_i - F_k{}^m{}_j T_m{}^h{}_i - F_i{}^m{}_j T_k{}^h{}_m.$$

$$\nabla_{\bar{j}} T_k{}^h{}_i := T_k{}^h{}_{i|\bar{j}} = \dot{\partial}_j T_k{}^h{}_i + C_m{}^h{}_j T_k{}^m{}_i - C_k{}^m{}_j T_m{}^h{}_i - C_i{}^m{}_j T_k{}^h{}_m.$$

2 Lift metrics and conformal vector fields.

2.1 Complete lift vector fields and Lie derivative.

Let $V = v^i \frac{\partial}{\partial x^i}$ be a vector field on M. Then V induces an infinitesimal point transformation on M. This is naturally extended to a point transformation of the tangent bundle TM which is called *extended point transformation*. Let V be a vector field on M and $\{\Phi_t\}$ the local one parameter groups of M generated by V. Let $\tilde{\Phi}_t$ be the extended point transformation of Φ_t and $\{\tilde{\Phi}_t\}$ be the local one-parameter groups of TM. If X^c is a vector field on TM induced by $\{\tilde{\Phi}_t\}$ it is called the *complete lift* vector field of V.
It can be shown that the extended point transformation is a transformation induced by the complete lift vector field of V, $X^c = v^i \delta_i + y^j \nabla_j v^i \dot{\partial}_i$ with respect to the decomposition (1.1.1).

Let M be an n-dimensional manifold, V a vector field on M and $\{\phi_t\}$ a 1-parameter local group of local transformations of M generated by V. Take any tensor field S on M, and denote by $\phi_t{}^*(S)$ the pulled back of S by ϕ_t. Then the *Lie derivation* of S with respect to V is a tensor field $\pounds_V S$ on M defined by:

$$\pounds_V S = \frac{\partial}{\partial t} \phi_t{}^*(S)|_{t=0} = \lim_{t \longrightarrow 0} \frac{\phi_t{}^*(S) - (S)}{t},$$

on the domain of ϕ_t. The mapping $\pounds_V$ which map S to $\pounds_V(S)$ is called the Lie derivation with respect to V.

In Finsler geometry the Lie derivative of an arbitrary tensor, $T_{ij}{}^k$ is given locally by [Yan1]:

$$\pounds_V T_i{}^k = v^a \nabla_a T_i{}^k + v^a \nabla_a v^b \nabla_{\bar{b}} T_i{}^k - T_i{}^a \nabla_a v^k + T_a{}^k \nabla_i v^a,$$

or equivalently,

(2.2.1) $$\pounds_V T_i{}^j = v^a \partial_a T_i{}^j + y^a \partial_a v^b \dot{\partial}_b T_i{}^j - T_i{}^a \partial_a v^j + T_a{}^j \partial_i v^a.$$

So we have

(2.2.2) $$\pounds_V y^i = v^a \partial_a y^i + y^b \partial_b v^j \dot{\partial}_j y^i - y^a \partial_a v^i = y^b \partial_b v^i - y^a \partial_a v^i = 0,$$

(2.2.3) $$\pounds_V g_{ij} = v^a \partial_a g_{ij} + y^a \partial_a v^b \dot{\partial}_b g_{ij} + g_{aj}\partial_i v^a + g_{ia}\partial_j v^a.$$

We have also this interchanging formula between Cartan covariant derivatives and Lie derivatives.

(2.2.4) $$\nabla_k \pounds_V g_{ij} - \pounds_V \nabla_k g_{ij} = g_{aj}\pounds_V F^a_{ik} + g_{ai}\pounds_V F^a_{jk}.$$

2.2 A lift metric on tangent bundle.

Let (M,g) be a Finsler manifold. In this section we define a new Riemannian or Pseudo-Riemannian metric on TM derived from the Finsler metric. This metric is in some senses more general than the other lift metrics defined previously on TM. By mean of the dual basis $\{dx^i, \delta y^i\}$ analogously to the Riemannian geometry the tensors; $g_1 := g_{ij}dx^i \otimes dx^j$ $g_2 := 2g_{ij}dx^i \otimes \delta y^j$ $g_3 := g_{ij}\delta y^i \otimes \delta y^j$ are all quadratic differential tensors defined globally on TM, see [9]. Now let's consider the Finsler metric tensor g with the components $g_{ij}(x,y)$ defined on TM. The tensor field $\widetilde{g} = \alpha g_1 + \beta g_2 + \gamma g_3$ on TM, where the coefficient α, β, γ are real numbers, has the components

$$\begin{pmatrix} \alpha g & \beta g \\ \beta g & \gamma g \end{pmatrix}$$

with respect to the dual basis of TM. From the linear algebra we have $det\widetilde{g} = (\alpha\gamma - \beta^2)^n detg^2$. Therefore $\widetilde{g}$ is nonsingular if $\alpha\gamma - \beta^2 \neq 0$ and it is positive definite if α, γ are positive and $\alpha\gamma - \beta^2 > 0$. Indeed $\widetilde{g}$ define respectively a Pseudo-Riemannian or a Riemannian lift metric on $T(M)$.

Definition 1. *Let (M,g) be a Finsler manifold. Consider tensor field $\widetilde{g} = \alpha g_1 + \beta g_2 + \gamma g_3$ on TM, where the coefficient α, β, γ are real numbers. If $\alpha\gamma - \beta^2 \neq 0$ then $\widetilde{g}$ is non-singular and it can be regarded as a Pseudo-Riemannian metric on TM. If α and γ are positive such that $\alpha\gamma - \beta^2 > 0$ then $\widetilde{g}$ is positive definite and consequently can be regarded as a Riemannian metric on TM.*
$\widetilde{g}$ is called the lift metric of g on TM.

2.3 Conformal vector fields.

Let $(TM, G(x,y))$ be a Riemannian (or pseudo-Riemannian) manifold. A vector field $\widetilde{X} \in \mathcal{X}(TM)$ on TM is called a *conformal vector field on TM* if there is a scalar function Ω on TM such that

$$\pounds_{\widetilde{X}} G = 2\Omega G.$$

If Ω is constant then the vector field $\widetilde{X}$ is called *homothetic* and if Ω is zero then its called an *isometric* or a *Killing* vector field .
Now let we consider $(TM, \tilde{g}(x,y))$ with the complete lift vector field X^c of an arbitrary vector field V on M. Then by above definition we call X^c a conformal vector field on TM if

$$\pounds_{X^c}\tilde{g} = 2\Omega\tilde{g}.$$

3 Main results

Analogous to the Riemannian geometry [7], by straight forward calculation we have the following lemmas in Finsler geometry.

Lemma 1. *: Let (M, g) be a Finsler manifold with Cartan connection, then we have;*
(1) $[X_i, X_j] = R^h{}_{ij} X_{\overline{h}}$,
(2) $[X_i, X_{\overline{j}}] = \dot{\partial}_j N^h_i X_{\overline{h}}$,
(3) $[X_{\overline{i}}, X_{\overline{j}}] = 0$.
where we put $X_i = \delta_i$ *and* $X_{\overline{i}} = \dot{\partial}_i$ *for simplicity.*

Let's denote by $\pounds_{X^c}$ the lie derivative with respect to the complete lift vector field X^c. Then we obtain the following lemma :

Lemma 2. *: Let (M, g) be a Finsler manifold with Cartan connection, then we have;*
(1)$\pounds_{X^c} X_i = -\partial_i v^h X_h - \pounds_V N^h_i X_{\overline{h}}$,
(2)$\pounds_{X^c} X_{\overline{i}} = -\partial_i v^h X_{\overline{h}}$,
(3)$\pounds_{X^c} dx^h = \partial_m v^h dx^m$,
(4)$\pounds_{X^c} \delta y^h = \pounds_V N^h_m dx^m + \partial_m v^h \delta y^m$.

Proof. (1) $\pounds_{X^c} X_i = [X^c, X_i]$
$= [v^h X_h + y^m v^h{}_{|m} X_{\overline{h}} \ \ , X_i]$
$= v^h [X_h, X_i] - X_i(v^h) X_h + y^m v^h{}_{|m} [X_{\overline{h}}, X_i] - X_i(y^m v^h{}_{|m}) X_{\overline{h}}$
$= -\partial_i v^h X_h - \pounds_V N^h_i X_{\overline{h}}$.
Thus we get (1). We can prove (2) by the same way as the proof of (1). Next we prove (3). Since $(dx^h, \delta y^h)$ is the dual basis of $(X_h, X_{\overline{h}})$, if we put

$$\pounds_{X^c} dx^h = \alpha^h_m dx^m + \beta^h_m \delta y^m.$$

Then we have

$$0 = \pounds_{X^c}(dx^h(X_i)) = (\pounds_{X^c} dx^h) X_i + dx^h(\pounds_{X^c} X_i) = \alpha^h_i - \partial_i v^h,$$

and

$$0 = \pounds_{X^c}(dx^h(X_{\overline{i}})) = (\pounds_{X^c} dx^h) X_{\overline{i}} + dx^h(\pounds_{X^c} X_{\overline{i}}) = \beta^h_i.$$

Thus we get (3). By the same way as the proof of (3), we can prove (4). □

Lemma 3. *: Let (M, g) be a Finsler manifold with Cartan connection, then we have;*
(1)$\pounds_{X^c}(g_{ij} dx^i dx^j) = (\pounds_V g_{ij}) dx^i dx^j$,
(2)$\pounds_{X^c}(g_{ij} dx^i \delta y^j) = g_{mi}(\pounds_V N^m_j) dx^i dx^j + (\pounds_V g_{ij}) dx^i \delta y^j$,
(3)$\pounds_{X^c}(g_{ij} dx^i \delta y^j) = 2(g_{mi} \pounds_V N^m_j) dx^i \delta y^j + (\pounds_V g_{ij}) \delta y^i \delta y^j$.

Proof. By mean of above lemma, we get
$\pounds_{X^c}(g_{ij} dx^i dx^j) = X^c(g_{ij}) dx^i dx^j + 2 g_{ij} (\pounds_{X^c} dx^i) dx^j$
$= (v^h X_h + y^m v^h{}_{|m} X_{\overline{h}})(g_{ij}) dx^i dx^j + 2 g_{ij} (\partial_m v^i dx^m) dx^j$
$= (\pounds_V g_{ij}) dx^i dx^j$.
Thus we have (1). (2) and (3) are easily proof by the same way as the proof of (1). □

Theorem 1. *Let (M,g) be a C^∞ connected Finsler manifold, TM its tangent bundle and $\tilde{g}$ the Riemannian (or Pseudo-Riemannian) metric on TM derived from g. Then every complete lift conformal vector field on TM is homothetic.*

Proof. Let V be a vector field on M, X^c the complete lift vector field of V which is conformal and g̃ be a Pseudo-Riemannian metric on TM derived from g. We have by definition $\pounds_{X^c}\tilde{g} = 2\Omega\tilde{g}$. The Lie derivative of $\tilde{g}$ gives

$$\pounds_{X^c}\tilde{g} = \alpha(\pounds_V g_{ij})dx^i dx^j + 2\beta(\pounds_V g_{ij})dx^i\delta y^j + 2\beta g_{ai}(\pounds_V N^a_j)dx^i dx^j + \gamma(\pounds_V g_{ij})\delta y^i\delta y^j + 2\gamma g_{aj}(\pounds_V N^a_i)dx^i\delta y^j. \tag{3 .1}$$

So we have

$$\begin{aligned}\pounds_{X^c}\tilde{g} &= [\alpha\pounds_V g_{ij} + 2\beta g_{ai}\pounds_V N^a_j]dx^i dx^j\\ &+[2\beta\pounds_V g_{ij} + 2\gamma g_{aj}\pounds_V N^a_i]dx^i\delta y^j\\ &+\gamma(\pounds_V g_{ij})\delta y^i\delta y^j\\ &= 2\Omega\tilde{g}.\end{aligned}$$

Comparing with the definition of $\tilde{g}$, we find;

$$\alpha\pounds_V g_{ij} + \beta(g_{ai}\pounds_V N^a_j + g_{aj}\pounds_V N^a_i) = 2\alpha\Omega g_{ij}. \tag{3 .2}$$

$$\beta\pounds_V g_{ij} + \gamma g_{aj}\pounds_V N^a_i = 2\beta\Omega g_{ij}. \tag{3 .3}$$

$$\gamma\pounds_V g_{ij} = 2\gamma\Omega g_{ij}. \tag{3 .4}$$

I) If $\gamma \neq 0$ then from (3.4) we have

$$\pounds_V g_{ij} = 2\Omega g_{ij},$$

and from (3.3) we have

$$\pounds_V N^a_i = 0.$$

Using this and $N^h_i = y^m F^{\ h}_{m\ i}$ we get

$$0 = \pounds_V N^h_i = \pounds_V(y^m F^{\ h}_{m\ i}) = y^m\pounds_V F^{\ h}_{m\ i}. \tag{3 .5}$$

Where the last equality holds from equation (2.2.2).
II) If $\gamma = 0$ since $\alpha\gamma - \beta^2 \neq 0$ we have $\beta \neq 0$ so from (3.3) we have

$$\pounds_V g_{ij} = 2\Omega g_{ij},$$

and from (3.2) we have

$$g_{ai}\pounds_V N^a_j + g_{aj}\pounds_V N^a_i = 0.$$

Using this and equation (2.2.2) and $N^a_i = y^k F^{\ a}_{k\ i}$, we have

$$y^k(g_{ai}\pounds_V F^{\ a}_{k\ j} + g_{aj}\pounds_V F^{\ a}_{k\ i}) = 0. \tag{3 .6}$$

In each case I) and II) we have

$$\pounds_V g_{ij} = 2\Omega g_{ij}, \tag{3 .7}$$

or from equation (1.6)

$$v^m \partial_m g_{ij} + g_{mj} \partial_i v^m + g_{im} \partial_j v^m + y^a \partial_a v^m \dot{\partial}_m g_{ij} = 2\Omega g_{ij}.$$

Applying $\dot{\partial}_k$ to the both side of the above equation, we find;

$$2v^m \partial_m C_{ijk} + 2C_{mjk} \partial_i v^m + 2C_{imk} \partial_j v^m + 2\partial_k v^m C_{ijm} + 2y^a \partial_a v^m \dot{\partial}_k C_{ijm}.$$
$$= 2g_{ij} \dot{\partial}_k \Omega + 4\Omega C_{ijk}.$$

By using $y^i C_{ijk} = 0$, we obtain $\dot{\partial}_k \Omega = 0$. Therefore Ω is a function of x alone. From (2.2.4) we have

$$y^k (\nabla_k \pounds_V g_{ij} - \pounds_V \nabla_k g_{ij}) = y^k (g_{ai} \pounds_V F^a_{jk} + g_{aj} \pounds_V F^a_{ik}).$$

By using (3.5),(3.6) and (3.7) in each case I) and II) we find that

$$y^k \nabla_k \Omega = 0.$$

Since Ω is a function of x alone, we obtain $\partial_i \Omega = 0$. This together with connectedness of M, shows that Ω is constant. □

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