Mond-Weir duality in vector programming with generalized invex functions on differentiable manifolds

Massimiliano Ferrara and Ştefan Mititelu

Abstract. The main purpose of this paper is to develop a duality of Mond-Weir type for a vector mathematical program on a differentiable manifold. The components of the program objective are ρ-pseudoinvex functions and the constraint functions are ρ-quasiconvex and ρ-inquasimonotonic all defined on a differential manifold. The developed duality in this paper is based on weak, direct and converse duality theorems.

Mathematics Subject Classification: 26B25, 58A05, 58B20, 90C29, 90C46.
Key words: manifold, vector optimization, duality, generalized invexity.

1 Introduction

Let M be a differentiable manifold. We denote by T_pM the tangent space to M at p. Let also

$$TM = \bigcup_{p\in M} T_pM$$

be the tangent bundle of M.

Let N be another differentiable manifold and $\varphi : M \longrightarrow N$ a differentiable application.

Definition 1.1.[5, 10] The linear application defined by $d\varphi(v) = \varphi'(p)v$ is called the *differential* of φ at the point p.

We consider now an application $\eta : M \times M \longrightarrow TM$ such that $\eta(p, q) \in T_qM$ for every $q \in M$, where $p \in M$.

Let $F : M \to \mathbf{R}$ be a differentiable function. The differential of F at p, namely $dF_p : T_pM \to T_{F(p)}\mathbf{R} \equiv \mathbf{R}$, is introduced by

Balkan Journal of Geometry and Its Applications, Vol.11, No.1, 2006, pp. 80-87.

$$dF_p(v) = dF(p)v, \quad v \in T_pM$$

and for the Riemannian manifold (M, g) by

$$dF_p(v) = g_p(dF(p), v) \quad v \in T_pM,$$

where g is the Riemannian metric.

Let $\rho \in \mathbf{R}$ and d a distance function on M. If (M, g) is a Riemannian manifold, then d is the distance induced by the metric g.

Definition 1.2. The differentiable function F is said to be ρ - *invex* at $u \in M$ if there exists an application η such that (shortly F is called $(\rho, \eta)-$invex)

$$\forall x \in M : F(x) - F(u) \geqq dF(u)(\eta(x, u)) + \rho d^2(x, u).$$

Definition 1.3. The differentiable function F is said to be $\rho-$ *pseudoinvex* at $u \in M$ if there exists an application η such that (shortly F is (ρ, η)-*pseudo-invex*)

$$\forall x \in M : dF(u)(\eta(x, u)) + \rho d^2(x, u) \geqq 0 \Longrightarrow \quad F(x) \geqq F(u).$$

Definition 1.4. The differentiable function F is said to be $\rho-$*quasiinvex* at $u \in M$ if there exists an application η such that (shortly F is named $(\rho, \eta)-$ *quasiinvex*)

$$\forall x \in M : F(x) \leqq F(u) \Longrightarrow dF(u)(\eta(x, u)) + \rho d^2(x, u) \leqq 0.$$

Definition 1.5. [8] The differentiable function F is said to be $\rho-$ *inquasimonotonic* at $u \in M$ if there exists an application η such that (shortly F is $(\rho, \eta)-$ *inquasimonotonic*)

$$\forall x \in M : F(x) = F(u) \Longrightarrow dF(u)(\eta(x, u)) + \rho d^2(x, u) = 0.$$

The invex and generalized invex functions have the property that every local minimum point is a global minimum point [4].

Everywhere in this paper the relations $u = v, u < v, u \leqq v, u \leq v$ etc between two vectors $u = (u_1, ..., u_n)'$ and $v = (v_1, ..., v_n)'$ are equivalent to

$$\begin{aligned} u &= v \Longleftrightarrow u_i = v_i, i = \overline{1, n}; \\ u &< v \Longleftrightarrow u_i < v_i, i = \overline{1, n}; \\ u &\leqq v \Longleftrightarrow u_i \leqq v_i, i = \overline{1, n}; \\ u &\leq v \Longleftrightarrow u \leqq v, u \neq v, \end{aligned}$$

respectively, where we denoted by $'$ the transposition sign.

The paper is divided in three sections. Sections 1 is an introduction. Section 2 presents the study object of the paper that is the multiobjective mathematical program (PV) on a differentiable manifold. An efficiency solution is defined and efficiency conditions for the program (PV) are given. Section 3 contains the main result of the paper. Here is developed a duality of Mond-Weir-type through weak, direct and converse duality theorems.

2 Main results: efficiency conditions on manifolds

Let us consider the vector functions $f = (f_1, ..., f_p)' : M \to \mathbf{R}^p, g = (g_1, ..., g_m)' : M \to \mathbf{R}^m$ and $h = (h_1, ..., h_q)' : M \to \mathbf{R}^q$, all differentiable on M. A minimization vector program on M is the following Pareto extremum problem:

$$(VP) \qquad \begin{cases} Minimize & f(x) = (f_1(x), ..., f_p(x))' \\ subject\,to & g(x) \leqq 0, h(x) = 0, x \in M. \end{cases}$$

The domain of this program is the set

$$D_{VP} = \{x \in M \mid g(x) \leqq 0, h(x) = 0\}.$$

Definition 2.1. [2] A feasible point $x^0 \in D_{VP}$ is said to be a Pareto minimum point, or an *efficiency solution* (minimum) of (*VP*) if there exists no other point $x \in D_{VP}$ such that $f(x) \leq f(x^0)$.

In this paper we develop a Mond-Weir duality [8] for the program (*VP*). In order to achieve this aim necessary efficiency conditions of Kuhn-Tucker type relative la (PV), are used. Mititelu established necessary efficiency conditions for vector programs in a locally convex space [6]. But the manifold M can be organized as a particular locally convex space as follows. First, using the distance d, the pair (M, d) is a metric space (M, d). We endow this space with the topology τ which is generated by open balls with respect to d. It follows a topological space that is Hausdorff separated. Now, we define on this space an algebraic structure of linear space that is compatible to τ and then the manifold M becomes locally a local convex space. Within this mathematical framework we consider the program (VP) and for a point $x^0 \in D_{VP}$ we define the set $J^0 = \{\{1, ..., m\} \mid g_j(x^0) = 0\}$.

Definition 2.2. The point x^0 is *regular* for (VP) if the domain D_{VP} verifies at x^0 the constraint

$$R(x^0) : d(g_{j^0})_{x^0}(v) \leqq 0, dh_{x^0}(v) = 0, \quad \forall j \in J^0.$$

Here $d(g_{J^0})_{x^0}(v)$ is the vector of components $d(g_j)_{x^0}(v), \forall j \in J^0$, taken in the increasing order of j and $dh_{x^0}(v) = (d(h_1)_{x^0}(v), ..., d(h_q)_{x^0}(v))'$.

Now we can introduce necessary efficiency conditions for (VP) at x^0, above announced:

Theorem 2.1.(Corollary 2.2.[6]). *Let* $x^0 \in D_{VP}$ *be an efficient solution of* (VP), *where the functions* f, g *and* h *are differentiable.*

We also suppose that the constraint qualification $R(x^0)$ *is satisfied.*

Then there are vectors $t^0 = (t^{01}, ..., t^{0p})' \in \mathbf{R}^p, y^0 = (y^{01}, ..., y^{0m})' \in \mathbf{R}^m$ *and* $z^0 = (z^{01}, ..., z^{0q})' \in \mathbf{R}^q$ *such that the following efficiency conditions of Kuhn-Tucker type at* x^0 *are satisfied by* (VP)*:*

$$(KT) \qquad \begin{cases} t^{0^i} df_i(x^0) + y^{0j} dg_j(x^0) + z^{0k} dh_k(x^0) = 0 \\ y^{0j} g_j(x^0) = 0, \ y^0 \geqq 0 \\ t^0 \geq 0, e't^0 = 1, \quad e = (1, ..., 1)' \in R^p. \end{cases}$$

3 A Mond-Weir duality for the program (VP)

We define the sets $P = \{1, ..., p\}$, $S = \{1, ..., m\}$ and $Q = \{1, ..., q\}$. Let $\{S_0, S_1, ..., S_r\}$ be a partition of S, that is

$$S_\alpha \subseteq S, S_\alpha \cap S_\beta = \varnothing \text{ if } \alpha \neq \beta, \bigcup_{\alpha=0}^{r} S_\alpha = S$$

and $\{Q_0, Q_1, ..., Q_r\}$ be a similar defined partition of Q.

We remind that all the functions of the program (VP) are differentiable on M. The generalized Mond-Weir dual program associated to (VP) is the following Pareto extremum problem on manifold M:

$$(WMD) \quad \begin{cases} Maximize & L(u, y, z) = f(u) + \left[y'_{S_0} g_{S_0} + z'_{Q_0} h_{Q_0}\right] e \\ subject\ to & t^i df_i(u) + y^j dg_j(u) + z^k dh_k(u) = 0 \\ & y'_{S_\alpha} g_{S_\alpha}(u) + z'_{Q_\alpha} h_{Q_\alpha}(u) \geqq 0, \alpha = \overline{1, r} \\ & u \in M, \quad t \geq 0, \quad e't = 1, \quad y \geqq 0. \end{cases}$$

where for $\alpha = \overline{1, r}$ we introduce the notations:

$$y'_{S_\alpha} g_{S_\alpha}(u) = \sum_{j \in S_\alpha} y^j g_j(u) \quad , \quad z'_{Q_\alpha} h_{Q_\alpha}(u) = \sum_{k \in Q_\alpha} z^k h_k(u).$$

We denote by D_{WMD} the domain of dual program (WMD). For the pair of vector programs (VP) and (WMD) we develop a duality theory through weak, direct and converse duality theorems.

Theorem 3.1. (Weak duality). *Let* x *and* (u, t, y, z) *be arbitrary feasible solutions of the dual programs (VP) and (WMD).*

Assume that following conditions are satisfied:

a) for each $i \in P$, f_i *is* $(\rho'_i, \eta) - pseudoinvex$ *at* u;

b) for each $j \in S$, g_j *is* $(\rho''_j, \eta)-$quasiinvex *at* u;

c) for each $k \in Q$, h_k *is* $(\rho'''_k, \eta)-$inquasimonotonic at u;

d) $t^i\rho'_i + y^j\rho''_j + z^k\rho'''_k \geqq 0.$

Then the relation $f(x) \leq L(u,y,z)$ *is false.*

Proof. We suppose, by absurdum, that the relation $f(x) \leq L(u,y,z)$ is true. Then it follows

$$t^i f_i(x) \leqq t^i f_i(u) + y'_{S_0} g_{S_0}(u) + z'_{Q_0} h_{Q_0}(u).$$

From this inequality and $x \in D_{VP}$ and $(u,t,y,z) \in D_{WMD}$, we obtain

$$t^i f_i(x) + y^j g_j(x) + z^k h_k(x) \leqq t^i f_i(u) + y^j g_j(u) + z^k h_k(u).$$

From a), b) and c) we obtain, respectively:

$$df_i(u)(\eta(x,u)) + \rho d^2(x,u) \geqq 0 \Longrightarrow f(x) \geqq f(u),$$

or equivalently,

$$f_i(x) < f_i(u) \Longrightarrow df_i(u)(\eta(x,u)) + \rho'_i d^2(x,u) < 0, \tag{3.2}$$

$$g_j(x) \leqq g_j(u) \Longrightarrow dg_j(u)(\eta(x,u)) + \rho''_j d^2(x,u) \leqq 0, \tag{3.3}$$

$$h_k(x) = h_k(u) \Longrightarrow dh_k(u)(\eta(x,u)) + \rho'''_k d^2(x,u) = 0. \tag{3.4}$$

Multiplying now (3.2), (3.3) and (3.4) by t^i, y^j and z^k respectively, summing by i, j and k and then, summing side by side the obtained relations, it results

$$t^i f_i(x) + y^j g_j(x) + z^k \leq t^i f_i(u) + y^j g_j(u) + z^k h_k(u) \Longrightarrow \tag{3.5}$$

$$\Longrightarrow (t^i df_i(u) + y^j dg_j(u) + z^k dh_k(u))(\eta(x,u)) + (t^i\rho'_i + y^j\rho''_j + z^k\rho'''_k)d^2(x,u) < 0$$

Taking into account the first constraint of (WMD) and of the condition d) of the theorem, we infer that (3.5) implies $0 < 0$, that is a contradiction.

It follows that the supposition, above made, is false. □

Corollary 3.1. (Weak duality). *Let* x *and* (u,t,y,z) *be arbitrary feasible solutions of the dual programs (VP) and (WMD).*

Assume that the following conditions are satisfied:

a) *for each* $i \in P, f_i$ *is* $(\rho'_i, \eta)-$*pseudoinvex at* u*;*

b) *for each* $\alpha \in \overline{1,r}, y'_{S_\alpha} g_{S_\alpha} + z'_{Q_\alpha} h_{Q_\alpha}$ *is* $(\overline{\rho}_\alpha, \eta)-$*quasiinvex at* u*;*

c) $t^i\rho'_i + \sum_{\alpha=1}^{r}\overline{\rho}_\alpha \geqq 0.$

Then the relation $f(x) \leq L(u,y,z)$ is false.

Theorem 3.2. (Direct duality). *Let* x^0 *be a regular efficient solution of (VP) and suppose satisfied the hypotheses of Theorem 3.1. Then there are vectors* $t^0 \in \mathbf{R}^p, y^0 \in$

$\mathbf{R}^m$ *and* $z^0 \in \mathbf{R}^q$ *such that* (x^0, t^0, y^0, z^0) *is an efficient solution for the dual (WMD) and moreover,* $f(x^0) = L(x^0, y^0, z^0)$.

Proof. Because x^0 is a regular efficient solution of (VP) then, according to Theorem 2.1, there are vectors $t^0 \in \mathbf{R}^p, y^0 \in \mathbf{R}^m$ and $z^0 \in \mathbf{R}^q$ such that the following efficiency conditions of Kuhn-Tucker type are satisfied:

$$\begin{cases} t^{0i} df(x^0) + y^{0j} dg(x^0) + z^{0k} dh(x^0) = 0 \\ y^{0j} g_j(x^0) = 0, y^0 \geq 0 \\ t^0 \geqq 0, e' t^0 = 1. \end{cases}$$

From the relations $y^{0j} g_j(x^0) = 0$ and $z^{0k} h_k(x^0) = 0$ it follows

$$y^{0j} g_j(x^0) + z^{0k} h_k(x^0) = 0, \quad \forall j \in S_\alpha, \forall k \in Q_\alpha,$$

or equivalently,

$$y^0_{S_\alpha}{'} g_{S_\alpha}(x^0) + z^0_{Q_\alpha}{'} h_{Q_\alpha}(x^0) = 0.$$

Therefore $(x^0, t^0, y^0 z^0) \in D_{WMD}$ and moreover, $f(x^0) = L(x^0, y^0, z^0)$.

By using the hypotheses of Theorem 3.1 it results that the relation $f(x^0) \leq L(u, y, z), \forall (u, t, y, z) \in D_{WMD}$ is false. Since $y^0_{S_\alpha}{'} g_{S_\alpha}(x^0) \leqq 0, \quad z^0_{Q_\alpha}{'} h_{Q_\alpha}(x^0) = 0$ we infer that doesn't exist $(u, t, y, z) \in D_{WMD}$ such that $L(x^0, y^0, z^0) \leq L(u, y, z)$. Therefore $(x^0, t^0 y^0 z^0)$ is a (maximally) efficient solution for the dual program (WMD). □

Corollary 3.2. (Direct duality). *Let* x^0*be a regular efficient solution of (VP) and suppose satisfied the hypotheses of Corollary 3.1. Then there are vectors* $t^0 \in \mathbf{R}^p, y^0 \in \mathbf{R}^m$ *and* $z^0 \in \mathbf{R}^q$ *such that* (x^0, t^0, y^0, z^0) *is an efficient solution for the dual (WMD) and moreover,* $f(x^0) = L(x^0, y^0 z^0)$.

Theorem 3.3. (Converse duality). *Let* (x^0, t^0, y^0, z^0) *be an efficient solution of (WMD). We suppose that the following conditions are satisfied:*

(i) $\overline{x}$ *is a regular efficient solution of (VP);*

(a^0) *for each* $i \in P$, *the function* f_i *is* $(\rho'_i, \eta)-$*pseudoinvex at* x^0;

(b^0) *for each* $j \in S$, *the function* g_j *is* $(\rho''_j, \eta)-$*quasiinvex at* x^0;

(c^0) *for each* $k \in Q$, *the function* h_k *is* $(\rho'''_k, \eta)-$*inquasimonotonic at* x^0*;*

(d^0) $t^{0^i} \rho'_i + y^{0j} \rho''_j + z^{0k} \rho'''_k \geqq 0.$

Then $\overline{x} = x^0$ *and moreover,* $f(x^0) = L(x^0, y^0 z^0)$.

Proof. We suppose, by absurdum, that $\overline{x} \neq x^0$. Because $\overline{x}$ is a regular efficient function of (VP), according to Theorem 2.1, there are vectors $\overline{t} \in \mathbf{R}^p, \overline{y} \in \mathbf{R}^m$ and $\overline{z} \in \mathbf{R}^q$ such that the following efficiency conditions of Kuhn-Tucker type are satisfied:

$$\begin{cases} \overline{t} df_i(\overline{x}) + \overline{y}^j dg_j(\overline{x}) + \overline{z}^k dh_k(\overline{x}) = 0 \\ \overline{y}^j g_j(\overline{x}) = 0, \quad \overline{y} \geqq 0 \\ \overline{t} \geq 0, \quad e' \overline{t} = 1. \end{cases}$$

From these conditions we obtain

$$\overline{y}'_{S_\alpha} g_{S_\alpha}(\overline{x}) + \overline{z}'_{Q_\alpha} h_{Q_\alpha}(\overline{x}) = 0, \quad \alpha = \overline{1, r}. \tag{3.6}$$

Therefore $(\overline{x}, \overline{t}, \overline{y}, \overline{z}) \in D_{WMD}$ and moreover,

$$f(\overline{x}) = L(\overline{x}, \overline{y}, \overline{z}). \tag{3.7}$$

According to Theorem 3.1 it follows that the relation

$$f(\overline{x}) \leq L(x^0, y^0, z^0) \tag{3.8}$$

is false.

Multiplying (3.6) by e and summing side by side the obtained relations and then, summing side by side the obtained relation with (3.8), it results that the following relation

$$L(\overline{x}, \overline{y}, \overline{z}) \leq L(x^0, y^0, z^0) \tag{3.9}$$

is false.

But (x^0, t^0, y^0, z^0) is a (maximally) efficient solution of (WMD) and then, the relation

$$L(\overline{x}, \overline{y}, \overline{z}) \geq L(x^0, y^0, z^0) \tag{3.10}$$

is false, too.

We remark that relations (3.9) and (3.10) are contradictory. Consequently, $\overline{x} = x^0$ and in addition, $L(\overline{x}, \overline{y}, \overline{z}) = L(x^0, y^0, z^0)$. By using now relation (3.7) we obtain

$$f(x^0) = L(x^0, y^0, z^0).$$

□

Corollary 3.3. (Converse duality). *Let $(x^0, t^0, y^0 z,^0\,)$ be an efficient solution of (WMD). We suppose that the next conditions are satisfied:*

(i) *$\overline{x}$ is a regular efficient solution of (VP);*

(a^0) *for each $i \in P, f_i$ is $(\rho'_i, \eta)-$pseudoinvex at x^0;*

(b^0) *for each $\alpha = \overline{1, r}$, $\;y^0_{S_\alpha}{}' g_{S_\alpha} + z^0_{Q_\alpha}{}' h_{Q_\alpha}$ is $(\overline{\rho}_\alpha, \eta)-$quasiinvex at x^0;*

(c^0) $t^{0^i} \rho'_i + \sum_{\alpha=1}^{r} \overline{\rho}_\alpha \geqq 0.$

Then $\overline{x} = x^0$ and $f(x^0) = L(x^0, y^0, z^0)$.

References

[1] M. Ferrara, *η-invex-type functions on differentiable manifolds in optimization problems,* Rendiconti del Seminario Matematico di Messina, Serie II, 6 (1999), 155-163.

[2] A. M. Geoffrion, *Proper efficiency and theory of vector maximization,* J. Math. Anal. Appl. 22 (1968), 618-630.

[3] E. Miglierina, *Invex functions on differentiable manifold.* "In Generalized convexity and optimization for economic and financial decisions", edited by G. Giorgi and A. Guerraggio, Pitagora Editrice Bologna, 1999.

[4] S. Mititelu, *Invex functions,* Rev. Roumaine Math. Pures Appl., 49 (2005), 5-6, 529-544.

[5] S. Mititelu, *Generalized invexity and vector optimization on differentiable manifolds,* Differential Geometry-Dynamical Systems, 3 (2001), 1, 21-31; http://www.mathem.pub.ro/dgds/v3n1/v3n1.htm

[6] S. Mititelu, *Efficiency and duality in multiobjective nonsmooth programming,* Proceedings of The 3-rd International Colloquium of Mathematics in Engineering and Numerical Physics (MENP-3) 7-9 october 2004, Politehnica University of Bucarest, Romania.

[7] S. Mititelu, I. M. Stancu-Minasian, *Invexity at a point: generalisations and classifications,* Bull. Austral. Math. Soc., 48 (1993), 117-126.

[8] S. Mititelu, I. M. Stancu-Minasian, *Optimality and algorithm for generalized invex Chebyshev problem,* Invextigação Operational, 17(1997), 151-161.

[9] B.Mond-T. Weir, *Generalized concavity and duality,* in "Generalized Concavity in Optimization and Economic", edited by S. Schaible and T. Ziemba, pp. 263-279, Academic Press, New York, 1981.

[10] R. Pini, *Convexity along curves and invexity,* Optimization, 29 (1994), 301-309.

[11] T. Rapcsák, *Geodesic convexity in nonlinear optimization,* JOTA, 69 (1991), 1, 169-183.

[12] C. Udrişte, *Convex functions and optimization methods on Riemannian manifolds,* Mathematics and Applications, 297, Kluwer Academic Publishers, 1994.

[13] C. Udriste, M. Ferrara, D. Opris, *Economic Geometric Dynamics,* Geometry Balkan Press, 6, 2004.

Authors' addresses:

Massimiliano Ferrara
Department of Economics, Statistics, Mathematics and Sociology,
"V. Pareto", Faculty of Economics, University of Messina,
Via dei Verdi, 75, 98122 Messina, Italy.
email: mferrara@unime.it

Stefan Mititelu
Technical University of Civil Engineering of Bucharest,
Department of Mathematics and Informatics,
124 Lacul Tei, 020396 Bucharest, Romania.
email: st_mititelu@yahoo.com