Generalization of Hashiguchi–Ichijyō's Theorems to Wagner–type manifolds

M.M. Rezaii and M. Barzegari

Abstract. We introduced a class of conformally invariant Ehresmann connections so–called L-horizontal endomorphism in [7]. Using this class, we define conformally invariant manifolds: Wagner–type manifold and locally Minkowski–type manifold as special generalized Berwald manifolds. Then a generalization of Hashiguchi–Ichijyō's Theorems to Wagner–type manifolds is presented.

Mathematics Subject Classification: 53C60.
Key words: L–horizontal endomorphism, generalized Berwald maifold, Wagner–type manifold, locally Minkowski–type manifold.

1 Introduction

In [5] M. Hashiguchi and Y. Ichijyō have explored the significance of Wagner manifolds relating them to the conformal change. One of the most important observations in [5] is that the class of Wagner manifolds is closed under the conformal change. In [4], Hashiguchi suggested and (in some sense!) solved the problem: under what conditions does a Finsler manifold become conformal to a Berwald (or a locally Minkowski) manifold. In [14] Cs. Vincze presents intrinsic version of Hashiguchi–Ichijyō's theorem for Wagner manifolds. In this paper, we introduce Wagner–type manifolds as a generalization of Wagner manifolds by using a class of Ehresmann connections so–called L–horizontal endomorphisms which are closed under conformal change. Then we prove generalization of Hashiguchi–Ichijyō's Theorems to Wagner–type manifolds. In last section, we introduce and study locally Minkowski–type manifolds. The main result of this section is to show the conformally closeness of the locally Minkowski–type manifolds.

2 Preliminary

We work on an n-dimensional connected smooth manifold M whose topology is Hausdorff and has a countable base. $C^\infty(M)$ denotes the ring of smooth real-valued functions on M, $\mathfrak{X}(M)$ stands for the $C^\infty(M)$-module of (smooth) vector fields on M.

Balkan Journal of Geometry and Its Applications, Vol.11, No.1, 2006, pp. 125-133.

$\Omega(M) := \oplus_{i=0}^{n}\Omega^k(M)$ is the graded algebra of differential forms on M, with multiplication given by the wedge product. The symbols d, i_X, $\mathcal{L}_X$ $(X \in \mathfrak{X}(M))$ denote the exterior derivative, the substitution operator and the Lie derivative.

TM is the $2n$-dimensional tangent manifold of M, $\mathcal{T}M \subset TM$ is the open submanifold of the non-zero tangent vectors to M. f^v and f^c stand for the vertical and the complete lift of a smooth function f on M into TM.

For any vector field X on M there exist unique vector fields X^v,X^c on TM such that

$$X^v f^c = (Xf)^v, \quad X^c f^c = (Xf)^c \quad (f \in C^\infty(M)). \tag{2.2.1}$$

X^v is the *vertical lift*, X^c is the *complete lift* of X. The $C^\infty(TM)-$module of vertical fields on TM will be denoted by $\mathfrak{X}^v(TM)$. The *Liouville vector field* $C \in \mathfrak{X}^v(TM)$ is generated by the flow of positive dilatation $\mathfrak{p}_t : v \in TM \longmapsto \mathfrak{p}_t(v) := e^t v \in TM \quad (t \in \mathbb{R})$. Notice that

$$[C, X^v] = -X^v, \quad (X \in \mathfrak{X}(M)). \tag{2.2.2}$$

By a vector $k-$form on TM we mean a skew symmetric $C^\infty(TM)$-multilinear map $K : (\mathfrak{X}(TM)^k \to \mathfrak{X}(TM)$ if $k \in \{1, 2, \dots, 2n\}$, and a vector field on TM, if $k = 0$. In particular, a vector $1-$form on TM is just a type $(1,1)$ tensor field. The $C^\infty(TM)-$module of vector $k-$forms on TM will be denoted by $\Psi^k(TM)$. There is a unique vector $1-$form $J \in \Psi^1(TM)$ such that

$$JX^v = 0, \quad JX^c = X^v, \quad (X \in \mathfrak{X}(M). \tag{2.2.3}$$

J is called the *vertical endomorphism*. Clearly, J is of rank n and $J^2 = 0$. A vector form $K \in \Psi^k(TM)$ is *semibasic*, if $i_{J\xi}K = 0$ and $J \circ K = 0$ $(k \geq 1, \xi \in \mathfrak{X}(TM))$.

We recall that if θ_r and θ_s are graded derivation of degree r and s, resp. of a graded algebra, then their *graded commutator* is defined by

$$[\theta_r, \theta_s] := \theta_r \circ \theta_s - (-1)^{rs}\theta_s \circ \theta_r. \tag{2.2.4}$$

Then $[\theta_r, \theta_s]$ is a graded derivation of degree $r + s$. By the *Frölicher-Nijenhuis theory* of vector forms to any vector k–form $K \in \Psi^k(TM)$ two graded derivations of $\Omega(TM)$ are associated, denoted by i_K and d_K. i_K is of degree $k - 1$, d_K is of degree k, and the following rules are prescribed:

$$i_K \restriction C^\infty(TM) = 0; \quad i_K \circ \alpha = \alpha \circ K, \quad \text{if } \alpha \in \Omega^1(TM); \tag{2.2.5}$$

$$d_K := [i_K, d] = i_K \circ d - (-1)^{k-1} d \circ i_K. \tag{2.2.6}$$

Then, in particular, for all $F \in C^\infty(TM), K \in \Psi^k(TM)$ we have $d_K F = dF \circ K$. For vector 0-forms $\xi \in \Psi^0(TM) = \mathfrak{X}(TM)$, i.e., for vector fields on TM, i_ξ and d_ξ reduce to the usual substitution operator and Lie derivative, respectively. To any vector forms $K \in \Psi^k(TM)$, $L \in \Psi^\ell(TM)$ there is a unique vector $(k+l)-$form $[K, L] \in \Psi^{k+l}(TM)$, the *Frölicher-Nijenhuis bracket* of K and L such that

$$d_{[K,L]} = [d_K, d_L].$$

In this paper we are going to systematically use the Frölicher-Nijenhuis calculus of vector forms. A detailed account on the theoretical background can be found e.g.

in monographs ([6]), ([11]), and (of course) in the original source ([2]). Let K be a vector 1-form and β a differential 1-form. Then the following important formula can be deduced:

$$[K, \beta \otimes X] = d_K\beta \otimes X - d\beta \otimes KX - \beta \wedge [K, X], \tag{2.2.7}$$

$(X \in \mathfrak{X}(M))$.

3 Conformal change of L-horizontal endomorphisms

A vector 1–form $\mathbf{h} \in \Psi^1(TM)$, smooth –in general– only over $\mathcal{T}M$ is said to be a *horizontal endomorphism* (or *Ehresmann connection*) over M if it is a projector (i.e., $\mathbf{h}^2 = \mathbf{h}$) and $\ker \mathbf{h} = \mathfrak{X}^v(TM)$. $\mathbf{h}$ is called homogeneous if $[C, \mathbf{h}] = 0$. The (*strong*) *torsion* of $\mathbf{h}$ is the vector 2–form $\Omega := -\frac{1}{2}[\mathbf{h}, \mathbf{h}]$. The mapping

$$X \in \mathfrak{X}(TM) \mapsto X^{\mathbf{h}} := \mathbf{h}X^c \in \mathfrak{X}(\mathcal{T}M) \tag{3.3.1}$$

is called the *horizontal lifting* determined by horizontal endomorphism $\mathbf{h}$.

Suppose that ∇ is a linear connection on the manifold M. It is well–known that ∇ induces a homogeneous horizontal structure $\mathbf{h}_\nabla \in \Psi^1(TM)$, which is smooth on the whole tangent manifold TM. In this case

$$\forall\ X, Y \in \mathfrak{X}(M):\ \left(\nabla_X Y\right)^v = [X^{\mathbf{h}_\nabla}, Y^v].$$

By Lemma 1.5 of ([8]), If two homogeneous horizontal endomorphisms $\mathbf{h}_1$ and $\mathbf{h}_2$ on M satisfy the relation

$$[X^{\mathbf{h}_1}, Y^v] = [X^{\mathbf{h}_2}, Y^v], \tag{3.3.2}$$

for any vector fields X, Y on M, then $\mathbf{h}_1 = \mathbf{h}_2$. Thus $\mathbf{h}_\nabla$ is unique.

Let a function $E : TM \to \mathbb{R}$ be given. The pair (M, E) is said to be a Finsler manifold if the following conditions are satisfied:

(F 1) For any vector $v \in \mathcal{T}M, E(v) > 0,\ E(0) = 0$.

(F 2) E is of class C^1 on TM and smooth over $\mathcal{T}M$.

(F 3) $CE = 2E$, i.e, E is homogeneous of degree 2.

(F 4) The *fundamental form* $\omega := d\,d_J E$ is symplectic.

Due to the nondegeneracy of ω, for any 1–form $\beta \in \Omega^1(TM)$ there is unique vector field $\beta^\#$ on TM (smooth, in general, only on $\mathcal{T}M$) such that

$$i_{\beta^\#}\,\omega = \beta. \tag{3.3.3}$$

This map $\# : \beta \to \beta^\#$ is called the (Finslerian) *sharp operator*. In particular, the *gradient* of a function $F \in C^\infty(TM)$ is the vector field *grad* $F := (dF)^\#$.

For every Finsler manifold there is a horizontal endomorphism $\mathbf{h}_0$ on M, called the *Barthel endomorphism*. The Barthel endomorphism is homogeneous, conservative (i.e., $d_{\mathbf{h}_0}E = 0$) and torsion free (i.e., $[J, \mathbf{h}_0] = 0$).

Let L be a semibasic vector 1–form on TM. The horizontal endomorphism

$$\mathbf{h}_L := \mathbf{h}_0 + L + [J, (d_L E)^{\#}] \tag{3.3.4}$$

is called *L–horizontal endomorphism* on Finsler manifold (M, E).

Wagner endomorphism $\overline{\mathbf{h}}$ on Finsler manifold (M, E) is conservative and

$$\overline{\mathbf{h}} = \mathbf{h}_{\frac{1}{2}\alpha^c J - \frac{1}{2}d\alpha^v \otimes C} = h_0 + \alpha^c J - E[J, grad\,\alpha^v] - d_J E \otimes grad\,\alpha^v, \tag{3.3.5}$$

(see [7], [15]).

Let α be a smooth function on M and define a positive function on TM by

$$\varphi := exp \circ \alpha^v. \tag{3.3.6}$$

If $\widetilde{E} := \varphi E$, then $(M, \widetilde{E})$ is also a Finsler manifold (see [14] Lemma 1). We say that $(M, \widetilde{E})$ has been obtained by a *conformal change* of E given by the *scale function* φ. It is known ([7]) that the set of all conservative L–horizontal endomorphism is invariant under conformal change with scale function (3.3.6). L–horizontal endomorphism $\widetilde{\mathbf{h}}_L$ of $(M, \widetilde{E})$ are related to the corresponding data of (M, E) by

$$\widetilde{\mathbf{h}}_L = \mathbf{h}_L - \frac{1}{2}(\alpha^c J + d\alpha^v \otimes C) + \frac{1}{2}E[J, grad\,\alpha^v] + \frac{1}{2}d_J E \otimes grad\,\alpha^v. \tag{3.3.7}$$

4 Wagner–type manifolds

Suppose that (M, E) is a Finsler manifold and let ∇ be a linear connection on M. The triplet (M, E, ∇) is said to be a *generalized Berwald manifold* if horizontal endomorphism $\mathbf{h}_\nabla$ is conservative, i.e., $d_{\mathbf{h}_\nabla} E = 0$, ([8]).

Suppose that (M, E, ∇) and $(M, E, \overline{\nabla})$ are generalized Berwald manifolds. The linear connections ∇ and $\overline{\nabla}$ are equal if and only if $\mathbf{h}_\nabla = \mathbf{h}_{\overline{\nabla}}$ ([9]).

Sz. Szakál and J. Szilasi have shown in ([9]) that (M, E, ∇, α) is a Wagner manifold if and only if the horizontal endomorphism h_∇ is of form

$$\mathbf{h}_\nabla = \mathbf{h}_0 + \alpha^c J + E[J, grad\,\alpha^v] - d_J E \otimes grad\,\alpha^v. \tag{4.4.1}$$

Next we consider a quite natural generalization.

Definition 1. A quadruple (M, E, ∇, L) is said to be *Wagner–type manifold with respect to* L if (M, E, ∇) is a generalized Berwald manifold, and $\mathbf{h}_\nabla$ is the L–horizontal endomorphism.

Remark. The linear connection of a Wagner–type manifold with respect to the semibasic vector one-form L is clearly unique.

Lemma 1.

(i) A Berwald manifold (M, E) is a Wagner–type manifold with respect to 0.

(ii) *A Wagner manifold (M,E) with respect to α (α is smooth function on M) is a Wagner–type manifold with respect to*

$$w_\alpha := \frac{1}{2}\,(\,\alpha^c J - d\alpha^v \otimes C).$$

Proof. (i) By Definition 6.5 and Remarks 6.6(a) of ([12]) A Finsler manifold (M, E) is said to be a Berwald manifold if there is a linear connection ∇ on M such that the horizontal endomorphism induced by ∇ is just the Barthel endomorphism, i.e., $\mathbf{h}_\nabla = \mathbf{h}_0$.
(ii) By 4.2 Finsler manifold (M, E, ∇, α) is a Wagner manifold if and only if the horizontal endomorphism h_∇ is of form

$$\mathbf{h}_\nabla = \mathbf{h}_0 + \alpha^c J + E[J, \mathrm{grad}\,\alpha^v] - d_J E \otimes \mathrm{grad}\,\alpha^v \overset{(3.3.5)}{=} \mathbf{h}_{w_\alpha}. \tag{4.4.2}$$

It proves what we want. □

Next we gather together some equivalent definition for Wagner–type manifolds.

Proposition 1. *Let (M, E) be a Finslermanifold, L be a semibasic vector one–form on TM and $\mathbf{h}_L$ is consevative. Suppose ∇ is a linear connection on M. Then following conditions are equivalent:*

(1) *(M, E, ∇, L) is a Wagner–type manifold.*

(2) *For each vector fields X, Y on M,*

$$(\nabla_X Y)^v = [X^{\mathbf{h}_L}, Y^v].$$

(3) *For all vector fields X, Y on M, $[X^{\mathbf{h}_L}, Y^v]$ is a vertical lift.*

Proof. It is evident by Lemma 6.7 of ([12]), Definition 1 and (3.3.2). □

Cs. Vincze in ([14]) prove an intrinsic version of Hashiguchi–Ichijyō's Theorem for Wagner manifolds, here we state and prove our main result, generalization of this theorem for Wagner–type manifolds.

Theorem 1. *Let (M,E) be a Wagner–type manifold with respect to L and let us consider the conformal change given by the scale function (3.3.6). Then the Finsler manifold $(M, \widetilde{E})$ is a Wagner–type manifold with respect to $L + \frac{1}{2}\,w_\alpha$.*

Proof. We have

$$\mathbf{h}_{(L+\frac{1}{2}\,w_\alpha)} = \mathbf{h}_L + \frac{1}{2}\,(\mathbf{h}_{w_\alpha} - \mathbf{h}_0)$$

therefore $\mathbf{h}_{(L+\frac{1}{2}\,w_\alpha)}$ and consequently $\widetilde{\mathbf{h}}_{(L+\frac{1}{2}\,w_\alpha)}$ is conservative by (3.3.7). It is sufficient to show that for all $X, Y \in \mathfrak{X}(M)$ the vector field $[X^{\widetilde{\mathbf{h}}_{(L+\frac{1}{2}\,w_\alpha)}}, Y^v]$ is vertical lift.
A direct computation yields

$$
\begin{aligned}
\widetilde{\mathbf{h}}_{(L+\frac{1}{2}\,w_\alpha)} &\overset{(3.3.7)}{=} \mathbf{h}_{(L+\frac{1}{2}\,w_\alpha)} - \frac{1}{2}\,(\alpha^c J + d\alpha^v \otimes C) \\
&+ \frac{1}{2}\,E[J, grad\,\alpha^v] + \frac{1}{2}\,d_J E \otimes grad\,\alpha^v \\
&\overset{(3.3.4)}{=} \mathbf{h}_L + \frac{1}{2}\,\alpha^c J - \frac{1}{2}\,E[J, grad\,\alpha^v] \\
&- \frac{1}{2}\,d_J E \otimes grad\,\alpha^v - \frac{1}{2}\,(\alpha^c J + d\alpha^v \otimes C) \\
&+ \frac{1}{2}\,E[J, grad\,\alpha^v] + \frac{1}{2}\,d_J E \otimes grad\,\alpha^v \\
&= \mathbf{h}_L - \frac{1}{2}\,d\alpha^v \otimes C.
\end{aligned} \tag{4.4.3}
$$

By Proposition 1, the vector field $[X^{\mathbf{h}_L}, Y^v]$ is a vertical lift for all $X, Y \in \mathfrak{X}(M)$, since (M, E) is a Wagner–type manifold with respect to L. Then an straightforward computation implies that

$$
\begin{aligned}
[X^{\widetilde{\mathbf{h}}_{(L+\frac{1}{2}\,w_\alpha)}}, Y^v] &\overset{(3.3.1)}{=} [\widetilde{\mathbf{h}}_{(L+\frac{1}{2}\,w_\alpha)}(X^c), Y^v] \overset{(4.4.3)}{=} [\mathbf{h}_L(X^c), Y^v] \\
&- \frac{1}{2}\,[(d\alpha^v \otimes C)(X^c), Y^v] \overset{(3.3.1),(2.2.1)}{=} [X^{\mathbf{h}_L}, Y^v] \\
&- \frac{1}{2}\,[(X\alpha)^v C, Y^v] = [X^{\mathbf{h}_L}, Y^v] - \frac{1}{2}\,(X\alpha)^v [C, Y^v] \\
&\overset{(2.2.2)}{=} [X^{\mathbf{h}_L}, Y^v] + \frac{1}{2}\,(X\alpha)^v Y^v
\end{aligned}
$$

It means that $[X^{\widetilde{\mathbf{h}}_{(L+\frac{1}{2}\,w_\alpha)}}, Y^v]$ is a vertical lift. Applying Proposition 1, we conclude that $(M, \widetilde{E})$ is a Wagner–type manifold with respect to $L + \frac{1}{2}\,w_\alpha$. □

Definition 2. A Finsler manifold (M, E) is said to be conformal to a Wagner–type manifold with respect to L, if there is a conformal change $\widetilde{E} := \varphi E$ such that $(M, \widetilde{E})$ is a Wagner–type manifold with respect to L.

We are in the position to show the conforlmally closeness of Wagner–type manifolds.

Theorem 2. *A Finsler manifold is conformal to a Wagner–type manifold with respect to L if and only if it is a Wagner–type manifold with respect to* $L - \frac{1}{2}\,w_\alpha$ *for a smooth function* α *on* M.

Proof. Let us suppose that the Finsler manifold (M, E) is conformal to a Wagner–type manifold with respect to L, i.e., there is a conformal change $\widetilde{E} = \varphi E$ $(\varphi = exp \circ \alpha^v)$ such that $(M, \widetilde{E})$ is a Wagner–type manifold with respect to L. In view of Theorem 1 the conformal change $E = \frac{1}{\varphi}\,\widetilde{E}$ yields a Wagner–type manifold with respect to $L - \frac{1}{2}\,w_\alpha$. The converse is also true by Theorem 1. □

Let us mention some corollaries of the Theorem 1.

Corollary 1. *A Wagner–type manifold with respect to L is conformal to a Wagner–type manifold with respect to* $\bar{L}$ *with scale function (3.3.6) if and only if* $\bar{L} - L = \frac{1}{2}\,w_\alpha$.

Corollary 2. *A Finsler manifold is conformal to a Wagner manifold if and only if it is a Wagner manifold.*

Corollary 3. *A Finsler manifold is conformal to a Berwald manifold if and only if it is a Wagner manifold.*

5 Locally Minkowski–type manifolds

Definition 3. A Wagner–type manifold with respect to L is called *locally Minkowski–type manifold with respect to L* if

$$\Omega_L = 0, \tag{5.5.1}$$

where Ω_L is the (strong) torsion of $\mathbf{h}_L$.

Lemma 2. *A Locally Minkowski manifold is a locally Minkowski–type manifold with respect to* 0.

Proof. By Definition 7.1 of ([12]) a Berwald manifold (M, E) is said to be locally Minkowski manifold if $\Omega_0 = 0$. □

Theorem 3. *A Finsler manifold is conformal to a locally Minkowski–type manifold with respect to L if and only if for a smooth function α on M, it is a locally Minkowski–type manifold with respect to $L - \frac{1}{2}\, w_\alpha$.*

Proof. Let us suppose that the Finsler manifold (M, E) is conformal to a locally Minkowski–type manifold with respect to L, i.e., there is a conformal change $\widetilde{E} = \varphi E$ ($\varphi = exp \circ \alpha^v$) such that $(M, \widetilde{E})$ is a locally Minkowski–type manifold with respect to L. In view of Theorem 2, (M, E) is a Wagner–type manifold with respect to $L - \frac{1}{2}\, w_\alpha$. Since $(M, \widetilde{E})$ is a locally Minkowski–type manifold with respect to L thus $\widetilde{\Omega}_L = 0$, therefore we get

$$\begin{aligned}
-2\,\Omega_{(L-\frac{1}{2}\,w_\alpha)} &= \left[\mathbf{h}_{(L-\frac{1}{2}\,w_\alpha)}, \mathbf{h}_{(L-\frac{1}{2}\,w_\alpha)}\right] \\
&\overset{(4.4.3)}{=} \left[\widetilde{\mathbf{h}}_L + \frac{1}{2}\, d\alpha^v \otimes C, \widetilde{\mathbf{h}}_L + \frac{1}{2}\, d\alpha^v \otimes C\right] \\
&= \left[\widetilde{\mathbf{h}}_L, \widetilde{\mathbf{h}}_L\right] + \left[\widetilde{\mathbf{h}}_L, d\alpha^v \otimes C\right] + \frac{1}{4}\, \left[d\alpha^v \otimes C, d\alpha^v \otimes C\right] \\
&= \left[\widetilde{\mathbf{h}}_L, d\alpha^v \otimes C\right] + \frac{1}{4}\, \left[d\alpha^v \otimes C, d\alpha^v \otimes C\right] \\
&\overset{(2.2.7)}{=} d_{\widetilde{\mathbf{h}}_L} d\alpha^v \otimes C - d\alpha^v \wedge \left[\widetilde{\mathbf{h}}_L, C\right] \\
&\quad + \frac{1}{4}\, d_{\,d\alpha^v \otimes C}\, d\alpha^v \otimes C - \frac{1}{4}\, d\alpha^v \wedge \left[d\alpha^v \otimes C, C\right] = 0.
\end{aligned}$$

Thus (M, E) is a locally Minkowski–type manifold with respect to $L - \frac{1}{2}\, w_\alpha$. The converse is similar. □

Corollary 4. *A Finsler manifold is conformal to a locally Minkowski manifold if and only if for a smooth function α on M, it is a locally Minkowski–type manifold with respect to $\frac{1}{2}\, w_\alpha$.*

Corollary 5. *Following diagram is commutative.*

$$
\begin{array}{ccc}
\begin{array}{c} Wagner\text{–}type \\ w.r.t\ \ L \end{array} & \xrightarrow{\ \Omega_L = 0\ } & \begin{array}{c} Locally\ Minkowski\text{–}type \\ w.r.t\ \ L \end{array} \\
\varphi \downarrow \ \uparrow \varphi^{-1} & & \varphi^{-1} \uparrow \ \downarrow \varphi \\
\begin{array}{c} Wagner\text{–}type \\ w.r.t\ \ L + \frac{1}{2} w_\alpha \end{array} & \xrightarrow{\ \Omega_{(L + \frac{1}{2} w_\alpha)} = 0\ } & \begin{array}{c} Locally\ Minkowski\text{–}type \\ w.r.t\ \ L + \frac{1}{2} w_\alpha \end{array}
\end{array}
$$

where $\varphi := exp \circ \alpha^v$ *is the scale function of a conformal change.*

References

[1] M. de León and P. R. Rodrigues, *Methods of Differential Geometry in Analytical Mechanics* , North-Holland, Amsterdam, 1989.

[2] A. Frölicher and A. Nijenhuis, *Theory of vector–valued differential forms*, Proc. Kon. Ned. Akad. A. 59 (1956), 338-359.

[3] J. Grifone, *Structure presque tangente et connexions, I,* Ann. Inst. Fourier, Grenoble 22 no. 1 (1972), 287-334.

[4] M. Hashiguchi, *On conformal transformations of Finsler metrics*, Math. Kyoto Univ. 16 (1976), 25-50.

[5] M. Hashiguchi and Y. Ichijyō, *On conformal transformations of Wagner spaces*, Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.) no. 10 (1977), 19-25.

[6] I. Kolář, P. W. Michor and J. Slovák, *Natural Operations In Differential Geometry,* Springer-Verlag, Berlin, 1993.

[7] M. M. Rezaii, M. Barzegari and B. Najafi, *On a class of conformally invariant horizontal endomorphisms*, (to appear in Publ. Math. Debrecen).

[8] Sz. Szakál and J. Szilasi, *A new approach to generalized Berwald manifolds I*, SUT Journal of Mathematics 37 (2001), No. 1, 19-41.

[9] Sz. Szakál and J. Szilasi, *A new approach to generalized Berwald manifolds II*, Publ. Math. Debrecen 60/3-4 (2002), 429-453.

[10] J. Szilasi, *Notable Finsler connections on a Finsler manifold,* Lecturas Matemáticas 19 (1998), 7-34.

[11] J. Szilasi, *A Setting for Spray and Finsler Geometry*, in: Handbook of Finsler Geometry Vol. 2 (ed. P.L. Antonelli), Kluwer Academic Publishers, Dorecht, 2003.

[12] J. Szilasi and Cs. Vincze, *A new look at Finsler connections and special Finsler manifolds*, Acta Math. Acad. Paed. Nyiregyháziensis 16 (2000), 33-63.

[13] J. Szilasi and Cs. Vincze, *On conformal equivalence of Riemann-Finsler metrics*, Publ. Math. Debrecen 52 (1-2) (1998), 167-185.

[14] Cs.Vincze, *An instric version of Hashiguchi-Ichijyō's theorems for Wagner manifolds* , SUT Journal of Mathematics 35 (1999), No. 2, 263-270.

[15] Cs. Vincze, *On Wagner connections and Wagner manifolds*, Acta Math. Hungar 89 (2000), 111-133

Authors' address:

M.M. Rezaii and Mansoor Barzegari
Faculty of Mathematics and Computer Science,
Amirkabir University of Technology (Tehran Polytechnic),
424 Hafez Ave. 15914 Tehran, Iran.
email: mmreza@aut.ac.ir